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Fractal Analysis and Non-linear Multivariate Model of Keplers stars

Ingila Rahim
Mathematical Sciences Research Center,
Federal Urdu University of Arts Science and Technology, Pakistan,
Department of Mathematics,
Sir Syed University Of Engineering and Technology,
Email: ingila.rahim@gmail.com

M. Rashid Kamal Ansari,
Department of Mathematics,
Sir Syed University Of Engineering and Technology,
Email: rashidka2000@gmail.com

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Abstract. The stellar surface rotation is the indicator of internal dynamo process. The dynamo generates many active phenomena like stellar spots and also affect stellar evolution. In this study we have analyzed the surface rotation and evolutionary parameters such as mass, density, radius, effective temperature, surface gravity and metallicity of stars all together. Surface rotation is used in different forms such as rotational shear, relative differential rotation and average surface rotation. Approximately 12000 Keplers star data of rotation and above mentioned evolutionary parameters were used. Fractal dimension and the Hurst exponent technique is used to analyze the persistency and persistency correlation of data in different spectral classes. Strong persistency correlation is found between data of rotational shear vs effective temperature (0.992), rotational shear vs radius (-0.82) and average differential rotation vs metallicity (0.995). Additionally, correlation between other evolutionary parameters and rotational parameters are also found. On the basis of these correlations and physical relationship the nonlinear models between luminosity, radius and rotation (relative differential rotation and rotational shear) are established for main sequence.

AMS (MOS) Subject Classification Codes: 60E99; 60H99; 85A15
Key Words: Differential rotation, Stars spots, Steller evolution and Structure, Fractal Dimension..

1. INTRODUCTION

Observation asserts that the surface of a star is differentially rotating. The same is true in the particular case off Sun. It should be noted that the polar latitudes of the sun are rotating slower than the equator [1]. The internal rotation of the solar radiative core is like the rotation of a solid body. The differentially rotating radiative core and convective envelope create solar dynamo. This solar dynamo and convective process are the cause of differential rotation of solar latitudes [2]. The radiative core of the sun rotates with the average speed of the solar latitudes [3, 4]. This solar dynamo is the key of many solar activity phenomena [5, 6]. The stellar spots and differentially rotating surfaces are observed on other stars also. It indicates the existence of stellar dynamo and sun like internal structure (radiative core and convective envelop). Study of stellar rotation as parameter or factor of stellar dynamo and stellar evolution were introduced with different techniques, theories, ideas and are a wide research area of stellar astrophysics. Age and mass were used as function of rotation by Maeder 2009 [7] Bohem-Vetense 2007 and Reiner et.al 2012 used differential rotation to study the dynamo process [8, 9]. Behavior of rotation in spectral classes is also studied by Nielsen M. B. et.al. 2013 [10]. It raised many interesting questions regarding effects of stellar rotation on evolutionary parameters (mass, temperature, metallicity, radius, density, surface gravity) [11]. The relation and theories of differential rotation (DR) with these evolutionary parameters are still quiet unknown. In this study we have analyzed the persistency and persistency correlation [12, 13] of surface rotation and evolutionary parameters of stars. Further we formulated luminosity, radius and surface rotation on the basis of persistency correlation analysis and stellar evolutionary theory in the perspective of Hertzsprung-Russell Diagram [14, 15].

2. DATA AND METHODOLOGY

The Kepler satellite provides a large number of observational data of stars. The data is recorded in quarters. Reinhold et.al (2015) reanalyzed the rotational period of approximately 24000 stars sample from Reinhold et. al. (2013) [16, 17]. Quarter 1 (Q1) to 14 Quarter (Q14) were treated individually by different approaches using Lamb-Scargle periodgram. Surface rotation period of more than 18000 stars were found consistent. Multiple peaks of more than 12000 stars were analyzed and interpreted as differential rotation. In our study we are using DR of these Keplers stars. Further we are using evolutionary parameters data (mass, temperature, density, metallicity, surface gravity, radius) of these stars. All data is available online. The surface DR can be calculated as

$$\Omega = \Omega_{eq} - \triangle\Omega sin^2\phi \qquad (1)$$

where, Ω_{eq} is rotation rate at the equator,Ω_{pole} is the rotation rate at the pole, $\Delta\Omega = \Omega_{eq} - \Omega_{pole}$ is the difference between pole and equator and is called Rotational Shear(RS), $\alpha = \frac{\Delta\Omega}{\Omega_{eq}}$ is the Relative Differential Rotation (RDR) wheres ϕ is the heliographic latitude measured from equator [18]. RS, RDR and average DR of equator and pole are used as rotational parameter of surface differential rotation. The data is arranged according to temperature approximately in between $3300K < Teff < 8000K$. The statistical correlation in between rotational parameters and evolutionary parameters is weak that is not suitable for the selection of parameters for modeling. Most of the natural phenomena have complexity in the background. Stellar evolution is one of the most complex phenomena.

The complexity involved may dominate making it difficult to find their hidden correlation. Fractal dimension technique is used to analyze the persistency of data. Fractal methods initially measure the space filling property of the curves. Using these techniques data can be studied and compared through analyzing the persistency involved and the complexity of the background phenomena [12, 13, 19]. It is attempted to establish a relation between evolutionary parameters and differential rotation. Natural phenomena in particular may involve two prominent behaviors, the self-similarity and self-affinity [12, 13, 19]. For the study of persistency (self-similarity) Hurst exponent is used. For calculating Hurst exponents two methods are used, the Rescaled Range Method and Box Counting Fractal Dimension Method. Rescaled Range Method is used in terms of the following log linear power law.

$$logE[\frac{R}{S}] = H.logn + logc \qquad (2)$$

Where, R is the range of data set, S is its standard deviation, E[R/S] the expected value, n is the number of data and H is the Hurst exponent [12]. It is calculated by dividing the data in Nn parts (where N=1, 2, 3 is positive integer and n is number of data). Here we used N=2. Hurst exponent H is the global property that helps to calculate directly the long term behavior of data. Hurst exponent can also be calculated by fractal dimension (FD). Fractal dimension (FD) is the local property and is use to analyze the hidden complexity of data. The two are different properties [19] and the formula to connect H and FD is as follows.

H = 2 FD (3)

where, $FD < 1.5(orH > 0.5)$ indicates a positive correlation or a persistent process.
$FD > 1.5(orH < 0.5)$ indicates a negative correlation or anti-persistent process.
FD=1.5 (or H =0.5) indicates a zero correlation or a Brownian process.
FD is calculated by the box counting algorithm using Fractalyse the fractal analysis software. It helped to calculate Hurst exponent by formula [20], http://www.fractalyse.org/en-docall.html]. Fractal dimensions using the box counting method were calculated by the following logarithmic linear power law,

$$log(N(\epsilon)) = D.log(\epsilon) + c \qquad (4)$$

where, N = no. of counted boxes, ϵ = size of box, D= Fractal dimension Box counting technique is applied to the data by segregating it in different ways. We segregate 12000 Kepler stars data in main sequence and giant stars. Data is also segregated according to spectral classes A, F, G, K, M. Box counting Fractal dimension(FD) is further applied to the data by reduced by a factor of 2^n , where n=1, 2, 3, . (taking sample size by reducing original data into lags of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192). This segregation gives 13 FD of data in different portions. The division is made to study the persistency behavior of data in different portions by different approaches (other division may apply). Further H data of these 13 points is used to find the correlation between persistency of parameters. In the next section the parameters having strong correlation are used to develop the model.

3. RESULTS AND DISCUSSIONS

Hurst exponent (H) by Rescaled Range Method in all portions of data are found anti persistent. There is no self- similarity or correlation found in data. This is because the

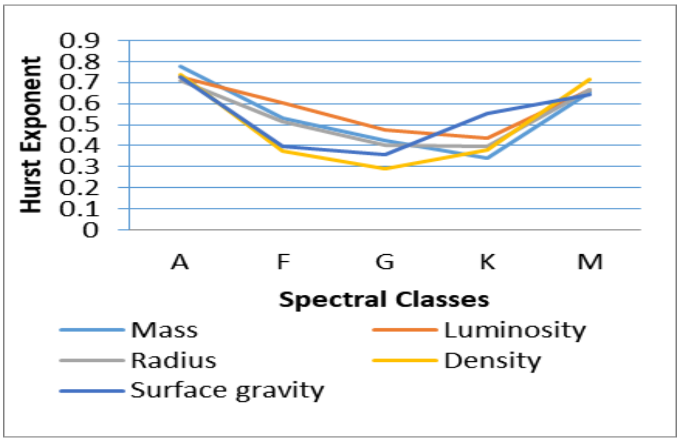

FIGURE 1. The persistency behavior of mass, radius, luminosity, density, surface gravity of MS stars in A, F, G, K and M spectral classes.

Rescaled Range Method is the global property that is used to analyze the long term behavior of data. But the data has hidden complexity that dominates the persistency of data. Fractal dimension helps to analyze the hidden complexity (roughness or smoothness) of data. Hurst exponents calculated by Box counting fractal dimension (H = 2 FD) show different behavior in different portions. This technique helps to give better analysis in the presence of complexity of data. All data is provided in Appendix A. Table 1. shows the correlation between H of evolutionary parameters and H of rotational parameters of main sequence stars. This procedure is adopted because the statistical correlation between the parameters is poor and the use of Hursts exponents helps to overcome the dominating complexity.

Relation	**Hurst Exponent Correlation.**		
Parameters	**RS**	**RDR**	**Avg.DR**
Temperature	0.957017	-0.75618	-0.64912
Mass	-0.11366	0.187514	0.242019
Density	0.26783	-0.49816	0.467491
Luminosity	-0.23198	0.147395	0.152204
Radius	-0.82021	0.597352	0.505079
Surface Gravity	-0.78484	0.676977	0.623995
Metallicity	-0.64788	0.992197	0.995132

TABLE 1. Correlation between Hurst's exponents of rotational and evolutionary parameters.

The segregation of data in spectral classes also show change of parametric behavior in different spectral classes of main sequence stars. The RS of main sequence stars is persistent in all spectral classes, RDR is persistent only in A spectral class and anti-persistent in F, G, K and M classes. Average DR is persistent in A and M spectral classes and anti-persistent in F, G, M spectral classes. Graphical presentation of H of main sequence evolutionary parameters shows the persistency behavior in Fig1. Table 1. depicts strong correlation between RS and temperature, RS and Radius, RS and SG, and RS and metallicity. Similarly, strong correlation occurs between RDR and Temperature, RDR and radius, RDR and SG, and RDR and metallicity. Further, strong correlation is exhibited between Avg DR and Temperature, Avg DR and Radius, Avg DR and SG, and Avg DR and metallicity. So we conclude that there exists an overall strong correlation between rotational parameters and evolutionary parameters. Thus our findings strengthen the work of [21] also which considers rotation as a crucial factor to study stellar structure and stellar evolution. But the work in [21] is particularly associated to massive stars. Such stars are hot stars. However, in view of the analysis of Keplers data performed above the study here establishes strong correlations between rotational parameters and evolutionary parameters in cool stars (less than 3 solar mass). Continuing our study of Keplers stars, now we will stick to two rotational parameters RS and RDR and two evolutionary parameters temperature and radius only. We observe that despite a low statistical correlation (0.21) between RS and effective temperature a strong correlation exists between RS and effective temperature (0.957) with reference to their Hurst exponents. This result is in accordance with the results obtained by J. R. Branes et al (2005) which establishes a strong dependency of RS on effective temperature (Teff) in the form of the power law $\Delta\Omega \propto T_{eff}^{P}(p \sim 8.9)$ by using Doppler Imaging (DI) technique [22]. L.A. Balona in 2011 also studied the dependency of RS and T_{eff} in different spectral classes [23]. The value of p was assigned to be 6.4, 3.5,1.0 and -2 for K, G, F and A spectral class respectively.

Stellar evolution is studied by Hertzsprung-Russell Diagram(HRD). It represents stellar evolution graphically in terms of luminosity and temperature. Luminosity is the function of radius and temperature [24]. Persistency correlation between luminosity and all rotational parameters is weak but the persistency correlation between temperature and all rotational parameters is strong. On the basis of our study we establish a model involving stellar evolutionary parameters (luminosity and radius) and rotational parameters (RS and RDR). In this model RS and RDR are used instead of temperature. The models are specific to main sequence stars. Two models, the multivariate regression model and least square non- linear multivariate second degree polynomial model are develope. Applying Akaike information criterion and Schwarz criterion tests the non-linear multivariate second degree polynomial model is found to be the best [25, 26, 27]. Multivariate second degree polynomial model used is represented as

$$z = c + ax + by + cx^2 + dy^2 + exy \qquad (5)$$

where, $z = logluminosity, x = logRadius, y = log\Delta\Omega$ and C, a, b, c, d, e are coefficients. For C=-159.68, a=38.76, b=-1.999, c=-1.999, d=0.023, e=0.13 the standard error was estimated to be 0.13, $R^2 = 94\%$.The model is formed using 10000 points data and tested on remaining approximately 1500 points data. The minimum and maximum difference between luminosities $< L_{star} - L_{model} >$ are 0.55 and 1.0266 respectively. The same model is formed for main sequence Keplers stars by using luminosity, radius and RDR. where z =log luminosity, x= log Radius, y=log RDR and C, a, b, c, d, e are coefficients.

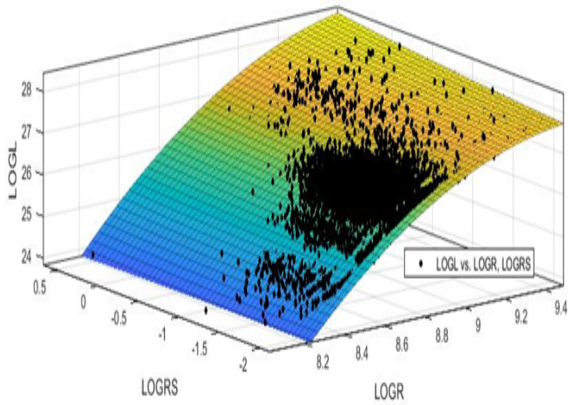

FIGURE 2. 3D diagram of stellar evolution of luminosity with respect of radius and rotational shear.

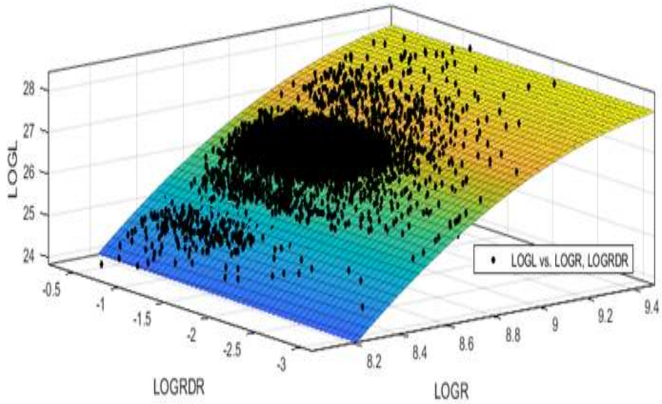

FIGURE 3. 3D diagram of stellar evolution of luminosity with respect of radius and relative differential rotation.

Where C=-145.1, a=35.49, b= -0.21, c=-1.82, d=-0.003, e=0.019. The standard error appeared to be 0.1225, R2= 87%. Fig 2 and 3 are the 3D diagrams representing the models. The minimum and maximum differences between luminosities viz. $< L_{star} - L_{model} >$ appear to be 0.44 and 1.0094 respectively. The black granules appearing in Figs 2 and 3 represent Keplers modeled stars. The blue area in the diagrams represent small radius-low luminosity region, green area represents the region of sun like stars and orange area represents region of high luminosity-large radius A spectral class stars.

4. Conclusion

Differential rotation is a very important factor that helps us to study internal structure and dynamo of the sun. The study of solar interior helps us to study the complex solar activity phenomena. DR is found on other stars too. The aim of this study was to evaluate the influence of surface differential rotation of stars on other parameters like mass, temperature, luminosity, radius, surface gravity, metallicity etc. There were to study stars evolution. Approximately 12000 stars data (Keplers satellite data) was used to study evolutionary and DR parameters. The stellar evolution is complicated and nonlinear in nature. It involves a complex of different forces, magnetic field and physical laws. Statistical correlation is not sufficient to resolve the complexity involved. So Fractal techniques were used to study the persistency of data, their correlation and analysis. Hurst exponents (self-similarity) of data were calculated by Rescaled range and Box counting Fractal dimension techniques. For a deeper insight different segregation of data techniques were used. First segregation of data was done according to the location on HR diagram (giants and main sequence stars), the other segregation was made according to spectral classes and the third using according to a partition of data with lags of 2n. With these segregations the persistency or self-similarity of data is studied computing Hurst exponents of the data for different stages of stellar evolution. It helped us to determine the dependence of luminosity on rotation and radius. HRD is the study of stellar evolution on the basis of the luminosity and temperature. Our model helped to study Keplers star evolution in main sequence on the basis of luminosity and rotation. The study of rotation with evolutionary parameters also helped us to study stellar interior as it is an important factor involved in internal phenomena. At the end it is to mention that it remains to study the relationship of rotation with many other parameters like mass, density, metallicity and surface gravity. These studies will appear in our future works. These findings will help us to achieve the objectives mentioned as above. The contents of this study are a part of authors doctoral thesis.

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