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Some Weighted Hermite-Hadamard-Noor Type Inequalities for Differentiable Preinvex and Quasi Preinvex Functions

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Abstract. In this paper, we derive several weighted Hermite-Hadmard-Noor type inequalities for the differentiable preinvex functions and quasi preinvex functions.

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1. Introduction

Any paper on Hermite-Hadamard type inequalities seems to be incomplete without mentioning the famous Hermite-Hadamard inequality, which states as follows:

Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping of one variable and $e, j \in I$ with $e < j$. Then

$$g\left(\frac{e+j}{2}\right) \leq \frac{1}{j-e}\int_e^j g(u)du \leq \frac{g(e)+g(j)}{2}. \tag{1. 1}$$

The inequalities in (1. 1) are turned over if g possesses concavity property. Inequalities (1.1) are distinguished in mathematical analysis due to its intense geometrical importance and usefulness (see [26]).

A number of papers have been written during the past few years which generalize, enrich and extend the inequalities (1. 1). For numerous results on Hermite-Hadamard type inequalities, the interested reader is suggested to read [1], [6], [7], [9]-[13], [17], [18], [25], [28]-[29], [31]-[32], [36], [38] and the references therein.

Approximation of the difference between the middle and the leftmost terms in (1. 1) has been a notable question in mathematical analysis see for instance [12, 13, 25, 38]. The most expressive work to give the answer of the above raised question are the articles of Kirmaci [12] and Pearce and Pečarić [25].

Now, we evoke that the concept of quasi-convex functions generalizes the concept of convex functions. More accurately, a function $g : [e, j] \to \mathbb{R}$ is said quasi-convex on $[e, j]$ if

$$g\left(u\alpha + (1-u)\beta\right) \leq \max\left\{g\left(\alpha\right), g\left(\beta\right)\right\}$$

for $u \in [0, 1]$ and $\forall \alpha, \beta \in [e, j]$. Evidently, the class of quasi-convex functions is broader than the class of convex functions (see [11]). For more results on Hermite-Hadamard type inequalities for quasi-convex functions we want to mention the concerned reader to [1], [9]-[11], [27] and the references stated in them.

Hwang [10], ascertained results for convex and quasi-convex functions, those results provide a weighted version of the findings given in [12] and [25].

The convex functions and convex sets have been generalized and extended in several directions using different techniques. Hanson [8], introduced the convex of invex functions which inspired its applications in optimization and related fields. Mond and Israel [5], introduced the concept of the preinvex functions and showed that preinvex implies invexity. Noor [22], proved that the minimum of the differentiable preinvex functions on the invex sets can be characterized by a class of variational inequalities which are called variational-like inequalities.

Let us recall the definitions of preinvexity and quasi preinvexity which are substantial generalizations of the notions of convexity and quasi-convexity respectively.

Definition 1. [35] Let $\emptyset \neq V \subseteq \mathbb{R}^n$ and $\phi : V \times V \to \mathbb{R}^n$. Let $\alpha \in V$, then V is exclaimed to be invex at α with regard to $\phi\left(\cdot, \cdot\right)$, if

$$\alpha + u\phi(\beta, \alpha) \in V, \forall \alpha, \beta \in V, u \in [0, 1].$$

The set V is known to be an invex set in connection to ϕ if V is invex at every $\alpha \in V$. The invex set V is also renamed as an ϕ-connected set.

Remark 2. [2] The Definition 1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point α which is contained in V. We do not require that the point β should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that β should be an end point of the path for every pair of points α, $\beta \in V$ then $\phi(\beta, \alpha) = \beta - \alpha$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\phi(\beta, \alpha) = \beta - \alpha$, but the converse is not necessarily true, see [20, 37] and the references therein.

Definition 3. [35] A function $g : V \to \mathbb{R}$ on an invex set $V \subseteq \mathbb{R}^n$ is defined to be preinvex with regard to ϕ, if

$$g(\alpha + u\phi(\beta, \alpha)) \leq (1-u)\, g(\alpha) + ug(\beta), \forall \alpha, \beta \in V, u \in [0,1].$$

The function g is said to be preincave iff $-g$ is preinvex.

Definition 4. [3] A function $g : V \to \mathbb{R}$ on an invex set $V \subseteq \mathbb{R}^n$ is considered to be quasi preinvex with respect to ϕ, if

$$g(\alpha + u\phi(\beta, \alpha)) \leq \max \{g(\alpha), g(\beta)\}, \forall \alpha, \beta \in V, u \in [0,1].$$

The concept of quasi preinvexity is more general than the concept of quasi-convexity, see for example [3].

Noor [21] has shown that the function g is preinvex function on $[e, e+\phi(j,e)]$ if and only if the following inequalities holds:

$$g\left(\frac{2e+\phi(j,e)}{2}\right) \leq \frac{1}{\phi(j,e)} \int_e^{e+\phi(j,e)} g(x)\, dx \leq \frac{g(e)+g(j)}{2}. \qquad (1.\ 2)$$

The inequality (1. 2) is called the Hermite-Hadamard-Noor type inequalities for preinvex functions. This result is basic and is analogous to the original Hermite-Hadamard inequalities.

Note that if $\phi(j,e) = j - e$, then the inequality (1. 2) reduces to inequalities (1. 1).

The result given by (1. 2) has been extended and generalized in several directions, see for instance [3], [4], [14]-[16], [19], [21], [23], [24], [30], [33], [34] and the references therein.

The current paper is about new weighted integral inequalities of Hermite-Hadamard-Noor type in which preinvex and quasi preinvex functions are involved. Our findings take a broad view of those results appeared in a very fresh article of Hwang [10] and also provide weighted version of those results for preinvex and quasi preinvex functions which gives new bounds of the deference between the middle and the leftmost terms in Hermite-Hadamard-Noor type inequalities for the preinvex functions given above by (1. 2).

2. Main Results

The results of this sections depends entirely on the following lemma and throughout in this section we will use the notations: $V \subseteq \mathbb{R}$ an invex set with respect to the mapping $\phi : V \times V \to \mathbb{R}$, $L^{'}(e,j,u) = e + \left(\frac{1-u}{2}\right)\phi(j,e)$ and $U^{'}(e,j,u) = e + \left(\frac{1+u}{2}\right)\phi(j,e)$, where $e, j \in V^{\circ}$ (the interior of V) with $\phi(j,e) > 0$.

Lemma 5. *Let $g : V \to \mathbb{R}$ be a differentiable mapping on V° and $g^{'} \in L_1([e, e+\phi(j,e)])$, where $e, j \in V^{\circ}$ with $\phi(j,e) > 0$. If $h : [e, e+\phi(j,e)] \to [0,\infty)$ be a differentiable mapping. Then*

$$\begin{aligned}
&\frac{h(e)}{2}\left[g(e) + g(e+\phi(j,e))\right] - h(e+\phi(j,e))\, g\left(e + \frac{1}{2}\phi(j,e)\right) \\
&\quad + \frac{\phi(j,e)}{4}\int_0^1 \left[g\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right) + g\left(e + \left(\frac{1+u}{2}\right)\phi(j,e)\right)\right] \\
&\qquad \times \left[h^{'}\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right) + h^{'}\left(e + \left(\frac{1+u}{2}\right)\phi(j,e)\right)\right] du \\
&= \frac{\phi(j,e)}{4}\left\{\int_0^1 \left[h\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right) - h\left(e + \left(\frac{1+u}{2}\right)\phi(j,e)\right)\right.\right.
\end{aligned}$$

$$+h\left(e+\phi\left(j,e\right)\right)\right]\times\left[-g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]du\right\} \tag{2. 3}$$

holds.

Proof. We note that

$$\begin{aligned}
I_1 &= -\int_0^1\left[h\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)-h\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right.\\
&\qquad \left.+h\left(e+\phi\left(j,e\right)\right)\right]g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)du\\
&= \frac{2}{\phi\left(j,e\right)}\left[h\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)-h\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right.\\
&\qquad \left.+h\left(e+\phi\left(j,e\right)\right)\right]\times g\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)\Big|_0^1\\
&\quad +\int_0^1\left[h^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+h^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]\\
&\qquad \times g\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)du\\
&= \frac{2}{\phi\left(j,e\right)}\left[h\left(e\right)g\left(e\right)-h\left(e+\phi\left(j,e\right)\right)g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right]\\
&\quad +\int_0^1\left[h^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+h^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]\\
&\qquad \times g\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)du.
\end{aligned} \tag{2. 4}$$

and

$$\begin{aligned}
I_2 &= \int_0^1\left[h\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)-h\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right.\\
&\qquad \left.+h\left(e+\phi\left(j,e\right)\right)\right]g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)du\\
&= \frac{2}{\phi\left(j,e\right)}\left[h\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)-h\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right.\\
&\qquad \left.+h\left(e+\phi\left(j,e\right)\right)\right]g\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\Big|_0^1\\
&\quad +\int_0^1\left[h^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+h^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]\\
&\qquad \times g\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\\
&= \frac{2}{\phi\left(j,e\right)}\left[h\left(e\right)g\left(e+\phi\left(j,e\right)\right)-h\left(e+\phi\left(j,e\right)\right)g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right]\\
&\quad +\int_0^1\left[h^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+h^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]
\end{aligned}$$

$$\times g\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)du. \quad (2.\ 5)$$

Thus from (2. 4) and (2. 5), we have

$$\frac{\phi\left(j,e\right)}{4}\left[I_1+I_2\right]=\frac{h\left(e\right)}{2}\left[g\left(e\right)+g\left(e+\phi\left(j,e\right)\right)\right]-h\left(e+\phi\left(j,e\right)\right)g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)$$
$$+\frac{\phi\left(j,e\right)}{4}\left\{\int_0^1\left[h\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)-h\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]\right.$$
$$\left.\times\left[-g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)\right]du\right\}.$$

which is the required result. □

Remark 6. Suppose $\phi\left(j,e\right)=j-e$, then Lemma 5 becomes Lemma 2.1 from [10].

Now using Lemma 5, we shall intend to prove new upper bounds for the difference between the leftmost and the middle terms of weighted version of the Hermite-Hadamard-Noor type inequality from [21] using preinvex and qusi preinvex mappings.

Theorem 7. *Let* $g:V\to\mathbb{R}$ *is a differentiable mapping on* V° *and* $z:[e,e+\phi(j,e)]\to[0,\infty)$ *be continuous and symmetric to* $e+\frac{1}{2}\phi\left(j,e\right)$*, where* e*,* $j\in V^{\circ}$ *with* $\phi\left(j,e\right)>0$*. If* $\left|g^{'}\right|$ *is preinvex on* $[e,e+\phi(j,e)]$*,*

$$\left|\int_e^{e+\phi(j,e)}g\left(x\right)z\left(x\right)dx-g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\int_e^{e+\phi(j,e)}z\left(x\right)dx\right|$$
$$\leq\frac{\phi\left(j,e\right)}{2}\left[\left|g^{'}\left(e\right)\right|+\left|g^{'}\left(j\right)\right|\right]\int_0^1 M^{'}\left(z;e,j,u\right)du \quad (2.\ 6)$$

holds, where $M^{'}\left(z;e,j,u\right)=\int_e^{L^{'}(e,j,u)}z\left(x\right)dx$ *for all* $u\in[0,1]$*.*

Proof. Let $h\left(u\right)=\int_e^u z\left(x\right)dx$ for all $u\in\left[e,e+\phi\left(j,e\right)\right]$ in Lemma 5, we have

$$-g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\int_e^{e+\phi(j,e)}z\left(x\right)dx$$
$$+\frac{\phi\left(j,e\right)}{4}\int_0^1\left[g\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+g\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]$$
$$\times\left[z\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+z\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]du$$
$$=\frac{\phi\left(j,e\right)}{4}\left\{\int_0^1\left[\int_e^{L^{'}(e,j,u)}z\left(x\right)dx+\int_{U^{'}(e,j,u)}^{e+\phi(j,e)}z\left(x\right)dx\right]\right.$$
$$\left.\times\left[-g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)+g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right]du\right\}. \quad (2.\ 7)$$

Since $z\left(x\right)$ is symmetric to $e+\frac{1}{2}\phi\left(j,e\right)$, we have

$$z\left(e+\left(\frac{1-u}{2}\right)\phi\left(j,e\right)\right)=z\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right) \quad (2.\ 8)$$

and

$$\int_{e}^{L^{'}(e,j,u)} z(x)\,dx = \int_{U^{'}(e,j,u)}^{e+\phi(j,e)} z(x)\,dx \tag{2. 9}$$

for all $u \in [0,1]$. Hence by using (2. 8), we have

$$\begin{aligned}
&\frac{\phi(j,e)}{4}\int_0^1 \left[g\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)+g\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right] \\
&\qquad\times\left[z\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)+z\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right]du \\
&\quad=\frac{\phi(j,e)}{2}\int_0^1 g\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)z\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)du \\
&\quad+\frac{\phi(j,e)}{2}\int_0^1 g\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)z\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)du \\
&=\int_e^{e+\frac{1}{2}\phi(j,e)} g(x)\,z(x)\,dx+\int_{e+\frac{1}{2}\phi(j,e)}^{e+\phi(j,e)} g(x)\,z(x)\,dx=\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx.
\end{aligned} \tag{2. 10}$$

Using (2. 9) and (2. 10) in (2. 7), we get

$$\begin{aligned}
&\left|\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx-g\left(e+\frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\,dx\right| \\
&\qquad\leq\frac{\phi(j,e)}{2}\int_0^1 M^{'}(z;e,j,u) \\
&\quad\times\left[\left|g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|+\left|g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right|\right]du.
\end{aligned} \tag{2. 11}$$

Now by using the preinvexity of $\left|g^{'}\right|$ on $[e, e+\phi(j,e)]$, we obtain

$$\begin{aligned}
&\left|g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|+\left|g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right| \\
&\qquad\leq\left|g^{'}(e)\right|+\left|g^{'}(j)\right|
\end{aligned} \tag{2. 12}$$

for all $u \in [0,1]$. From (2. 11) and (2. 12) we get the the required inequality (2. 6). □

Remark 8. In Theorem 7, if we take $z(x)=\frac{1}{\phi(j,e)}$ for all $x \in [e, e+\phi(j,e)]$, then (2.6) becomes the inequality proved in Corollary 3.2 from [33].

Remark 9. If $\phi(j,e)=j-e$ in Theorem 7, then (2. 6) reduces to the result from [10, Theorem 2.2, page 70].

Remark 10. If $\phi(j,e)=j-e$ and $z(x)=\frac{1}{j-e}$ for all $x\in[e,j]$ in Theorem 7, we get the inequality proved in Theorem 2.2, page 138 from [12].

Theorem 11. *Let $g: V \to \mathbb{R}$ is a differentiable mapping on V° and $z:[e,e+\phi(j,e)]\to[0,\infty)$ be continuous and symmetric to $e+\frac{1}{2}\phi(j,e)$, where $e, j \in V^{\circ}$ with $\phi(j,e)>0$. If $\left|g^{'}\right|^{q}$ is preinvex on $[e,e+\phi(j,e)]$ for $q>1$, we have*

$$\left|\int_{e}^{e+\phi(j,e)} g(x)\,z(x)\,dx-g\left(e+\frac{1}{2}\phi(j,e)\right)\int_{e}^{e+\phi(j,e)} z(x)\,dx\right|$$
$$\leq \phi(j,e)\left[\frac{\left|g'(e)\right|^{q}+\left|g'(j)\right|^{q}}{2}\right]^{\frac{1}{q}}\left(\int_{0}^{1}\left[M'(z;e,j,u)\right]^{p}du\right)^{\frac{1}{p}}, \quad (2.\ 13)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $M'(z;e,j,u)$ *is defined as in Theorem 7.*

Proof. From the inequality (2. 11) in the proof of Theorem 7 and using the Hölder's integral inequality, we have

$$\left|\int_{e}^{e+\phi(j,e)} g(x)\,z(x)\,dx-g\left(e+\frac{1}{2}\phi(j,e)\right)\int_{e}^{e+\phi(j,e)} z(x)\,dx\right|$$
$$\leq \frac{\phi(j,e)}{2}\left(\int_{0}^{1}\left[M'(z;e,j,u)\right]^{p}du\right)^{\frac{1}{p}}\left[\left(\int_{0}^{1}\left|g'\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^{q}du\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{0}^{1}\left|g'\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right|^{q}du\right)^{\frac{1}{q}}\right]. \quad (2.\ 14)$$

By applying the power-mean inequality $t^{r}+s^{r}\leq 2^{1-r}(t+s)^{r}$ for $t>0,\ s>0$ and $r\leq 1$ and by the the preinvexity of $\left|g'\right|^{q}$ on $[e,e+\phi(j,e)]$ for $q>1$, we observe that the following inequality

$$\left(\int_{0}^{1}\left|g'\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^{q}du\right)^{\frac{1}{q}}+\left(\int_{0}^{1}\left|g'\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right|^{q}du\right)^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_{0}^{1}\left|g'\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^{q}du+\int_{0}^{1}\left|g'\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right|^{q}du\right]^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_{0}^{1}\left\{\left(\frac{1+u}{2}\right)\left|g'(e)\right|^{q}+\left(\frac{1-u}{2}\right)\left|g'(j)\right|^{q}\right.\right.$$
$$\left.\left.+\left(\frac{1-u}{2}\right)\left|g'(e)\right|^{q}+\left(\frac{1+u}{2}\right)\left|g'(j)\right|^{q}\right\}du\right]^{\frac{1}{q}}=2^{1-\frac{1}{q}}\left[\left|g'(e)\right|^{q}+\left|g'(j)\right|^{q}\right]^{\frac{1}{q}}$$
(2. 15)

holds. Application of the inequality (2. 15) in (2. 14), we get the needed inequality. □

Remark 12. In Theorem 11, if we take $z(x)=\frac{1}{\phi(j,e)}$ for all $x\in[e,e+\phi(j,e)]$ with $\phi(j,e)>0$, then (2. 13) becomes the inequality stated below

$$\left|\frac{1}{\phi(j,e)}\int_{e}^{e+\phi(j,e)} g(x)\,dx-g\left(e+\frac{1}{2}\phi(j,e)\right)\right|$$
$$\leq \frac{\phi(j,e)}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|g'(e)\right|^{q}+\left|g'(j)\right|^{q}}{2}\right]^{\frac{1}{q}}, \quad (2.\ 16)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 13. If we take $\phi(j,e) = j - e$ in Theorem 11, then (2. 13) becomes the following inequality

$$\left|\int_e^j g(x)\,z(x)\,dx - g\left(\frac{e+j}{2}\right)\int_e^j z(x)\,dx\right| \leq (j-e)\left[\frac{\left|g'(e)\right|^q + \left|g'(j)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1 [M(z;e,j,u)]^p\,du\right)^{\frac{1}{p}}, \quad (2.\ 17)$$

where $M(z;e,j,u) = \int_e^{L(e,j,u)} z(x)\,dx$, $L(e,j,u) = \left(\frac{1+u}{2}\right)e + \left(\frac{1-u}{2}\right)j$ for all $u \in [0,1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

A comparable result may be asserted in the following theorem.

Theorem 14. *Let $g : V \to \mathbb{R}$ is a differentiable mapping on V° and $z : [e, e+\phi(j,e)] \to [0,\infty)$ be continuous and symmetric to $e + \frac{1}{2}\phi(j,e)$, where $e, j \in V^\circ$ with $\phi(j,e) > 0$. If $\left|g'\right|^q$ is preinvex on $[e, e+\phi(j,e)]$ for $q \geq 1$, we have*

$$\left|\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx - g\left(e + \frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\,dx\right| \leq \phi(j,e)\left[\frac{\left|g'(e)\right|^q + \left|g'(j)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1 M'(z;e,j,u)\,du, \quad (2.\ 18)$$

where $M'(w;e,j,u)$ is defined as in Theorem 7.

Proof. Resuming from inequality (2. 11) in the proof of Theorem 7 and using the well-known Hölder's integral inequality, we have

$$\begin{aligned}\left|\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx - g\left(e + \frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\,dx\right| &\leq \frac{\phi(j,e)}{2}\left(\int_0^1 M'(z;e,j,u)\,du\right)^{1-\frac{1}{q}} \\ &\times \left\{\left[\int_0^1 M'(z;e,j,u)\left|g'\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^q du\right]^{\frac{1}{q}}\right. \\ &\left. + \left[\int_0^1 M'(z;e,j,u)\left|g'\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^q du\right]^{\frac{1}{q}}\right\}. \quad (2.\ 19)\end{aligned}$$

A usage of the power-mean inequality $t^r + s^r \leq 2^{1-r}(t+s)^r$ for $t > 0$, $s > 0$, $r \leq 1$, and by the the preinvexity of $\left|g'\right|^q$ on $[e, e+\phi(j,e)]$ for $q > 1$, we notice that the following inequality

$$\left[\int_0^1 M'(z;e,j,u)\left|g'\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^q du\right]^{\frac{1}{q}} + \left[\int_0^1 M'(z;e,j,u)\left|g'\left(e + \left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^q du\right]^{\frac{1}{q}}\Bigg\}$$

$$\leq 2^{1-\frac{1}{q}}\left(\int_0^1 M'(z;e,j,u)\,du\right)^{\frac{1}{q}}\left[\left|g'(e)\right|^q+\left|g'(j)\right|^q\right]^{\frac{1}{q}} \quad (2.\ 20)$$

holds. Applying the inequality (2. 20) in (2. 19), we get the inequality (2. 18). □

Corollary 15. *Suppose all the assumptions of Theorem 14 are satisfied and if* $z(x)=\frac{1}{\phi(j,e)}$ *for all* $x\in[e,e+\phi(j,e)]$ *with* $\phi(j,e)>0$*. Then*

$$\left|\frac{1}{\phi(j,e)}\int_e^{e+\phi(j,e)} g(x)\,dx-g\left(e+\frac{1}{2}\phi(j,e)\right)\right| \leq \frac{\phi(j,e)}{4}\left[\frac{\left|g'(e)\right|^q+\left|g'(j)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (2.\ 21)$$

Remark 16. Assume that $\phi(j,e)=j-e$ in Theorem 14, then (2. 18) diminishes to a result stated in Theorem 2.4 from [10, page 70].

Remark 17. For $q=1$, (2. 21) becomes the inequality proved in [33, Corollary 3.2]. If $q=\frac{p}{p-1}$ ($p>1$), we have $2^p>p+1$ for $p>1$, consequently

$$\frac{1}{4}<\frac{1}{2(p+1)^{\frac{1}{p}}}.$$

This shows that the inequality (2. 21) is better estimate than the one given by (2.16). Moreover, for $\phi(j,e)=j-e$ the inequality (2. 21) takes the form of the inequality proved in [25, Theorem 2, page 53].

The subsequent results are about quasi preinvex functions.

Theorem 18. *Let* $g:V\to\mathbb{R}$ *is a differentiable mapping on* V° *and* $z:[e,e+\phi(j,e)]\to[0,\infty)$ *be continuous and symmetric to* $e+\frac{1}{2}\phi(j,e)$*, where* e*,* $j\in V^\circ$ *with* $\phi(j,e)>0$*. If* $\left|g'\right|$ *is quasi preinvex on* $[e,e+\phi(j,e)]$*, we have*

$$\left|\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx-g\left(e+\frac{1}{2}\phi(j,e)\right)\int_c^{c+\phi(j,e)} z(x)\,dx\right| \leq \frac{\phi(j,e)}{2}\left\{\max\left(\left|g'(e)\right|,\left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right)\right.$$
$$\left.+\max\left(\left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|,\left|g'(e+\phi(j,e))\right|\right)\right\}\int_0^1 M'(z;e,j,u)\,du \quad (2.\ 22)$$

holds, where $M'(z;e,j,u)$ *is defined as in Theorem 7.*

Proof. We start from the inequality (2. 11) given in the proof of Theorem 7. Since $\left|g'\right|$ is quasi preinvexity on $[e,e+\phi(j,e)]$, hence for every $u\in[0,1]$, we obtain

$$\left|g'\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|\leq\max\left(\left|g'(e)\right|,\left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right) \quad (2.\ 23)$$

and

$$\left|g'\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right|\leq\max\left(\left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|,\left|g'(e+\phi(j,e))\right|\right). \quad (2.\ 24)$$

Combining the inequalities (2. 11), (2. 23) and (2. 24) produces the asserted inequality (2. 22). □

Corollary 19. *If all the assumptions of Theorem 18 are met. Moreover,*

(1) *If* $\left|g^{'}\right|$ *is non-decreasing on* $[e, e+\phi(j,e)]$, *we have that*

$$\left|\int_e^{e+\phi(j,e)} g(x)\, z(x)\, dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\, dx\right| \leq \frac{\phi(j,e)}{2}\left[\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right| + \left|g^{'}(e+\phi(j,e))\right|\right]\int_0^1 M^{'}(z;e,j,u)\, du \quad (2.\ 25)$$

holds and

(2) *If* $\left|g^{'}\right|$ *is non-increasing on* $[e, e+\phi(j,e)]$, *we have that*

$$\left|\int_e^{e+\phi(j,e)} g(x)\, z(x)\, dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\, dx\right| \leq \frac{\phi(j,e)}{2}\left[\left|g^{'}(e)\right| + \left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right]\int_0^1 M^{'}(z;e,j,u)\, du \quad (2.\ 26)$$

holds true.

Remark 20. If in Theorem 18, we take $z(x) = \frac{1}{\phi(j,e)}$ for all $x \in [e, e+\phi(j,e)]$ with $\phi(j,e) > 0$, then the inequality

$$\left|\frac{1}{\phi(j,e)}\int_e^{e+\phi(j,e)} g(x)\, dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\right| \leq \frac{\phi(j,e)}{8}\left\{\max\left(\left|g^{'}(e)\right|, \left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right) + \max\left(\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|, \left|g^{'}(e+\phi(j,e))\right|\right)\right\} \quad (2.\ 27)$$

holds. The inequality (2. 27) stands for a new improvement of the bound

$$\left|\frac{1}{\phi(j,e)}\int_e^{e+\phi(j,e)} g(x)\, dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\right|$$

for quasi preinvex functions and hence for preinvex functions. Moreover,

(1) If $\left|g^{'}\right|$ is non-decreasing $[e, e+\phi(j,e)]$, observe that

$$\left|\frac{1}{\phi(j,e)}\int_e^{e+\phi(j,e)} g(x)\, dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\right| \leq \frac{\phi(j,e)}{8}\left[\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right| + \left|g^{'}(e+\phi(j,e))\right|\right] \quad (2.\ 28)$$

holds and

(2) If $\left|g^{'}\right|$ is non-increasing $[e, e+\phi(j,e)]$, we notice that

$$\left|\frac{1}{\phi(j,e)}\int_e^{e+\phi(j,e)} g(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\right| \le \frac{\phi(j,e)}{8}\left[\left|g'(e)\right| + \left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right] \quad (2.\ 29)$$

holds valid.

Remark 21. If $\phi(j,e) = j-e$ in Theorem 18, then (2. 22) takes the form of the inequality established in Theorem 2.8 from [10] and the inequalities (2. 28) and (2. 29) recapture the inequalities given in the corollaries and remarks related to Theorem 2.8 from [10].

Remark 22. If $\phi(j,e) = j-e$ in Remark 20, then (2. 27), we get the following new results

$$\left|\frac{1}{j-e}\int_e^j g(x)\,dx - g\left(\frac{e+j}{2}\right)\right| \le \frac{j-e}{8}\left\{\max\left(\left|g'(e)\right|, \left|g'\left(\frac{e+j}{2}\right)\right|\right) + \max\left(\left|g'\left(\frac{e+j}{2}\right)\right|, \left|g'(j)\right|\right)\right\}. \quad (2.\ 30)$$

Moreover,

(1) If $\left|g'\right|$ is non-decreasing $[e,j]$, we have that

$$\left|\frac{1}{j-e}\int_e^j f(x)\,dx - g\left(\frac{e+j}{2}\right)\right| \le \frac{j-e}{8}\left[\left|g'\left(\frac{e+j}{2}\right)\right| + \left|g'(j)\right|\right] \quad (2.\ 31)$$

holds and

(2) If $\left|g'\right|$ is non-increasing $[e,j]$. Then

$$\left|\frac{1}{j-e}\int_e^j g(x)\,dx - g\left(\frac{e+j}{2}\right)\right| \le \frac{j-e}{8}\left[\left|g'(e)\right| + \left|g'\left(\frac{e+j}{2}\right)\right|\right]. \quad (2.\ 32)$$

Theorem 23. *Let $g: V \to \mathbb{R}$ is a differentiable mapping on V° and $z: [e, e+\phi(j,e)] \to [0,\infty)$ be continuous and symmetric to $e+\frac{1}{2}\phi(j,e)$, where $e, j \in V^\circ$ with $\phi(j,e) > 0$. If $\left|g'\right|^q$ is quasi preinvex on $[e, e+\phi(j,e)]$ for $q > 1$. Then*

$$\left|\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\,dx\right|$$
$$\le \frac{\phi(j,e)}{2}\left(\int_0^1\left[M'(z;e,j,u)\right]^p du\right)^{\frac{1}{p}}\left\{\left[\max\left(\left|g'(e)\right|^q, \left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|^q\right)\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[\max\left(\left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|^q, \left|g'(e+\phi(j,e))\right|^q\right)\right]^{\frac{1}{q}}\right\}, \quad (2.\ 33)$$

where $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. We begin with the inequality (2. 14) in the proof of Theorem 11. By the quasi preinvexity of $\left|g'\right|^q$ on $[e, e+\phi(j,e)]$ for $q > 1$, we have for every $u \in [0,1]$

$$\left|g'\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^q \le \max\left\{\left|g'(e)\right|^q, \left|g'\left(e+\frac{1}{2}\phi(j,e)\right)\right|^q\right\} \quad (2.\ 34)$$

and

$$\left|g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi\left(j,e\right)\right)\right|^{q} \leq \max\left\{\left|g^{'}\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|^{q},\left|g^{'}\left(e+\phi\left(j,e\right)\right)\right|^{q}\right\}. \quad (2.\ 35)$$

A usage of (2. 14), (2. 34) and (2. 35) gives us the required inequality (2. 33). □

Corollary 24. *Suppose all the conditions of Theorem 23 are satisfied. Moreover*

(1) *If* $\left|g^{'}\right|^{q}$ *is non-decreasing on* $[e, e+\phi(j,e)]$*, we notice that*

$$\left|\int_{e}^{e+\phi(j,e)} g\left(x\right)z\left(x\right)dx - g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\int_{e}^{e+\phi(j,e)} z\left(x\right)dx\right|$$
$$\leq \frac{\phi\left(j,e\right)}{2}\left[\left|g^{'}\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|+\left|g^{'}\left(e+\phi\left(j,e\right)\right)\right|\right]\left(\int_{0}^{1}\left[M^{'}\left(z;e,j,u\right)\right]^{p}du\right)^{\frac{1}{p}} \quad (2.\ 36)$$

holds, and

(2) *If* $\left|g^{'}\right|^{q}$ *is non-increasing on* $[e, e+\phi(j,e)]$*, we get that*

$$\left|\int_{e}^{e+\phi(j,e)} g\left(x\right)z\left(x\right)dx - g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\int_{e}^{e+\phi(j,e)} z\left(x\right)dx\right|$$
$$\leq \frac{\phi\left(j,e\right)}{2}\left[\left|g^{'}\left(e\right)\right|+\left|g^{'}\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|\right]\left(\int_{0}^{1}\left[M^{'}\left(z;e,j,u\right)\right]^{p}du\right)^{\frac{1}{p}} \quad (2.\ 37)$$

holds true, where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 25. If in Theorem 23, we take $z\left(x\right)=\frac{1}{\phi(j,e)}$ for all $x\in\left[e,e+\phi\left(j,e\right)\right]$ with $\phi\left(j,e\right)>0$, then we have the following inequality:

$$\left|\frac{1}{\phi\left(j,e\right)}\int_{e}^{e+\phi(j,e)} g\left(x\right)dx - g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|$$
$$\leq \frac{\phi\left(j,e\right)}{4\left(p+1\right)^{\frac{1}{p}}}\left[\max\left(\left|g^{'}\left(e\right)\right|,\left|g^{'}\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|\right)\right.$$
$$\left.+\max\left(\left|g^{'}\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|,\left|g^{'}\left(e+\phi\left(j,e\right)\right)\right|\right)\right]. \quad (2.\ 38)$$

The inequality (2. 38) signifies as a new enhancement of the bound

$$\left|\frac{1}{\phi\left(j,e\right)}\int_{e}^{e+\phi(j,e)} g\left(x\right)dx - g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|$$

for quasi preinvex functions and hence for preinvex functions. Moreover,

(1) If $\left|g^{'}\right|$ is non-decreasing on $[e, e+\phi(j,e)]$. Then

$$\left|\frac{1}{\phi\left(j,e\right)}\int_{e}^{e+\phi(j,e)} g\left(x\right)dx - g\left(e+\frac{1}{2}\phi\left(j,e\right)\right)\right|$$

$$\leq \frac{\phi(j,e)}{4(p+1)^{\frac{1}{p}}}\left[\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|+\left|g^{'}(e+\phi(j,e))\right|\right] \quad (2.\ 39)$$

is valid and

(2) If $\left|g^{'}\right|$ is non-increasing on $[e, e+\phi(j,e)]$. Then

$$\left|\frac{1}{\phi(j,e)}\int_{e}^{e+\phi(j,e)} g(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\right|$$
$$\leq \frac{\phi(j,e)}{4(p+1)^{\frac{1}{p}}}\left[\left|g^{'}(e)\right|+\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right], \quad (2.\ 40)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 26. If we take $\phi(j,e)=j-e$ in Remark 25, we get the results for quasi-convex functions.

Theorem 27. *Let $g: V \to \mathbb{R}$ is a differentiable mapping on V° and $z: [e, e+\phi(j,e)] \to [0,\infty)$ be continuous and symmetric to $e+\frac{1}{2}\phi(j,e)$, where $e, j \in V^{\circ}$ with $\phi(j,e)>0$. If $\left|g^{'}\right|^{q}$ is quasi preinvex on $[e, e+\phi(j,e)]$ for $q \geq 1$. Then*

$$\left|\int_{e}^{e+\phi(j,e)} g(x)\,z(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\int_{e}^{e+\phi(j,e)} z(x)\,dx\right|$$
$$\leq \frac{\phi(j,e)}{2}\left\{\left[\max\left(\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|^{q},\left|g^{'}(e+\phi(j,e))\right|^{q}\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\max\left(\left|g^{'}(e)\right|^{q},\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|^{q}\right)\right]^{\frac{1}{q}}\right\}\int_{0}^{1} M^{'}(z;e,j,u)\,du, \quad (2.\ 41)$$

where $M^{'}(z;e,j,u)$ is defined as in Theorem 7.

Proof. Beginning with the inequality (2. 19) in the proof of Theorem 14 and using the quasi preinvexity of $\left|g^{'}\right|^{q}$ on $[e, e+\phi(j,e)]$ for $q \geq 1$, we have

$$\left|g^{'}\left(e+\left(\frac{1-u}{2}\right)\phi(j,e)\right)\right|^{q} \leq \max\left\{\left|g^{'}(e)\right|^{q},\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|^{q}\right\} \quad (2.\ 42)$$

and

$$\left|g^{'}\left(e+\left(\frac{1+u}{2}\right)\phi(j,e)\right)\right|^{q} \leq \max\left\{\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|^{q},\left|g^{'}(e+\phi(j,e))\right|^{q}\right\} \quad (2.\ 43)$$

for every $u \in [0,1]$. Taking (2. 19), (2. 42) (2. 43) into consideration, we get the required inequality (2. 41). □

Corollary 28. *Suppose all the conditions of Theorem 27 are satisfied. Moreover*

(1) *If $\left|g^{'}\right|^{q}$ is non-decreasing on $[e, e+\phi(j,e)]$. Then*

$$\left|\int_{e}^{e+\phi(j,e)} g(x)\,z(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\int_{e}^{e+\phi(j,e)} z(x)\,dx\right|$$

$$\leq \frac{\phi(j,e)}{2}\left[\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|+\left|g^{'}(e+\phi(j,e))\right|\right]\int_0^1 M^{'}(z;e,j,u)\,du. \quad (2.\ 44)$$

and

(2) *If* $\left|g^{'}\right|^q$ *is non-increasing on* $[e, e+\phi(j,e)]$. *Then*

$$\left|\int_e^{e+\phi(j,e)} g(x)\,z(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\int_e^{e+\phi(j,e)} z(x)\,dx\right|$$
$$\leq \frac{\phi(j,e)}{2}\left[\left|g^{'}(e)\right|+\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|\right]\int_0^1 M^{'}(z;e,j,u)\,du. \quad (2.\ 45)$$

Remark 29. If in Theorem 27, we take $z(x)=\frac{1}{\phi(j,e)}$ for all $x\in[e,e+\phi(j,e)]$ with $\phi(j,e)>0$, the following inequality

$$\left|\frac{1}{\phi(j,e)}\int_e^{e+\phi(j,e)} g(x)\,dx - g\left(e+\frac{1}{2}\phi(j,e)\right)\right|$$
$$\leq \frac{\phi(j,e)}{8}\left\{\left[\max\left(\left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|^q, \left|g^{'}(e+\phi(j,e))\right|^q\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\max\left(\left|g^{'}(e)\right|^q, \left|g^{'}\left(e+\frac{1}{2}\phi(j,e)\right)\right|^q\right)\right]^{\frac{1}{q}}\right\}. \quad (2.\ 46)$$

holds. Moreover,

(1) If $\left|g^{'}\right|^q$ is non-decreasing on $[e, e+\phi(j,e)]$, the inequality (2. 28) holds and

(2) If $\left|g^{'}\right|^q$ is non-increasing on $[e, e+\phi(j,e)]$, the inequality (2. 29) holds.

Remark 30. If $\phi(j,e)=j-e$ in Theorem 27, then (2. 41) reduces to the inequality proved in Theorem 2.12 from [9] and the inequalities (2. 44) and (2. 45) takes the the form of the related inequalities mentioned in the remark followed by Theorem 2.12 from [10].

Remark 31. If $\phi(j,e)=j-e$ in Remark 29, we get results for qusi-convex functions.

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