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Some Congruences on CA-AG-groupoids

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Abstract. An AG-groupoid S satisfying the identity $x(yz) = z(xy)$ for all $x, y, z \in S$ is called a CA-AG-groupoid. In this article the notions of equivalence relation and congruence is extended to CA-AG-groupoids and various congruences on CA-AG-groupoid and inverse CA-AG-groupoid are defined and investigated. Furthermore, it is shown that a suitably defined relation ρ on inverse CA-AG-groupoid S is a maximal idempotent-separating congruence, that S/ρ is fundamental and that the semilattice of idempotents of S is isomorphic to the semilattice of idempotents on S/ρ.

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1. Introduction

A groupoid S satisfy $(xy)z = (zy)x$ for all $x, y, z \in S$ (known as the left invertive law [15]) is called an Abel-Grassmann groupoid (in short AG-groupoid [25]). This structure is introduced in 1972 by Kazim and Naseeruddin [15]. The said structure is called upon by different names by different authors, such as left almost semigroup (in short LA-semigroup) [15], right modular groupoid [7] and left invertive groupoid [9]. It is a non-associative algebraic structure midway between a groupoid and a commutative semigroup, and generalize the class of commutative semigroups. AG-groupoid is a well worked area of research having a variety of applications in various fields like flocks theory [15], matrix theory [6, 3], geometry [29] and topology [16] etc.

Various aspects of AG-groupoids are investigated by different researchers and many results are available in literature (see, e.g., [3, 33, 28, 34, 18, 2, 30, 14] and the references herein). Some new classes of AG-groupoids are discovered and investigated in [32, 24, 17, 31, 26, 1]. Iqbal et al. [10] introduced the notion of CA-AG-groupoid and enumerated it upto order 6. Further, they introduced CA-test for verification of arbitrary AG-groupoid to be cyclic associative and studied some fundamental properties of CA-AG-groupoids. The same authors in [11] discussed a different aspect of cancellativity of an element in CA-AG-groupoid and provided a partial solution to an open problem mentioned in [29]. For detail study of CA-AG-groupoids we recommend [10, 11, 12].

Mushtaq and Iqbal [21] defined the notion of partial ordering and congruence on LA-semigroup. They defined a congruence relation η on an inverse LA-semigroup S with a weak associative law and proved that η is idempotent-separating and also proved that $\mu = \{(a,b) \in S \times S : (\forall f \in E(S))\,(a'f)a = (b'f)b\}$, where a', b' are the unique inverses of a and b respectively, is the maximal idempotent-separating congruence on S. In [22] Protić and Božinović defined some congruences on AG**-groupoid, while in [23] Protić defined congruences on inverse AG**-groupoid via the natural partial order. Dudek and Gigoń [8] defined some congruences on completly inverse AG**-groupoid. Božinović et al. [4] discussed the notion of natural partial order on AG-groupoids and defined some congruences on inverse and completly inverse AG**-groupoid. Mushtaq and Yusuf [20] defined a congruence relation ρ on a locally associative LA-semigroup S and investigated that ρ is separative and S/ρ is maximal separative homomorphic image of S.

Motivated by this consideration, our main focus in the present article is to extend the notions of equivalence relation and congruence to CA-AG-groupoids, and to define different congruences on CA-AG-groupoids and on inverse CA-AG-groupoids and explore different aspects of these relations. We generalize the result given in [5, Lemma 1] to the whole class of AG-groupoids. Moreover, we explore some fundamental characteristic of an inverse CA-AG-groupoid.

2. Preliminaries

A magma $(S, \cdot)$ or simply S satisfying $xy \cdot z = zy \cdot x$ for every $x, y, z \in S$ is called an AG-groupoid [25]. Through out the article we will denote an AG-groupoid simply by S otherwise stated else. The medial law: $xy \cdot zt = xz \cdot yt$ always holds in S [13, Lemma 1.1(i)]. Left identity may or may not be contained in S; however, if S contains a left identity then it is unique [19] and S with left identity always satisfies the paramedial law: $xy \cdot zt = ty \cdot zx$ [13, Lemma 1.2(ii)]. Now, we define some elementary aspects and quote few definitions which are essential to step up this study.

An element $f \in S$ is called idempotent if $f^2 = f$. The set of all idempotents is represented by $E(S)$. S having all elements as idempotent is called AG-2-band (in short AG-band) [33]. If S is an AG-band then $S^2 = S$. A commutative AG-band is called a semilattice. S is called AG* [17] if for all $x, y, z \in S$, $xy \cdot z = y \cdot xz$ (known as weak associative law), AG** if $x \cdot yz = y \cdot xz$ [22] and is called cyclic associative AG-groupoid (in short CA-AG-groupoid) if $x \cdot yz = z \cdot xy$ [10]. An AG-groupoid S is called inverse AG-groupoid [21], if for every $x \in S$ there exists $x' \in S$ such that $x = xx' \cdot x$ and $x' = x'x \cdot x'$. Henceforth, by x' we shall mean an inverse of x and by $V(x)$ we shall mean the set of all inverses of x, i.e. $V(x) = \{x' \in S : x = xx' \cdot x \text{ and } x' = x'x \cdot x'\}$. An AG-groupoid

S is called completely inverse AG-groupoid if it satisfies the identity $xx' = x'x$ for all $x \in S$. The notion of an inverse AG-groupoid is a natural generalization of the notion of an AG-group, where an inverse element ($x \cdot x^{-1} = e$ and $x^{-1} \cdot x = e$, where e is the left identity) of AG-group is substituted by a generalized inverse ($xx' \cdot x = x$ and $x'x \cdot x' = x$). This is why the inverse AG-groupoids are called generalized AG-groups.

A relation ρ is called equivalence relation on AG-groupoid S if it satisfies the conditions: (i). ρ is reflexive, i.e. $x\rho x$ for every $x \in S$ (ii). ρ is symmetric, i.e. $x\rho y \Rightarrow y\rho x$ for all $x, y \in S$ (iii). ρ is transitive, i.e. $x\rho y$ and $y\rho z \Rightarrow x\rho z$ for all $x, y, z \in S$. A relation ρ is right compatible if $x\rho y \Rightarrow xz\rho yz$, for all $x, y, z \in S$ and is left compatible if $x\rho y \Rightarrow zx\rho zy$. A relation which is left and right compatible is called compatible. A (left/right) compatible equivalence relation is called (left/right) congruence. A congruence ρ on an AG-groupoid is called idempotent-separating if each $\rho-$class contains atmost one idempotent, i.e. if $(e, f) \in \rho$, then $e = f$ $\forall e, f \in E(S)$. An inverse AG-groupoid is called fundamental if $(\forall b \in S)$ $x'b \cdot x = y'b \cdot y \Rightarrow x = y$.

3. INVERSE CA-AG-GROUPOID

To start with, we prove the existence of inverse CA-AG-groupoid by providing supporting example. We also verify by counterexamples that a CA-AG-groupoid is not necessarily an inverse CA-AG-groupoid and an inverse AG-groupoid is not necessarily an inverse CA-AG-groupoid.

Example 3.1. (i) *Let $S = \{1, 2, 3\}$ and the binary operation on S be defined by the Cayley's Table 1. Then S is an inverse CA-AG-groupoid having $1' = 1,\ 2' = 2$ and $3' = 3$.*
(ii) *CA-AG-groupoid presented in Cayley's Table 2 is not an inverse CA-AG-groupoid, since for every $a \in S$ there exists no $x \in S$ such that $ax \cdot a = a$ and $xa \cdot x = x$.*
(iii) *The set of integers $\mathbb{Z}$ is an inverse AG-groupoid under the binary operation defined by $x \diamond y = y - x,\ \forall x, y \in \mathbb{Z}$, as $(x \diamond y) \diamond z = (z \diamond y) \diamond x$. But since $z - y - x \neq y - x - z$, so $x \diamond (y \diamond z) \neq z \diamond (x \diamond y)$, thus $(\mathbb{Z}, \diamond)$ is not an inverse CA-AG-groupoid.*

$\cdot$	1	2	3
1	1	1	1
2	1	2	3
3	1	3	2

Table 1

$\cdot$	1	2	3
1	2	2	1
2	2	2	2
3	1	2	3

Table 2

Mushtaq and Iqbal [21] proved that if x' is an inverse of x and y' is an inverse of y in an AG-groupoid, then by the medial law

$$(xy \cdot x'y')xy = (xx' \cdot yy')xy = (xx' \cdot x)(yy' \cdot y) = xy,$$
$$\text{and } (x'y' \cdot xy)x'y' = (x'x \cdot y'y)x'y' = (x'x \cdot x')(y'y \cdot y') = x'y'.$$

Thus $(xy \cdot x'y')xy = xy$ and $(x'y' \cdot xy)x'y' = x'y'$. Hence in an inverse AG-groupoid the inverse of xy is $x'y'$, i.e.

$$(xy)' = x'y' \tag{3. 1}$$

Remark 3.2. *If S is an AG-groupoid and $e, f \in E(S)$, then $ef \cdot ef = ee \cdot ff = ef$ by medial law. Thus ef is an idempotent and so $ef \in E(S)$. Hence in AG-groupoid holds: the product of two idempotents is an idempotent.*

The following example illustrates that in an AG-groupoid, idempotent elements can be mutually non-commutative.

Example 3.3. *Table 3 represents an AG-band. As $1 \cdot 2 \neq 2 \cdot 1$, so 1 and 2 does not commute. Similarly $xy \neq yx \; \forall x, y \in E(S)$ when $x \neq y$.*

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Table 3

However:

Lemma 3.4. *In CA-AG-groupoid idempotents commute with each other.*

Proof. Let S be a CA-AG-groupoid and $e, f \in E(S)$. Then by Remark 3.2, medial law, cyclic associativity and left invertive law we have $ef = ef \cdot ef = ee \cdot ff = f(ee \cdot f) = f(fe \cdot e) = e(f \cdot fe) = e(e \cdot ff) = ff \cdot ee = fe$, so $ef = fe$. Hence in CA-AG-groupoid idempotents commute with each other. □

As commutativity of an AG-groupoid implies associativity [10], thus from Lemma 3.4 we have.

Corollary 3.5. *For any CA-AG-groupoid S, $E(S)$ is a semilattice.*

Note that in [10] it is shown that every CA-AG-groupoid is paramedial. The following example depicts that in an inverse AG-groupoid, the elements xx' and $x'x$ are not necessarily idempotents and may not be equal.

Example 3.6. *Let $S = \{a, b, c, d\}$ and the binary operation on S be defined by the following Cayley's table 4.*

$\cdot$	a	b	c	d
a	b	c	a	d
b	d	a	c	b
c	c	b	d	a
d	a	d	b	c

Table 4

Then S is an inverse AG-groupoid. Further $ad \cdot a = a$, $da \cdot d = d$, $bc \cdot b = b$, $cb \cdot c = c$, thus $a' = d$, $d' = a$, $b' = c$ and $c' = b$. Now $(aa')(aa') = (ad)(ad) = c \neq d$, $(dd')(dd') = b \neq a$, thus aa' and dd' are not idempotent. Also $(bb')(bb') = d \neq c$, $(cc')(cc') = a \neq b$, so bb' and cc' are not idempotent. Moreover, $ad = d \neq a = da$ and $bc = c \neq b = cb$. Note that $E(S) = \phi$.

It is proved by M. Božinović et al. [5, Lemma 1] that in an inverse AG**-groupoid S, if $V(x) = \{x'\}$ then $xx' = x'x$ if and only if $xx', x'x \in E(S)$. However, there is no clue given whether in AG-groupoid xx', $x'x$ belong to $E(S)$ implies $xx' = x'x$ or not. Similarly if in an AG-groupoid $xx' = x'x$ then whether $xx', x'x \in E(S)$ or not. We claim that in an AG-groupoid S, $xx' = x'x$ if and only if xx' and $x'x$ belong to $E(S)$. We proceed to prove our claim in the following lemma, which definitely generalize the result of [5, Lemma 1] to the whole class of AG-groupoids instead of AG**.

Lemma 3.7. *Let S be an inverse AG-groupoid. Then for every $x \in S$,*

$$xx',\, x'x \in E(S) \Longleftrightarrow xx' = x'x.$$

Proof. Let $x \in S$ and $x' \in V(x)$ such that $xx', x'x \in E(S)$. Then by definition of inverse, left invertive law, definition of idempotent and medial law

$$\begin{aligned} x'x &= (x'x \cdot x')x = xx' \cdot x'x = (xx' \cdot xx')(x'x) = ((xx' \cdot x')x)(x'x) \\ &= (x'x \cdot x)(xx' \cdot x') = (x'x \cdot xx')(xx') = ((xx' \cdot x)x')(xx') \\ &= (xx')(xx') = xx'. \end{aligned}$$

Conversely, suppose $xx' = x'x$. Then by definition of idempotent, left invertive law and definition of inverse $(xx')^2 = xx' \cdot xx' = (xx' \cdot x')x = (x'x \cdot x')x = x'x = xx'$, imply that xx' is an idempotent, i.e. $xx' \in E(S)$. Similarly $(x'x)^2 = x'x \cdot x'x = (x'x \cdot x)x' = (xx' \cdot x)x' = xx' = x'x$, thus $x'x \in E(S)$. □

If in an AG-groupoid S, $xx' \neq x'x$, then xx' and $x'x$ may not be in $E(S)$, also it is not necessary that $xx' = x'x$. To justify this we provide an example.

Example 3.8. *Table 5 represents an inverse AG-groupoid in which $1' = 2$, $2' = 1$ and $3' = 3$. As $(1 \cdot 1')(1 \cdot 1') = (1 \cdot 2)(1 \cdot 2) = 2 \cdot 2 = 3 \neq 2 = 1 \cdot 1'$, thus $1 \cdot 1' \notin E(S)$. Similarly $1' \cdot 1$, $2 \cdot 2'$, $2' \cdot 2 \notin E(S)$. Also as $1 \cdot 1' = 1 \cdot 2 = 2 \neq 1 = 2 \cdot 1 = 1' \cdot 1$, thus $1 \cdot 1' \neq 1' \cdot 1$. Note that $3 \cdot 3' = 3 = 3' \cdot 3$. Also $3 \cdot 3' \in E(S)$ as $(3 \cdot 3')(3 \cdot 3') = 3$.*

$\cdot$	1	2	3
1	3	2	3
2	1	3	3
3	3	3	3

Table 5

However, in inverse CA-AG-groupoids both xx' and $x'x$ are idempotents and also $xx' = x'x$, as it is proved in the following lemma.

Lemma 3.9. *Let S be an inverse CA-AG-groupoid and $V(x) = \{x'\}$, then xx', $x'x \in E(S)$ and S is completely inverse CA-AG-groupoid.*

Proof. As $x' \in V(x)$, so $xx' \cdot x = x$ and $x'x \cdot x' = x'$. Now by the paramedial and medial laws and cyclic associativity

$$xx' \cdot xx' = x'x' \cdot xx = x'x \cdot x'x = x(x'x \cdot x') = xx'.$$

Thus, xx' is idempotent. Similarly

$$x'x \cdot x'x = xx \cdot x'x' = xx' \cdot xx' = x'(xx' \cdot x) = x'x.$$

This shows that $x'x$ is also idempotent. Hence xx', $x'x \in E(S)$. Now, it is remaining to show that $xx' = x'x$. As

$$\begin{aligned} xx' &= x(x'x \cdot x') = x'(x \cdot x'x) = x'(x \cdot xx') \\ &= xx' \cdot x'x = (x'x \cdot x')x = x'x. \end{aligned}$$

Consequently, S is completely inverse CA-AG-groupoid. □

Remark 3.10. *If a is an idempotent element of an AG-groupoid, then $a^2 = a$, $a^3 = a^2 a = aa = a$, $a^4 = a^3 a = (aa \cdot a)a = (aa)a = aa = a$ and in general $a^n = a$ for $n \in \mathbb{N}$. By Lemma 3.9, xx' and $x'x$ are idempotents in CA-AG-groupoid, so $(xx')^n = xx'$ and $(x'x)^n = x'x$ for all $n \in \mathbb{N}$. Also, since $xx' = x'x$, so $(xx')^n = (x'x)^n$.*

Now, we proceed to prove that the inverse of an element in an inverse CA-AG-groupoid is unique.

Lemma 3.11. *The inverse of an element in inverse CA-AG-groupoid is unique.*

Proof. Assume the contrary. Let a and b be the inverses of an element x of an inverse CA-AG-groupoid, then by definition $x = xa \cdot x$, $a = ax \cdot a$, $x = xb \cdot x$ and $b = bx \cdot b$. Now by cyclic associativity, Lemma 3.9, left invertive law and medial law we have

$$\begin{aligned} a &= ax \cdot a = (a(xb \cdot x)a) = (a(xb(xb \cdot x)))a = ((xb \cdot x)(a \cdot xb))a \\ &= ((xb \cdot x)(b \cdot ax))a = (ax((xb \cdot x)b))a = (b(ax(xb \cdot x)))a \\ &= (a(ax(xb \cdot x)))b = ((xb \cdot x)(a \cdot ax))b = ((bx \cdot x)(a \cdot ax))b \\ &= ((xx \cdot b)(a \cdot ax))b = ((xx \cdot a)(b \cdot ax))b = ((ax \cdot x)(b \cdot ax))b \\ &= ((xa \cdot x)(b \cdot ax))b = (x(b \cdot ax))b = (ax \cdot xb)b = (b(ax \cdot x))b \\ &= (b(xa \cdot x))b = (bx)b = b. \end{aligned}$$

Thus, inverse of an element in inverse CA-AG-groupoid is unique. □

In the following we provide an example to verify that in case of semigroup the inverse of element may not be unique.

Example 3.12. *Table 6 represents an inverse semigroup having $V(a) = \{a, b, c, d\} = V(b) = V(c) = V(d)$.*

$\cdot$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Table 6

The following example clarify that in CA-AG-groupoid S, $S \nsubseteq S^2$, thus $S^2 \neq S$.

Example 3.13. *$S = \{a, b, c, d\}$ with the following Cayley's Table 7 is a CA-AG-groupoid. As $S^2 = \{a, b, c\}$, so $S \nsubseteq S^2$.*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	c

Table 7

However:

Lemma 3.14. *If S is an inverse CA-AG-groupoid, then $S^2 = S$.*

Proof. Since S is inverse CA-AG-groupoid, then for for all $x \in S$ there exists $y \in S$ such that $x = xy \cdot x \in S \cdot S = S^2$. Thus for each $x \in S$ we have $x \in S^2$. This means that $S \subseteq S^2$. But since $S^2 \subseteq S$ holds in general. It follows that $S^2 = S$. □

Now we provide an example to verify that in inverse AG-groupoid $(xx')' \neq xx'$.

Example 3.15. *An inverse AG-groupoid is represented in Table 8 having $a' = b$, $b' = a$ and $c' = c$. As $(aa')' = (ab)' = b' = a$ and $aa' = ab = b$, then $(aa')' \neq aa'$.*

$\cdot$	a	b	c
a	c	b	c
b	a	c	c
c	c	c	c

Table 8

However:

Lemma 3.16. *In inverse CA-AG-groupoid $(xx')' = xx'$.*

Proof. By cyclic associativity, medial, left invertive and paramedial laws

$$\begin{aligned}(xx' \cdot xx')(xx') &= (x'(xx' \cdot x))(xx') = (x'x)((xx' \cdot x)x') \\ &= (x'x)(x'x \cdot xx') = (xx' \cdot x)(x'x \cdot x') = xx'.\end{aligned}$$

Thus, xx' is the inverse of xx'. As by Lemma 3.11, the inverse of an element in CA-AG-groupoid is unique, so $(xx')' = xx'$. □

Lemma 3.17. *If S is an inverse CA-AG-groupoid, then $(x')' = x$.*

Proof. Clearly x is the solution of the equations $x' = x'y \cdot x'$ and $y = yx' \cdot y$. As by Lemma 3.11, inverse of an element in CA-AG-groupoid is unique so $(x')' = x$. □

Lemma 3.18. *Let S be an inverse CA-AG-groupoid. Then $A(S) = \{xx' \mid x \in S\}$ is a semilattice.*

Proof. Let $x_1x_1', x_2x_2' \in A(S)$. Then by Lemma 3.9, paramedial and medial laws $x_1x_1' \cdot x_2x_2' = x_1'x_1 \cdot x_2'x_2 = x_2x_1 \cdot x_2'x_1' = x_2x_2' \cdot x_1x_1'$, thus commutative law holds in $A(S)$. As in AG-groupoid, commutativity implies associativity [10], thus $A(S)$ is associative. Hence $A(S)$ is a semilattice. □

Corollary 3.19. *Let S be an inverse CA-AG-groupoid. Then $A_1(S) = \{x'x \mid x \in S\}$ is a semilattice.*

Lemma 3.20. *Let S be an inverse CA-AG-groupoid. Then $e \in E(S)$ implies $e' \in E(S)$.*

Proof. As S is an inverse CA-AG-groupoid, so for all $x \in S$ there exists $x' \in S$ such that $x = xx' \cdot x$ and $x' = x'x \cdot x'$. As $E(S) \subseteq S$, so in particular for $e \in E(S)$, $e = ee' \cdot e$ and $e' = e'e \cdot e'$. We will show that e' is an idempotent. Using the medial, left invertive and paramedial laws, cyclic associativity, the definition of inverse and Lemma 3.9 we have

$$\begin{aligned} e'^2 &= e'e' = (e'e \cdot e')(e'e \cdot e') = (e'e \cdot e'e)(e'e') = (e(e'e \cdot e'))(e'e') \\ &= (ee')(e'e') = (ee \cdot e')(e'e') = (e'e \cdot e)(e'e') = (ee' \cdot e)(e'e') \\ &= e(e'e') = (ee)(e'e') = (e'e)(e'e) = (e'e')(ee) = (ee \cdot e')e' \\ &= (ee')e' = (e'e)e' = e'. \end{aligned}$$

Thus $e'^2 = e'$. Hence $e' \in E(S)$. □

Lemma 3.21. *If S is an inverse AG-groupoid and $e \in E(S)$. Then:*
(i) $e \cdot xe' = e \cdot xe$, *for all* $x \in S$,
(ii) $ee' = e$,
(iii) $e'e = e$,
(iv) $e' = e$.

Proof. As $e \in E(S)$, so $e^2 = e$. Also $ee' \cdot e = e$ and $e'e \cdot e' = e'$.
(i) By left invertive and medial laws

$$\begin{aligned} e \cdot xe' &= ee \cdot xe' = ((ee' \cdot e)e)(xe') = (ee \cdot ee')(xe') = (xe' \cdot ee')(ee) \\ &= ((ee' \cdot e')x)(ee) = ((ee' \cdot e')e)(xe) = (ee' \cdot ee')(xe) \\ &= (ee \cdot e'e')(xe) = ((e'e' \cdot e)e)(xe) = ((e'e' \cdot ee)e)(xe) \\ &= ((e'e \cdot e'e)e)(xe) = (((e'e \cdot e)e')e)(xe) = ((ee')(e'e \cdot e))(xe) \\ &= ((e \cdot e'e)(e'e))(xe) = ((ee \cdot e'e)(e'e))(xe) = ((ee' \cdot ee)(e'e))(xe) \\ &= ((ee' \cdot e)(e'e))(xe) = (e(e'e))(xe) = (ee \cdot e'e)(xe) \\ &= (ee' \cdot ee)(xe) = (ee' \cdot e)(xe) = e \cdot xe. \end{aligned}$$

(ii) By part (i) and left invertive law

$$\begin{aligned} ee' &= e(e'e \cdot e') = e(e'e \cdot e) = e(ee \cdot e') \\ &= e(ee') = e(ee) = ee = e. \end{aligned}$$

(iii) By left invertive law and part (ii)

$$\begin{aligned} e'e &= (e'e \cdot e')e = ee' \cdot e'e = e \cdot e'e \\ &= ee \cdot e'e = ee' \cdot ee = ee' \cdot e = e. \end{aligned}$$

(iv) By part (iii) and part (ii)

$$e' = e'e \cdot e' = ee' = e,$$

or equivalently said the inverse of $e \in E(S)$ is e. □

In the following we provide an example to verify that an AG-groupoid S, element of S may not commute with element of $E(S)$.

Example 3.22. *Cayley's Table* 9 *represented an AG-groupoid on* $S = \{1, 2, 3\}$ *having* $E(S) = \{1\}$. *The element* $1 \in E(S)$ *does not commute with* $2 \in S$, *as* $1 \cdot 2 \neq 2 \cdot 1$. *However, as* $1 \cdot 3 = 3 \cdot 1$, *thus* $1 \in E(S)$ *commute with* $3 \in S$.

$\cdot$	1	2	3
1	1	1	1
2	3	1	1
3	1	1	1

Table 9

Lemma 3.23. *In CA-AG-groupoid* S, *elements of* S *and* $E(S)$ *commute with each other.*

Proof. Let x be an arbitrary element of S and $f \in E(S)$, then by cyclic associativity, paramedial and left invertive laws

$$xf = x(ff) = x(ff \cdot f) = f(x \cdot ff) = ff \cdot fx = xf \cdot ff = xf \cdot f = ff \cdot x = fx.$$

Thus, elements of S commute with elements of $E(S)$. □

Lemma 3.24. *Let* S *be an inverse CA-AG-groupoid and* $e \in E(S)$. *Then for any* $x \in S$, *the following* $x'e \cdot x \in E(S)$ *is holds.*

Proof. By the left invertive, medial and paramedial laws, Lemma 3.23 and cyclic associativity

$$\begin{aligned}
(x'e \cdot x)^2 &= (x'e \cdot x)(x'e \cdot x) = (xe \cdot x')(x'e \cdot x) = (xe \cdot x'e)(x'x) \\
&= (ex \cdot ex')(x'x) = (x'x \cdot ee)(x'x) = (x'x \cdot x')(ee \cdot x) \\
&= x((x'x \cdot x')(ee)) = x((ee \cdot x')(x'x)) = (x'x)(x(ee \cdot x')) \\
&= (ee \cdot x')(x'x \cdot x) = (ee \cdot x'x)(x'x) = (x \cdot x'x)(x' \cdot ee) \\
&= (ee)((x \cdot x'x)x') = (ee)((x' \cdot x'x)x) = x((ee)(x' \cdot x'x)) \\
&= (x' \cdot x'x)(x \cdot ee) = (ee \cdot x'x)(xx') = (ee \cdot x)(x'x \cdot x') \\
&= (ex)(x'x \cdot x') = (xe)x' = (x'e)x.
\end{aligned}$$

Thus, $x'e \cdot x$ is an idempotent, i.e. $x'e \cdot x \in E(S)$. □

By using Remark 3.2 and Lemma 3.24 we have.

Corollary 3.25. *Let* S *be an inverse CA-AG-groupoid and* $e \in E(S)$. *Then for any* $x \in S$ *and any natural* n *the following* $(x'e \cdot x)^n \in E(S)$ *holds.*

4. Congruences on inverse CA-AG-groupoids

Congruences play an important role in associative and non-associative structures. Here, we extend the notions of equivalence relation and congruence to CA-AG-groupoids and define different congruences on CA-AG-groupoids and on inverse CA-AG-groupoids.

Lemma 4.1. *Let S be an AG-groupoid. Then*
(i) $\gamma_1 = \{(x, y) \in S \times S : (\forall a \in S)\, ax = ay\}$,
(ii) $\gamma_2 = \{(x, y) \in S \times S : (\forall a \in S)\, xa = ya\}$,
are equivalence relations on S.

Proof. (i) As $ax = ax$ for all $a \in S$, so γ_1 is reflexive. Also, if $x\gamma_1 y$ then $ax = ay$ which implies $ay = ax$, so γ_1 is symmetric. To show that γ_1 is transitive, let $x\gamma_1 y$ and $y\gamma_1 z$ where $x, y, z \in S$ then for all $a \in S$, $ax = ay$ and $ay = az$, which implies $ax = az$, thus $x\gamma_1 z$, hence γ_1 is transitive.
(ii) Similarly to (i). □

Lemma 4.2. *Let S be a CA-AG-groupoid. Then the relation γ_1 as defined in Lemma 4.1 is right compatible.*

Proof. For if $x\gamma_1 y$ then $ax = ay$, for every $a \in S$. Now for any $z \in S$, by cyclic associativity

$$a(xz) = z(ax) = z(ay) = y(za) = a(yz).$$

This implies $xz\gamma_1 yz$. □

Remark 4.3. *γ_1 is not left compatible. The relation γ_2 as defined in Lemma 4.1 is neither left compatible nor right compatible.*

Lemma 4.4. *Let S be an inverse AG-groupoid. Then the relations*
(i) $\gamma_3 = \{(x, y) \in S \times S : (\forall x, y \in S)\, x'x = y'y\}$,
(ii) $\gamma_4 = \{(x, y) \in S \times S : (\forall x, y \in S)\, xx' = yy'\}$,
are idempotent-separating congruences on S. Moreover, if $x'x \in E(S)$ for every $x \in S$, then γ_3 and γ_4 are maximal.

Proof. (i) Clearly γ_3 is reflexive and symmetric. If $x\gamma_3 y$ and $y\gamma_3 z$, then $x'x = y'y$ and $y'y = z'z$, which implies $x'x = z'z$, thus $x\gamma_3 z$ and γ_3 is transitive. Hence γ_3 is an equivalence relation. Now, if $x\gamma_3 y$ then $x'x = y'y$, let $z \in S$ then by medial law $(x'x)(z'z) = (y'y)(z'z) \Rightarrow (x'z')(xz) = (y'z')(yz)$ which by virtue of equation (3. 1) gives $(xz)'(xz) = (yz)'(yz)$, so $xz\gamma_3 yz$, thus γ_3 is right compatible. Similarly γ_3 is left compatible. Hence γ_3 is a congruence on S. To show that γ_3 is idempotent-separating, let $e, f \in E(S)$ such that $e\gamma_3 f$, then by Lemma 78 $e = ee = e'e = f'f = ff = f \Rightarrow e = f$. Thus γ_3 is idempotent-separating congruence. To show that γ_3 is maximal, let μ be another idempotent-separating congruence. Let $x\mu y$ then $x'\mu y'$. Also, as μ is compatible so from $x'\mu y'$ and $x\mu y$ we have $x'x\mu y'x$ and $y'x\mu y'y$. These by transitivity of μ implies $x'x\mu y'y$. As for all $x \in S$, $x'x \in E(S)$ (given) and since μ is idempotent-separating it follows that $x'x = y'y$, whence it follows that $x\gamma_3 y$. Hence $x\mu y$ implies $x\gamma_3 y$, thus $\mu \subseteq \gamma_3$. Hence, γ_3 is the maximal idempotent-separating congruence on S.
(ii) Similar to (i). □

Theorem 4.5. *Let S be an inverse CA-AG-groupoid. Then the relations*
(i) $\gamma_5 = \{(x, y) \in S \times S : (\forall x, y \in S)\, x'x = yy'\}$,
(ii) $\gamma_6 = \{(x, y) \in S \times S : (\forall x, y \in S)\, x'x = yy'\}$,
are maximal idempotent-separating congruences on S.

Proof. (i) As by Lemma 3.9 in inverse CA-AG-groupoid $x'x = xx'$, thus γ_5 is reflexive. Again, if $x\gamma_5 y$ then $x'x = yy'$, which implies $yy' = x'x$. By using Lemma 3.9, $y'y = xx'$, thus $y\gamma_5 x$, so γ_5 is symmetric. If $x\gamma_5 y$ and $y\gamma_5 z$, then $x'x = yy'$ and $y'y = zz'$, which by virtue of Lemma 3.9 implies $yy' = zz'$. Thus $x'x = yy'$ and $yy' = zz'$, which implies $x'x = zz'$, thus $x\gamma_5 z$, consequently γ_5 is transitive. Hence, γ_5 is an equivalence relation. Now, if $x\gamma_5 y$, then $x'x = yy'$, let $z \in S$ then $(x'x)(z'z) = (yy')(z'z)$, which by medial law and Lemma 3.9 implies $(x'z')(xz) = (yy')(zz')$, which by virtue of equation (3. 1) and medial law gives $(xz)'(xz) = (yz)(y'z')$ implies $(xz)'(xz) = (yz)(yz)'$. So $xz\gamma_5 yz$, thus γ_5 is right compatible. Similarly, γ_5 is left compatible. Hence, γ_5 is a congruence on S. To show that γ_5 is idempotent-separating, let $f, g \in E(S)$ such that $f\gamma_5 g$, then by Lemma 78 $f = ff = f'f = g'g = gg = g \Rightarrow f = g$. Thus γ_5 is idempotent-separating congruence. To show that γ_5 is maximal, let μ be another idempotent-separating congruence. Let $x\mu y$ then $x'\mu y'$. As μ is compatible so from $x'\mu y'$ and $x\mu y$ we have $x'x\mu y'x$ and $y'x\mu y'y$. By transitivity of μ implies $x'x\mu y'y$. This by virtue of Lemma 75 implies $x'x\mu yy'$. Since by Lemma 75 $x'x$ and yy' are idempotents, and as μ is idempotent-separating so $x'x = yy'$, hence $x\gamma_5 y$. Therefore $x\mu y$ implies $x\gamma_5 y$, thus $\mu \subseteq \gamma_5$. Hence, γ_5 is the maximal idempotent-separating congruence on S.
(ii) As by Lemma 3.9, in inverse CA-AG-groupoid $x'x = xx'$. Hence, the result follows. □

Theorem 4.6. *Let S be a CA-AG-groupoid and $E(S) \neq \phi$. Then the relation defined on S by $\eta = \{(x, y) \in S \times S : (\exists f \in E(S))\, (xf, yf \in E(S) \wedge xf = yf)\}$ is a congruence on S.*

Proof. Clearly η is reflexive and symmetric. To prove transitivity of η, let $x\eta y$ and $y\eta z$, then $xg = yg$ and $yf = zf$ for some $g, f \in E(S)$. Now by cyclic associativity, left invertive, paramedial and medial laws, assumption and Lemma 3.4

$$\begin{aligned} x \cdot gf &= f \cdot xg = f \cdot yg = g \cdot fy = gg.fy = gf \cdot gy = yf \cdot gg = zf \cdot gg \\ &= g(zf \cdot g) = g(gf \cdot z) = z(g \cdot gf) = z(f \cdot gg) = z(fg) = z(gf). \end{aligned}$$

As $g, f \in E(S)$, so by Remark 3.2, $gf \in E(S)$. Thus $x(gf) = z(gf)$, implies $x\eta z$. Hence η is an equivalence relation on S. To prove that η is right compatible, let $x\eta y$ and $z \in S$, then $xg = yg$ for some $g \in E(S)$. Now, by the medial law

$$xz \cdot g = xz \cdot gg = xg \cdot zg = yg \cdot zg = yz \cdot gg = yz \cdot g.$$

Thus, $xz\eta yz$. Hence, η is right compatible. Again, by left invertive law

$$\begin{aligned} &zx \cdot g = gx \cdot z = (gg \cdot x)z = (xg \cdot g)z = (yg \cdot g)z = (gg \cdot y)z = zy \cdot gg = zy \cdot g \\ &\Rightarrow zx\eta\, zy. \end{aligned}$$

Thus, η is also left compatible. Hence, η is a congruence on S. □

Using Lemma 3.23 and Theorem 4.6, we have the following.

Corollary 4.7. *Let S be a CA-AG-groupoid and $E(S) \neq \phi$. Then the relations defined on S by*
(i) $\eta_1 = \{(x, y) \in S \times S(\exists f \in E(S))\, (fx, fy \in E(S) \wedge fx = fy)\}$,
(ii) $\eta_2 = \{(x, y) \in S \times S(\exists f \in E(S))\, (xf, fy \in E(S) \wedge xf = fy)\}$,

(iii) $\eta_3 = \{(x,y) \in S \times S (\exists f \in E(S)) \, (fx, yf \in E(S) \wedge fx = yf)\}$,
are congruences on S.

In the following lemma we establish different relationships of η (where η is as defined in Theorem 4.6) on AG-groupoid and prove that if $x\eta y$ then $x^2\eta y^2$ and then prove in general $x^n\eta y^n$, where $n \in \mathbb{N}$. We further prove that if $x\eta y$ and $a\eta b$, then $xa\eta yb$ and $ax\eta by$.

Lemma 4.8. *Let* S *be an AG-groupoid and* $E(S) \neq \phi$. *Then*
(i) $x\eta y \Rightarrow x^2\eta y^2$,
(ii) $x\eta y \wedge a\eta b \Rightarrow xa\eta yb \wedge ax\eta by$,
(iii) $x\eta y \Rightarrow x^n\eta y^n$.

Proof. (i) As $x\eta y$, so $xg = yg$ for some $g \in E(S)$. Now by the medial law

$$x^2 g = xx \cdot gg = xg \cdot xg = yg \cdot yg = yy \cdot gg = y^2 g$$
$$\Rightarrow x^2 \eta \, y^2.$$

(ii) As $x\eta y$ and $a\eta b$, so $xg = yg$ and $af = bf$, for some $g, f \in E(S)$. Now by these results and medial law we have

$$xa \cdot gf = xg \cdot af = yg \cdot bf = yb \cdot gf.$$

As by Remark 3.2, $g, f \in E(S)$ implies $gf \in E(S)$, thus $xa\eta yb$. Similarly $ax\eta by$.
(iii) Let $x\eta y$ then by Part (i), $x^2\eta y^2$. Again by Part (ii), from $x^2\eta y^2$ and $x\eta y$ we have $x^3\eta y^3$. By repeated use of Part (i) and Part (ii), we get the desired result. □

Theorem 4.9. *Let* S *be an AG-groupoid and* $E(S) \neq \phi$. *Then, the relation defined on* S *by* $\beta = \{(x,y) \in S \times S : (\forall e \in E(S)) \, xe = ye\}$ *is a congruence on* S.

Proof. Clearly β is a reflexive and symmetric. To prove β is transitive, let $x\beta y$ and $y\beta z$, then for all e belongs to $E(S)$, $xe = ye$ and $ye = ze$, which implies $xe = ze$, thus $x\eta z$. Hence, β is an equivalence relation. To prove that β is right compatible, let $x\beta y$ then $xe = ye \; \forall e \in E(S)$. Now for $z \in S$, by medial law and assumption

$$xz \cdot e = xz \cdot ee = xe \cdot ze = ye \cdot ze = yz \cdot ee = yz \cdot e$$
$$\Rightarrow xz\beta \, yz.$$

Thus, β is right compatible. Similarly, β is left compatible. Hence the result follows. □

Note that if β (as defined in Theorem 4.9) is a congruence on CA-AG-groupoid S then S/β is a CA-AG-groupoid. Also, if β is a congruence on inverse CA-AG-groupoid S then S/β is an inverse CA-AG-groupoid and $x\beta y$ if and only if $x'\beta y'$. Using Lemma 3.23 and Theorem 4.9 we have the following.

Corollary 4.10. *Let* S *be a CA-AG-groupoid and* $E(S) \neq \phi$. *Then the relations defined on* S *by*
(i) $\beta_1 = \{(x,y) \in S \times S : (\forall e \in E(S)) \, ex = ey\}$,
(ii) $\beta_2 = \{(x,y) \in S \times S : (\forall e \in E(S)) \, xe = ey\}$,
(iii) $\beta_3 = \{(x,y) \in S \times S : (\forall e \in E(S)) \, ex = ye\}$,
are congruences on S.

Theorem 4.11. *Let S be an inverse CA-AG-groupoid and $E(S) \neq \phi$. Then the relation defined on S by $\delta = \{(x, y) \in S \times S : (\exists e \in E(S)) \, (x'e \cdot x, \, y'e \cdot y \in E(S) \wedge x'e \cdot x = y'e \cdot y)\}$ is a congruence on S.*

Proof. As $x'e \cdot x = x'e \cdot x$ so $x\delta x$, thus δ is reflexive. Again, if $x\delta y$ then $x'e \cdot x = y'e \cdot y$ which implies $y'e \cdot y = x'e \cdot x$, so $y\delta x$, thus δ is symmetric. To prove that δ is transitive, let $x\delta y$ and $y\delta z$ then $x'e \cdot x = y'e \cdot y$ and $y'f \cdot y = z'f \cdot z$ for some $e, f \in E(S)$. Now by cyclic associativity, left invertive, paramedial and medial laws, definition of idempotent, Lemma 3.4, Lemma 3.23 and assumption

$$\begin{aligned}(x' \cdot ef)x &= (f \cdot x'e)x = (x \cdot x'e)f = (e \cdot xx')f = (ee \cdot xx')f = (x'e \cdot xe)f \\ &= (e(x'e \cdot x))f = (e(y'e \cdot y))f = (f(y'e \cdot y))e = (y(f \cdot y'e))e \\ &= (y'e \cdot yf)e = (y'y \cdot ef)e = (y'y \cdot fe)e = (ey \cdot fy')e \\ &= (ey \cdot y'f)e = ((y'f \cdot y)e)e = ((z'f \cdot z)e)e = (ee)(z'f \cdot z) \\ &= (ze)(z'f \cdot e) = (z \cdot z'f)(ee) = (f \cdot zz')e = (z' \cdot fz)e \\ &= (e \cdot fz)z' = (z \cdot ef)z' = (z' \cdot ef)z.\end{aligned}$$

As by Remark 3.2, for $e, f \in E(S) \Rightarrow ef \in E(S)$. Thus from $(x' \cdot ef)x = (z' \cdot ef)z$, we get $x\delta z$. Hence, δ is an equivalence relation on S. Now to show that δ is compatible, let $x\delta y$ then $x'e \cdot x = y'e \cdot y$. Now for any $z \in S$, by equation (3. 1), medial law and the assumption

$$\begin{aligned}((xz)'e)(xz) &= (x'z' \cdot ee)(xz) = (x'e \cdot z'e)(xz) = (x'e \cdot x)(z'e \cdot z) \\ &= (y'e \cdot y)(z'e \cdot z) = (y'e \cdot z'e)(yz) = (y'z' \cdot ee)(yz) = ((yz)'e)(yz).\end{aligned}$$

Thus $xz\delta yz$, hence δ is right compatible. Similarly, one can easily shows that δ is left compatible. Hence δ is a congruence on S. □

Remark 4.12. *If x is an element of an inverse CA-AG-groupoid S and $e \in E(S)$, then by Lemma 3.23 the elements of S commute with elements of $E(S)$. By this result and left invertive law from $x'e \cdot x = xe \cdot x'$ we have $ex' \cdot x = ex \cdot x'$, which further implies $xx' \cdot e = x'x \cdot e$. Also by cyclic associativity and Lemma 3.23 $x'x \cdot e = e \cdot x'x = x \cdot ex' = x \cdot x'e = e \cdot xx'$. This by cyclic associativity and Lemma 3.23 implies $e \cdot xx' = x' \cdot ex = x' \cdot xe$. Also by cyclic associativity and Lemma 3.23, $x' \cdot ex = x \cdot x'e$ and $x \cdot x'e = x \cdot ex'$. Again by cyclic associativity, Lemma 3.23 and left invertive law $e \cdot xx' = x' \cdot ex = x' \cdot xe = e \cdot x'x$ and $xx' \cdot e = ex' \cdot x = x'e \cdot x = xe \cdot x' = ex \cdot x' = x'x \cdot e$. Similarly all other possibility of x, x' and e are equal to $x'e \cdot x$. Also $x'e \cdot x = x' \cdot ex$ clarify that in x, x', e any two can by operate by "$\cdot$" first and then with the third one from left or from right. Similarly all other cases can be tackle on similar way.*

By Remark 4.12 and Theorem 4.11, the following corollary is now obvious:

Corollary 4.13. *Let S be an inverse CA-AG-groupoid and $E(S) \neq \phi$. Then the relation defined on S by $\delta_k = \{(x, y) \in S \times S : (\exists e \in E(S)) \, (x_{p_1}x_{p_2}x_{p_3}, \, y_{q_1}y_{q_2}y_{q_3} \in E(S) \wedge x_{p_1}x_{p_2}x_{p_3} = y_{q_1}y_{q_2}y_{q_3})\}$ is a congruence on S, where $x_{p_1}x_{p_2}x_{p_3}$ is any permutation of the elements x', e, x and $y_{q_1}y_{q_2}y_{q_3}$ is any permutation of the elements y', e, y.*

In the following, we define a relation ρ on an inverse CA-AG-groupoid S and prove that ρ is a maximal idempotent-separating congruence. We also define a generalized form of ρ, denoted by ρ_k . Furthermore, we prove that S/ρ is fundamental and $E(S)$ is isomorphic to $E(S/\rho)$.

Theorem 4.14. *Let S be an inverse CA-AG-groupoid and $E(S) \neq \phi$. Then the relation defined on S by $\rho = \{(x, y) \in S \times S : (\forall e \in E(S))\, x'e \cdot x = y'e \cdot y\}$ is the maximal idempotent-separating congruence on S.*

Proof. Clearly ρ is reflexive, as $x'e \cdot x = x'e \cdot x$ for every $e \in E(S)$. Also if $x\rho y$, then $x'e \cdot x = y'e \cdot y$, which implies $y'e \cdot y = x'e \cdot x$, thus $y\rho x$, hence ρ is also symmetric. Now to show that ρ is transitive, let $x\rho y$ and $y\rho z$, then $x'e \cdot x = y'e \cdot y$ and $y'e \cdot y = z'e \cdot z$ $\forall e \in E(S)$. Thus $x'e \cdot x = z'e \cdot z$ and $x\rho z$, hence ρ is transitive. Therefore ρ is an equivalence relation on S. To prove that ρ is left compatible, let $x\rho y$, then $x'e \cdot x = y'e \cdot y$ $\forall e \in E(S)$. Now for any $z \in S$, by equation (3. 1) and medial law

$$((zx)'e)(zx) = (z'x' \cdot ee)(zx) = (z'e \cdot x'e)(zx) = (z'e \cdot z)(x'e \cdot x)$$
$$= (z'e \cdot z)(y'e \cdot y) = (z'e \cdot y'e)(zy) = (z'y' \cdot ee)(zy) = ((zy)'e)(zy).$$

Thus $zx\rho zy$, therefore ρ is left compatible. It can be similarly shown that ρ is right compatible. Hence ρ is compatible and hence is a congruence on S. To show that ρ is idempotent-separating, let $f, g \in E(S)$ be such that $f\rho g$. Then $f'e \cdot f = g'e \cdot g$ $\forall e \in E(S)$. In particular for $e = f$ and $e = g$ we have $f'f \cdot f = g'f \cdot g$ and $f'g \cdot f = g'g \cdot g$. Now by definition of inverses, Lemma 3.21, Remark 3.4, cyclic associativity and definition of idempotent element

$$f = f'f \cdot f' = f'f \cdot f = g'f \cdot g = gf \cdot g = g \cdot gf = f \cdot gg = fg. \tag{4. 2}$$

Now by definition of inverses, Lemma 3.21, Lemma 3.4, left invertive law, definition of idempotent element and equation (4. 2)

$$g = g'g \cdot g' = g'g \cdot g = f'g \cdot f = fg \cdot f = gf \cdot f = ff \cdot g = fg = f.$$

Hence ρ is idempotent-separating. To show that ρ is maximal, let μ be another idempotent-separating congruence. If $x\mu y$, then $x'\mu y'$. As μ is right compatible, thus for $e \in E(S)$ and $x'\mu y'$ we have $x'e\mu y'e$. Also by Lemma 4.8 (ii), from $x'e\mu y'e$ and $x\mu y$ we have $x'e \cdot x\mu y'e \cdot y$. Now by medial, paramedial and left invertive laws, cyclic associativity, definition of inverse and Lemma 3.23

$$(x'e \cdot x)^2 = (x'e \cdot x)(x'e \cdot x) = (x'e \cdot x'e)(xx) = (ee \cdot x'x')(xx) = (xx \cdot x'x')e$$
$$= (x'x \cdot x'x)e = (x(x'x \cdot x'))e = (xx')e = (ex')x = (x'e)x.$$

Thus, $x'e \cdot x$ is an idempotent. Similarly, $y'e \cdot y$ is an idempotent. Since μ is idempotent-separating so $x'e \cdot x = y'e \cdot y$ for every $e \in E(S)$, which implies $x\rho y$. Hence $x\mu y$ implies $x\rho y$, thus $\mu \subseteq \rho$. Therefore, ρ is the maximum idempotent-separating congruence on S. □

Using Theorem 4.14 and Remark 4.12, we deduce the following.

Corollary 4.15. *Let S be an inverse CA-AG-groupoid and $E(S) \neq \phi$. Then the relation defined on S by $\rho_k = \{(x, y) \in S \times S : (\forall e \in E(S))\, x_{p_1}x_{p_2}x_{p_3} = y_{q_1}y_{q_2}y_{q_3}\}$ is the* *maximal idempotent-separating congruence on S, where $x_{p_1}x_{p_2}x_{p_3}$is any permutation of the elements x, e, x' and $y_{q_1}y_{q_2}y_{q_3}$ is any permutation of the elements y, e, y'.*

Theorem 4.16. *Let ρ be the maximal idempotent-separating congruence on an inverse CA-AG-groupoid S with $E(S) \neq \phi$. Then S/ρ is fundamental.*

Proof. Let $x \in S$ and $e \in E(S)$ such that $[(x'\rho)(e\rho)](x\rho) = [(y'\rho)(e\rho)](y\rho)$. Then $(x'e \cdot x)\rho = (y'e \cdot y)\rho$, i.e. $(x'e \cdot x)\rho(y'e \cdot y)$. Now by medial, paramedial and left invertive laws, cyclic associativity, definition of inverse and Lemma 3.23

$$(x'e \cdot x)^2 = (x'e \cdot x)(x'e \cdot x) = (x'e \cdot x'e)(xx) = (ee \cdot x'x')(xx) = (xx \cdot x'x')e$$
$$= (x'x \cdot x'x)e = (x(x'x \cdot x'))e = (xx')e = (ex')x = (x'e)x.$$

Thus, $x'e \cdot x$ is an idempotent. It can be similarly shown that $y'e \cdot y$ is an idempotent. As ρ is idempotent-separating so $(x'e \cdot x)\rho(y'e \cdot y)$ implies $x'e \cdot x = y'e \cdot y \, \forall e \in E(S)$, which by definition of ρ implies $x\rho y$. Thus S/ρ is fundamental. □

Theorem 4.17. *Let $E(S) \neq \phi$ be the semilattice of idempotents on an inverse CA-AG-groupoid S. If $E(S/\rho)$ is the semilattice of idempotents of S/ρ, where ρ is maximal idempotent-separating congruence on S, then $E(S)$ and $E(S/\rho)$ are isomorphic.*

Proof. Define $\hat{\rho} : E(S) \to E(S/\rho)$ by $e\hat{\rho} = e\rho \, \forall e \in E(S)$. Let $f, g \in E(S)$ such that $f = g$. Then $f\rho = g\rho$, which implies $f\hat{\rho} = g\hat{\rho}$, thus $\hat{\rho}$ is well-defined. Since $(fg)\hat{\rho} = (fg)\rho = (f\rho)(g\rho) = (f\hat{\rho})(g\hat{\rho}),$ it follows that $\hat{\rho}$ is homomorphism. For one-one, let $f\hat{\rho} = g\hat{\rho}$, then $f\rho = g\rho$. As ρ is idempotent-separating, so from $f\rho = g\rho$ we have $f = g$. Thus $\hat{\rho}$ is one-one. As elements of $E(S/\rho)$ are of the form $e\rho$, where $e \in E(S)$ and for each $e\rho \in E(S/\rho)$ there exists $e \in E(S)$ such that $e\hat{\rho} = e\rho$, thus ρ is onto. Hence $\hat{\rho}$ is an isomorphism from E(S) to $E(S/\rho)$, i.e. $E(S) \cong E(S/\rho)$. □

5. Conclusion

We demonstrated that inverse CA-AG-groupoids exist. We precisely discussed some fundamental characteristics of inverse CA-AG-groupoid and established various properties of this class. We also extended the notion of equivalence relation and congruences to CA-AG-groupoids and investigated various congruences on CA-AG-groupoid and inverse CA-AG-groupoid. Moreover, we defined a maximal idempotent-separating congruence ρ on inverse CA-AG-groupoid and proved that S/ρ is fundamental and $E(S) \cong E(S/\rho)$. We used the modern techniques of Prover-9, Mace-4 and GAP to produce illustrative examples and counterexamples to improve the standard of this research work.

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