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φ-Geometrically Log $\mathfrak{h}$-Convex Functions

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Abstract. In this paper, we present new classes of φ-geometrically log $\mathfrak{h}$-convex mappings in the first sense and in the second sense. We establish Hermite-Hadamard(H-H) type inequalities for these classes. It is proved that the class of generalized geometrically log $\mathfrak{h}$-convex mappings in both senses includes several new and known classes of log $\mathfrak{h}$- convex mappings. Results obtained in this article can be viewed as a new contributions in this area of research.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
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1. Introduction

Different new and innovative techniques brought the revolutionary results in the theory of convex analysis. Hence variant new classes of convex mappings has been introduced and investigated for the desirable results. Many researchers have been attracted to study different aspects of convex mappings, see [1, 3, 5, 7, 13, 14, 15, 20, 22, 25].

The theory of inequalities plays a vital role in the formation of many new inequalities in convex analysis and it has been remained a constant inspiration for many researchers. This is one of the reason which makes theory of convex analysis more attractive. The (H-H) inequality plays very important role in developing many new and important inequalities and it has triggered huge amount of attention and interest in recent years. Equally important is the field of variational inequalities associated with convex analysis, which is major source of applications, numerical analysis, dynamical systems and fixed points. For some recent developments, see [11, 12, 17, 18, 19, 24, 30, 31] and the references therein.

An important generalization of convex mappings was the introduction of $\mathfrak{h}$-convex funcions by Varosanec [33], which include s-convex [2], p convex [6] and Godunova-Levine [10] mappings as its special cases. For different properties and other aspects of $\mathfrak{h}$-convex mappings, see [22, 28, 29, 32]. Recently Gordji et al. [8] has introduced the notion of φ-convex. This class generalizes the class of convex mappings. For recent developments of these nonconvex mappings , see [4, 9, 21, 23, 26, 27].

Motivated by this ongoing research, we introduce a new class of φ-convex mappings, known as φ-geometrically log $\mathfrak{h}$-convex mappings in the first sense and in the second sense, respectively. We derive some new (H-H) integral inequalities for these nonconvex mappings. Some cases are discussed, which can be obtained as special cases from these new results.

2. Notations and Preliminaries

Let $\sigma = [\tilde{\varrho}, \eta]$ and J be the intervals in real line $\mathfrak{R}$, $[0,1] \subseteq J$. Let $\varsigma : \sigma \to \mathfrak{R}$ and $\mathfrak{h} : J \to \mathfrak{R}$ be two non negative and continuous mappings and $\varphi(\cdot,\cdot) : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ be a continuous bifunction.

Throughout this paper, we will use the following notations.
$\mathfrak{R} = (-\infty, +\infty)$, $\mathfrak{R}_+ = (0, \infty)$, $\mathfrak{R}_- = (-\infty, 0)$ and $\bar{L} = \bar{L}(\varrho, \eta) = \big[\varphi(\varsigma(\eta), \varsigma(\tilde{\varrho})) = \varsigma(\eta) - \varsigma(\tilde{\varrho})\big]$.

Let us recall some basic and new definitions as follows.

Definition 2.1. *A set $\sigma \subset \mathfrak{R}_+$ is said to be geometrically convex set, if*

$$\tilde{\varrho}^{1-\xi}\eta^{\xi} \in \sigma, \quad \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

Definition 2.2. [16] *A mapping $\varsigma : \sigma \subset \mathfrak{R}_+ \to \mathfrak{R}$ is said to be geometrically convex on σ, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \le (1-\xi)\varsigma(\tilde{\varrho}) + \xi\varsigma(\eta), \quad \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

Definition 2.3. [20] *A mapping $\varsigma : \sigma \subset \mathfrak{R} \to \mathfrak{R}$ is said to be a φ-geometrically convex mapping with respect to* (w.r.to) *a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \le (1-\xi)\varsigma(\tilde{\varrho}) + \xi[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))], \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

We now introduce some new classes which are φ-geometrically $\mathfrak{h}$ and φ-geometrically log $\mathfrak{h}$- convex mappings in the first sense and in the second sense.

Definition 2.4. *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A mapping $\varsigma : \sigma \to \mathfrak{R}$ is said to be φ-geometrically $\mathfrak{h}$-convex mapping in the first sense w.r.to a $\varphi(\cdot,\cdot)$ and a nonnegative mapping $\mathfrak{h}$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \le \mathfrak{h}(1-\xi)\varsigma(\tilde{\varrho}) + \mathfrak{h}(\xi)[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))], \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

If $\bar{L} = \bar{L}(\eta, \tilde{\varrho})$, then the Definition 2.4 reduces to

Definition 2.5. [25] *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A non-negative mapping $\varsigma : \sigma \to \mathfrak{R}$ is said to be geometrically $\mathfrak{h}$-convex, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(1-\xi)\varsigma(\tilde{\varrho}) + \mathfrak{h}(\xi)\varsigma(\eta), \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

Definition 2.6. *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ- geometrically log $\mathfrak{h}$-convex in the first sense w.r.to a $\varphi(\cdot,\cdot)$ and a nonnegative mapping $\mathfrak{h}$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq [\varsigma(\tilde{\varrho})]^{\mathfrak{h}(1-\xi)}[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]^{\mathfrak{h}(\xi)}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1]. \quad (2.\ 1)$$

If $\xi = \frac{1}{2}$, then

$$\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq [\varsigma(\tilde{\varrho})]^{\mathfrak{h}(\frac{1}{2})}[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]^{\mathfrak{h}(\frac{1}{2})}, \forall \tilde{\varrho}, \eta \in \sigma. \quad (2.\ 2)$$

The mapping ς is known as φ-geometrically Jensen log $\mathfrak{h}$-convex mapping.

From Definition 2.6, we have

$$\log \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(1-\xi)\log[\varsigma(\tilde{\varrho})] + \mathfrak{h}(\xi)\log[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))],$$

and

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(1-\xi)[\varsigma(\tilde{\varrho})] + \mathfrak{h}(\xi)[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))].$$

It shows that every φ-geometrically log $\mathfrak{h}$-convex mapping is a φ-geometrically $\mathfrak{h}$-convex mapping. However the converse is not true.

Now we will discuss some special cases of φ-geometrically log $\mathfrak{h}$-convex mappings in the first sense.

If $\bar{L} = \bar{L}(\eta, \tilde{\varrho})$, then Definition 2.6 becomes

Definition 2.7. . *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A mapping $\varsigma : \sigma \to (0, \infty)$ is said to be log $\mathfrak{h}$-convex or multiplicatively $\mathfrak{h}$-convex in the first sense, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq [\varsigma(\tilde{\varrho})]^{\mathfrak{h}(1-\xi)}[\varsigma(\eta)]^{\mathfrak{h}(\xi)}, \quad \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

Now we will discuss some special cases of φ-geometrically log $\mathfrak{h}$-convex mappings in the first sense.

I. If $\mathfrak{h}(\xi) = \xi$, then Definition 2.6 becomes

Definition 2.8. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ-geometrically log convex w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq [\varsigma(\tilde{\varrho})]^{1-\xi}[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]^{\xi}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

II. If $\mathfrak{h}(\xi) = \xi^s$, then Definition 2.6 becomes

Definition 2.9. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be Φ-geometrically log s-convex for $s \in (0,1)$ w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq [\varsigma(\tilde{\varrho})]^{(1-\xi)^s}[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]^{\xi^s}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

III. If $\mathfrak{h}(\xi) = 1$, then Definition 2.6 becomes

Definition 2.10. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be a φ-geometrically log P-convex w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq [\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))], \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

IV. If $\mathfrak{h}(\xi) = \frac{1}{\xi}$, then Definition 2.6 becomes

Definition 2.11. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be a φ-geometrically log Godunova-Levine convex w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq [\varsigma(\tilde{\varrho})]^{\frac{\xi}{1-\xi}}[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]^{\frac{1}{\xi}}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in (0,1).$$

We now introduce a new class of φ-geometrically convex mappings, which is called the φ-geometrically $\mathfrak{h}$-convex mappings and in the second sense.

Definition 2.12. *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A mapping $\varsigma : \sigma \to \mathfrak{R}$ is said to be φ-geometrically $\mathfrak{h}$-convex mapping in the second sense w.r.to a φ and non negative mapping $\mathfrak{h}$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(1-\xi)\mathfrak{h}(\xi)[2\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))], \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

If $\bar{L} = \bar{L}(\eta, \tilde{\varrho})$, then Definition 2.12 reduces to the following new concept.

Definition 2.13. *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A non-negative mapping $\varsigma : \sigma \to (0,\infty)$is said to be h-convex in the second sense, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(1-\xi)\mathfrak{h}(\xi)[\varsigma(\tilde{\varrho}) + \varsigma(\eta)], \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

Definition 2.14. *Let $\mathfrak{h} : J \to \mathfrak{R}$ be a non-negative mapping. A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ-geometrically log $\mathfrak{h}$-convex in the second sense w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}^{\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1]. \tag{2. 3}$$

If $\xi = \frac{1}{2}$, then

$$\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq \left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}^{h^2(\frac{1}{2})}, \forall \tilde{\varrho}, \eta \in \sigma.$$

The mapping ς is known as the φ-geometrically Jensen log $\mathfrak{h}$-convex mappings in the second sense.

From (2. 3), we have

$$\log \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\left\{ \log[\varsigma(\tilde{\varrho})] + \log[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}.$$

Now we will discuss some special cases of φ-geometrically log $\mathfrak{h}$-convex mappings in the second sense.

I. If $\mathfrak{h}(\xi) = \xi$, then Definition 2.14 becomes

Definition 2.15. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ-geometrically log tgs-convex w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}^{\xi(1-\xi)}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

II. If $\mathfrak{h}(\xi) = \xi^s$, then Definition 2.14 becomes

Definition 2.16. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ-geometrically log (tgs, s)-convex in the second sense for $s \in (0,1]$ w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}^{\xi^s(1-\xi)^s}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

III. If $\mathfrak{h}(\xi) = \xi^p$ and $\mathfrak{h}(1-\xi) = (1-\xi)^q$ then, Definition 2.14 becomes

Definition 2.17. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ- geometrically log beta-convex in the second sense w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}^{\xi^p(1-\xi)^q}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

IV. If $\mathfrak{h}(\xi) + \mathfrak{h}(1-\xi) = 1$ and $\mathfrak{h}(\xi) = \xi^p$, then Definition 2.14 becomes

Definition 2.18. *A mapping $\varsigma : \sigma \to \mathfrak{R}_+$ is said to be φ- geometrically log Toader-convex w.r.to a $\varphi(\cdot,\cdot)$, if*

$$\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho}))]\right\}^{\xi^p(1-\xi^p)}, \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0,1].$$

We would like to point out that for suitable choice of the bifunction $\varphi(.,.)$ and the non-negative mapping $\mathfrak{h}(.)$, one can obtain several known and new classes of convex mappings as special cases. This clearly shows that these concepts are quite flexible and unifying one.

3. MAIN RESULTS

We develop several new (H-H) type integral inequalities for φ-geometrically log $\mathfrak{h}$-convex mapping in the first sense and in the second sense in this section.

Theorem 3.1. *Let ς be a φ-geometrically log $\mathfrak{h}$-convex mapping in the first sense on σ and $\mathfrak{h}(\frac{1}{2}) \neq 0$. Then*

$$\begin{aligned}\log \varsigma(\sqrt{\tilde{\varrho}\eta})^{\frac{1}{\mathfrak{h}(\frac{1}{2})}} \quad - \quad & \frac{1}{(\log \eta - \log \tilde{\varrho})} \int_{\tilde{\varrho}}^{\eta} \log[\frac{\varsigma(x) + \varphi(\varsigma(\frac{\varrho\eta}{x}), \varsigma(x))}{x}]\mathrm{d}x \\ & \leq \frac{1}{(\log \eta - \log \tilde{\varrho})} \int_{\tilde{\varrho}}^{\eta} \log[\frac{\varsigma(x)}{x}]\mathrm{d}x \\ & \leq \log \left\{ [\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho})] \right\} \int_0^1 \mathfrak{h}(\xi)\mathrm{d}\xi.\end{aligned}$$

Proof. Let ς be φ-geometrically log $\mathfrak{h}$-convex mapping in the first sense. Then

$$\begin{aligned}\log \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \big\{\mathfrak{h}(1-\xi)\log[\varsigma(\tilde{\varrho})] + \mathfrak{h}(\xi)\log[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho})]\big\}, \\ \forall \tilde{\varrho}, \eta \in \sigma, \xi \in [0, 1]. \qquad (3.\ 4)\end{aligned}$$

Integrating (3. 4) w.r.to ξ on [0,1], we have

$$\begin{aligned}\int_0^1 \log \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})\mathrm{d}\xi \quad & \leq \quad \int_0^1 \big\{\mathfrak{h}(1-\xi)\log[\varsigma(\tilde{\varrho})] + \mathfrak{h}(\xi)\log[\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho})]\big\}\mathrm{d}\xi \\ & = \quad \log\big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho})]\big\} \int_0^1 \mathfrak{h}(\xi)\mathrm{d}\xi.\end{aligned}$$

Thus

$$\frac{1}{\log \eta - \log \tilde{\varrho}} \int_{\tilde{\varrho}}^{\xi} \log[\frac{\varsigma(x)}{x}]\mathrm{d}x \leq \log\big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho}) + \varphi(\varsigma(\eta), \varsigma(\tilde{\varrho})]\big\} \int_0^1 \mathfrak{h}(\xi)\mathrm{d}\xi. \qquad (3.\ 5)$$

Consider the Jensen form of the φ-geometrically log convex mapping in the first sense and subsituting $x = (\tilde{\varrho}^{1-\xi}\eta^{\xi})$ and $y = (\tilde{\varrho}^{\xi}\eta^{1-\xi})$, we have

$$\begin{aligned}\varsigma(\sqrt{\tilde{\varrho}\eta}) \quad & \leq \quad [\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})]^{\mathfrak{h}(\frac{1}{2})}[\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) + \varphi(\varsigma(\tilde{\varrho}^{\xi}\eta^{1-\xi}), \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}))]^{\mathfrak{h}(\frac{1}{2})} \\ & = \quad \left\{[\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})][\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) + \varphi(\varsigma(\tilde{\varrho}^{\xi}b^{1-\xi}), \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}))]\right\}^{\mathfrak{h}(\frac{1}{2})}.\end{aligned}$$

This implies that

$$\log \varsigma(\sqrt{\tilde{\varrho}\eta}) \quad \leq \quad \mathfrak{h}(\frac{1}{2}) \log\left\{[\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})][\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) + \varphi(\varsigma(\tilde{\varrho}^{\xi}\eta^{1-\xi}), \varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}))]\right\}. \qquad (3.\ 6)$$

Integrating (3. 6) on [0,1], we have

$$\frac{1}{\mathfrak{h}(\frac{1}{2})}\log\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\varrho}^{\eta}\Big\{\log[\frac{\varsigma(x)}{x}]+\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\varrho\eta}{x}),\varsigma(x))}{x}]\Big\}\mathrm{d}x.$$

Thus

$$\begin{aligned}\frac{1}{\mathfrak{h}(\frac{1}{2})}\log\varsigma(\sqrt{\tilde{\varrho}\eta}) \quad - \quad & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\tilde{\varrho}\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x\\ &\leq \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x.\end{aligned} \tag{3. 7}$$

Combining(3. 5) and (3. 7), we have

$$\begin{aligned}\log\varsigma(\sqrt{\tilde{\varrho}\eta})^{\frac{1}{\mathfrak{h}(\frac{1}{2})}} \quad - \quad & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\varrho\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x\\ &\leq \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\\ &\leq \log\Big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\Big\}\int_0^1\mathfrak{h}(\xi)\mathrm{d}\xi,\end{aligned}$$

which is the proof. □

Corollary 3.2. *If* $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ *and using the assumptions of* Theorem 3.1, *we obtain a new result.*

$$\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^{\frac{1}{2\mathfrak{h}(\frac{1}{2})}} \leq \exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x \leq \big[\varsigma(\tilde{\varrho})\varsigma(\eta)\big]\int_0^1\mathfrak{h}(\xi)\mathrm{d}\xi.$$

Now we will have some special cases of Theorem 3.1.

(I) If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ and $\mathfrak{h}(\xi)=\xi$ in Theorem 3.1, then we obtain a following new result.

Theorem 3.3. *Let* $\varsigma:\sigma\to\mathfrak{R}_+$ *be a* φ*-geometrically log* $\mathfrak{h}$*-convex mapping on* σ*. Then*

$$\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq \exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x \leq \sqrt{\big[\varsigma(\tilde{\varrho})\varsigma(\eta)\big]}.$$

(II) If $\mathfrak{h}(\xi)=\xi^s$, then Theorem 3.1 becomes

Theorem 3.4. *Let $\varsigma : \sigma \to \Re_+$ be a φ-geometrically log s-convex mapping on σ with $s \in (0,1)$. Then*

$$\begin{aligned}\log\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^{2^s} &- \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\varrho\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x\\ &\le \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\\ &\le \big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\big\}\int_0^1\xi^s\mathrm{d}\xi\\ &= \big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\big\}\frac{1}{(s+1)}.\end{aligned}$$

Corollary 3.5. *If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$, then, under the assumptions of* Theorem 3.4, *we have a new result.*

$$\begin{aligned}\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^{2^{s-1}} &\le \exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\\ &\le \big\{[\varsigma(\tilde{\varrho})][\varsigma(\eta)]\big\}\int_0^1\xi^s\mathrm{d}\xi\\ &= \big\{[\varsigma(\tilde{\varrho})][\varsigma(\eta)]\big\}\frac{1}{(s+1)}.\end{aligned}$$

Corollary 3.6. *If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ and $s=1$ in* Theorem 3.4, *we have a new result.*

$$\varsigma(\sqrt{\tilde{\varrho}\eta}) \le \exp\frac{1}{(\log\eta-\log\varrho)}\int_{\varrho}^{\eta}\log\varsigma(x)\mathrm{d}x \le \sqrt{\big[\varsigma(\tilde{\varrho})\varsigma(\eta)\big]}.$$

(III) If $\mathfrak{h}(\xi)=1$, then the Theorem 3.1 becomes the following new result.

Theorem 3.7. *Let $\varsigma : \sigma \to (0,\infty)$ be a φ-geometrically log P-convex mapping on σ. Then*

$$\begin{aligned}\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big) &- \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\tilde{\varrho}\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x\\ &\le \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\\ &\le \big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\big\}.\end{aligned}$$

Corollary 3.8. *If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ in* Theorem 3.7, *we have a new result.*

$$\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big) \le \exp\frac{2}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x \le \big[\varsigma(\tilde{\varrho})\varsigma(\eta)\big]^2.$$

(IV) If $\mathfrak{h}(\xi)=\frac{1}{\xi}$, then, Theorem 3.1 reduces to the following new result.

Theorem 3.9. *Let $\varsigma : \sigma \to (0,\infty)$ be a φ-geometrically log Godunova-Levin-convex mapping on σ. Then*

$$\begin{aligned}\frac{1}{2}\log\varsigma(\sqrt{\tilde{\varrho}\eta}) \quad - \quad & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\tilde{\varrho}\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x \\ & \leq \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\mathrm{d}x,\end{aligned}$$

and

$$\frac{1}{\log\eta-\log\tilde{\varrho}}\int_{\tilde{\varrho}}^{\eta}\tau x\log[\frac{\varsigma(x)}{x}]\mathrm{d}x \leq \frac{1}{2}\log\left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\right\},$$

where $\tau x = (\frac{x-a}{b-a})(\frac{b-x}{b-a})$.

Corollary 3.10. *If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ in* Theorem 3.9, *then we obtain a following new result.*

$$\varsigma(\sqrt{\tilde{\varrho}\eta})^{\frac{1}{4}} \leq \exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x,$$

and

$$\frac{1}{\log\eta-\log\tilde{\varrho}}\int_{\tilde{\varrho}}^{\eta}\tau x\log[\frac{\varsigma(x)}{x}\mathrm{d}x \leq \frac{1}{2}\log\left\{[\varsigma(\tilde{\varrho})][\varsigma(\eta)]\right\},$$

where $\tau x = (\frac{x-a}{b-a})(\frac{b-x}{b-a})$.

Theorem 3.11. *Let ς be a φ-geometrically log $\mathfrak{h}$-convex mapping in the second sense. Then*

$$\begin{aligned}\log\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^{\frac{1}{\mathfrak{h}^2(\frac{1}{2})}} \quad - \quad & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\tilde{\varrho}\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x \\ \leq \quad & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x \\ \leq \quad & \log\left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\right\}\int_0^1\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\mathrm{d}\xi.\end{aligned}$$

Proof. Let ς be φ-geometrically log $\mathfrak{h}$-convex mapping in the second sense. Then

$$\log\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi}) \leq \mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\big\{\log[\varsigma(\tilde{\varrho})]+\log[\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\big\}. \tag{3. 8}$$

Integrating (3. 8) w.r.to ξ on [0,1], we have

$$\begin{aligned}& \int_0^1\log\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})\mathrm{d}\xi \\ \leq \quad & \int_0^1\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\big\{\log[\varsigma(\tilde{\varrho})]+\log[\varsigma(\tilde{\varrho})+\Phi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\big\}\mathrm{d}\xi \\ = \quad & \log\big\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\Phi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\big\}\int_0^1\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\mathrm{d}\xi.\end{aligned}$$

Thus

$$\frac{1}{\log\eta-\log\tilde{\varrho}}\int_{\tilde{\varrho}}^{\eta}\log\frac{\varsigma(x)}{x}\mathrm{d}x\leq\log\left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\Phi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\right\}\int_0^1\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\mathrm{d}\xi. \tag{3. 9}$$

Consider the Jensen form of the φ-geometrically log$\mathfrak{h}$-convex mapping in the second sense and substituting $x=(\tilde{\varrho}^{1-\xi}\eta^{\xi})$ and $y=(\tilde{\varrho}^{\xi}\eta^{1-\xi})$, we have

$$\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq \left\{[\varsigma[(\tilde{\varrho}^{1-\xi}\eta^{\xi})][\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})+\varphi(\varsigma(\tilde{\varrho}^{\xi}b^{1-\xi})),\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})]\right\}^{\mathfrak{h}^2(\frac{1}{2})}.$$

This implies

$$\begin{aligned}\log\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq\ & \mathfrak{h}^2(\frac{1}{2})\log\left\{[\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})][\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})\right.\\ &\left.+\varphi(\varsigma(\tilde{\varrho}^{\xi}b^{1-\xi})),\varsigma(\tilde{\varrho}^{1-\xi}\eta^{\xi})]\right\}.\end{aligned} \tag{3. 10}$$

Integrating (3. 10) w.r.to ξ on [0,1], we have

$$\frac{1}{\mathfrak{h}^2(\frac{1}{2})}\log\varsigma(\sqrt{\tilde{\varrho}\eta}) \leq \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\varrho}^{\eta}\Big\{\log[\frac{\varsigma(x)}{x}]+\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\varrho\eta}{x}),\varsigma(x))}{x}]\Big\}\mathrm{d}x.$$

Thus

$$\begin{aligned}\frac{1}{\mathfrak{h}^2(\frac{1}{2})}\log\varsigma(\sqrt{\tilde{\varrho}\eta}) - & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\tilde{\varrho}\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x\\ &\leq\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x.\end{aligned} \tag{3. 11}$$

Combining (3. 9) and (3. 11), we have

$$\begin{aligned}\log\varsigma(\sqrt{\tilde{\varrho}\eta})^{\frac{1}{\mathfrak{h}^2(\frac{1}{2})}} - & \frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)+\varphi(\varsigma(\frac{\tilde{\varrho}\eta}{x}),\varsigma(x))}{x}]\mathrm{d}x\\ &\leq\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\\ &\leq\log\left\{[\varsigma(\tilde{\varrho})][\varsigma(\tilde{\varrho})+\varphi(\varsigma(\eta),\varsigma(\tilde{\varrho})]\right\}\int_0^1\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\mathrm{d}\xi,\end{aligned}$$

which is the proof. □

Corollary 3.12. *If* $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$*, then, under the assumption of* Theorem 3.11*, we have*

$$\begin{aligned}\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^{\frac{1}{2\mathfrak{h}^2(\frac{1}{2})}} &\leq \exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log[\frac{\varsigma(x)}{x}]\mathrm{d}x\\ &\leq \big[\varsigma(\tilde{\varrho})\varsigma(\eta)\big]\int_0^1\mathfrak{h}(\xi)\mathfrak{h}(1-\xi)\mathrm{d}\xi.\end{aligned}$$

Now we will discuss some special cases of Theorem 3.11.

I. If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ and $\mathfrak{h}(\xi)=\xi$, then we have a following new result.

Theorem 3.13. *Let ς be φ-geometrically log tgs-convex mapping in the second sense on σ. Then*

$$\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^2\leq\exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log\varsigma(x)\mathrm{d}x\leq\big[\{\varsigma(\tilde{\varrho})\}\{\varsigma(\eta)\}\big]^{\frac{1}{6}}.$$

II. If $\bar{L}=\bar{L}(\eta,\tilde{\varrho})$ and $\mathfrak{h}(\xi)=\xi^P$, then we have a following new result.

Theorem 3.14. *Let ς be φ-geometrically log Toader-convex mapping in the second sense on σ. Then*

$$\varsigma\big(\sqrt{\tilde{\varrho}\eta}\big)^{2^{p-1}}\leq\exp\frac{1}{(\log\eta-\log\tilde{\varrho})}\int_{\tilde{\varrho}}^{\eta}\log\varsigma(x)\mathrm{d}x\leq\big\{[\varsigma(\tilde{\varrho})][\varsigma(\eta)]\beta(p+1,p+1)\big\}.$$

4. Conclusion

In this article, we have introduced two new classes of generalized convex functions, which are known as generalized geometrically log h-convex functions in the first sense and in the second sense. We have shown that these classes unifies several new classes of generalized geometrically log-convex functions. We have also obtained several new integral inequalities of Hermite-Hadamard type for these classes. Some applications are also discussed. Applications of these concepts in different areas is an interesting problem for further research.

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6. Authors contributions

All the authors worked jointly and contributed equaly. They all read and approved the final manuscript.

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