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On Soft BCK-Modules

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Abstract. In the current work, the Molodtsov's idea of soft sets [14] is applied on the theory of BCK-modules [1]. The aim here, is to introduce the notion of soft BCK-modules and discuss its basic properties. In this regard, three theorems for soft BCK-modules isomorphism are developed. The notion of soft $X-$exactness of BCK-modules is introduced and its relation with soft $X-$isomorphism is studied. A transitivity between two soft $X-$exact sequences is also established.

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1. Introduction

The limitations of classical methods in dealing with uncertainties in economics, environmental sciences, engineering models and other fields persuade researchers to think otherwise. This result in development of fuzzy sets [20], rough set theory [16], probability theory, and other mathematical tools. However, these methods inherited their own difficulties and limitations. Consequently, in [14], Molodtsov proposed a new approach to deal with these difficulties, which is referred as the soft set theory. The idea attracted many researchers and the theory developed rapidly. A detailed theoretical study of soft sets and their implementation on decision making is discussed by Maji et al. in [12]. The application of soft sets is not limited to these areas only but it also motivated people working in more abstract areas of mathematics to apply soft sets in their areas. In this regard, Aktas et al. [3], introduced the notion of soft groups and developed its basic theory. Jun. applied soft set theory to BCK/BCI-algebras in [10]. Soft rings were introduced by Acar et al. in [2]. Atag$\ddot{u}$n and Sezgin, discussed soft substructures of rings, fields and modules in [4]. For other developments in soft set theory, we refer [15, 18, 19].

This paper is intended to apply the theory of soft sets on BCK-modules and thereby introducing the notion of soft BCK-modules. A BCK-module was presented in [1] as an action of a BCK-algebra on an abelian group. It has been explored by many researchers for various ventures (see [5, 6, 7, 11, 17]).

The paper begins with the preliminary concepts from the theories of soft sets, BCK-algebras and BCK-modules. The presentation of the notion soft BCK-module and developing its basic theory is one of the prime motives of the current work. Several examples and results have been presented in this regard. The three isomorphism theorems of soft BCK-modules are established. Finally, soft exactness of BCK-modules is introduced and a relationship between soft $X-$exactness and soft $X-$isomorphism aswell as transitivity of soft $X-$exact sequences of BCK-modules is established.

2. PRELIMINARIES

In this section, some preliminaries from the soft set theory, BCK-algebras and BCK-modules are included. All through the section, U is referred an initial universe, E is a parameters set, $A \subseteq E$ and $P(U)$ is the power set of U.

Definition 2.1. [13] **(Soft Set)**
A pair (F, A) is called a soft set($S-$set) over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.2. [13] *Let (F_1, A_1) and (F_2, A_2) are $S-$sets over U, (F_1, A_1) is called a soft subset of (F_2, A_2) if*

(i): *$A_1 \subset A_2$ and*
(ii): *$\forall \varepsilon \in A_1, F_1(\varepsilon)$ and $F_2(\varepsilon)$ are identical approximations.*

The above relation is denoted by $(F_1, A_1) \tilde{\subset} (F_2, A_2)$. Similarly, the notation $(F_1, A_1) \tilde{\supset} (F_2, A_2)$ denotes that (F_1, A_1) is a soft superset of (F_2, A_2).
Also, $(F_1, A_1) = (F_2, A_2)$, if $(F_1, A_1) \tilde{\subset} (F_2, A_2)$ and $(F_2, A_2) \tilde{\subset} (F_1, A_1)$.

Definition 2.3. [13] *Let (F_1, A_1) and (F_2, A_2) be $S-$sets over U*

(1) *The intersection of (F_1, A_1) and (F_2, A_2) is the $S-$set $(\tilde{F}, \tilde{A})$, where $\tilde{A} = A_1 \cap A_2$ and $\forall \varepsilon \in \tilde{A}, \tilde{F}(\varepsilon) = F_1(\varepsilon) \bigcap F_2(\varepsilon)$. This relationship is denoted by $(F_1, A_1) \tilde{\cap} (F_2, A_2) = (\tilde{F}, \tilde{A})$.*
(2) *The union of (F_1, A_1) and (F_2, A_2) is the $S-$set $(\tilde{\tilde{F}}, \tilde{\tilde{A}})$, where $\tilde{\tilde{A}} = A_1 \cup A_2$ and $\forall \varepsilon \in \tilde{\tilde{A}}$*

$$\tilde{\tilde{F}}(\varepsilon) = \begin{cases} F_1(\varepsilon) & \varepsilon \in A_1 \setminus A_2 \\ F_2(\varepsilon) & \varepsilon \in A_2 \setminus A_1 \\ F_1(\varepsilon) \cup F_2(\varepsilon) & \varepsilon \in A_1 \cap A_2 \end{cases}$$

This relationship is denoted by $(F_1, A_1) \mathbin{\tilde{\cup}} (F_2, A_2) = (\tilde{\tilde{F}}, \tilde{\tilde{A}})$.

Example 2.4. *Let* $U = \{1, 2, 3\}$ *and* E =*Set of Colours. Suppose that* $A_1 = \{red, green, blue\}$ *and* $A_2 = \{green, blue\}$ *are two subsets of* E. *Define* $F_1 : A_1 \to P(U)$ *and* $F_2 : A_2 \to P(U)$ *by*

$$F_1(\varepsilon) = \begin{cases} \{1\} & if\ \varepsilon = red \\ \{1,2\} & if\ \varepsilon = green \\ \{1,2,3\} & if\ \varepsilon = blue \end{cases} \qquad F_2(\varepsilon) = \begin{cases} \{1,3\} & if\ \varepsilon = green \\ \{2,3\} & if\ \varepsilon = blue \end{cases}$$

Then (F_1, A_1) *and* (F_2, A_2) *are a* $S-$*sets over* U. *Here,* $A_2 \subset A_1$, *but* $F_2(\varepsilon)$ *is not equal to* $F_1(\varepsilon)$ *for all* ε *in* A_2, *therefore,* (F_2, A_2) *is not* $S-$*subset of* (F_1, A_1).
Also, $(F_1, A_1) \mathbin{\tilde{\cap}} (F_2, A_2) = (\tilde{F}, \tilde{A})$ *and*

$$\tilde{F}(\varepsilon) = \begin{cases} \{1\} & if\ a = green \\ \{2,3\} & if\ a = blue \end{cases}$$

Similarly, $(F_1, A_1) \mathbin{\tilde{\cup}} (F_2, A_2) = (\tilde{\tilde{F}}, \tilde{\tilde{A}})$ *and*

$$\tilde{\tilde{F}}(\varepsilon) = \begin{cases} \{1\} & if\ a = red \\ \{1,2,3\} & if\ a = green \\ \{1,2,3\} & if\ a = blue \end{cases}$$

Next we define a BCK-algebra. It is an important class of logical structure introduced as a natural generalization of propositional calculus by K. Iseki and S. Tanaka in [9]. Several researchers have been investigating it since then.

Definition 2.5. [9] **(BCK-Algebra)**
A BCK-algebra is an algebraic system $(X, *, 0)$ *that satisfies the following axioms for all* $x_1, x_2, x_3 \in X$:

(1) $\big((x_1 * x_2) * (x_1 * x_3)\big) * (x_3 * x_2) = 0$
(2) $\big(x_1 * (x_1 * x_2)\big) * x_2 = 0$
(3) $x_1 * x_1 = 0$
(4) $0 * x_1 = 0$
(5) $x_1 * x_2 = 0, x_2 * x_1 = 0$ *implies* $x_1 = x_2$
(6) $x_1 * x_2 = 0$ *iff* $x_1 \leq x_2$

It can be noted that $(X, \leq)$ forms a poset. In sequel, the BCK-algebra $(X, *, 0)$ is denoted by X. If $\exists 1 \in X$ such that $x_1 \leq 1$ for all x_1 in X, then X is called bounded, X is called commutative if $x_1 \wedge x_2 = x_2 \wedge x_1$ holds for all x_1, x_2 in X, where $x_1 \wedge x_2 = x_2 * (x_2 * x_1)$. We refer [8, 9] for undefined terms and more details of BCK-algebras.
Here, we present some examples of BCK-algebra.

Example 2.6. *Let* $X_1 = \{0, 1, 2, 3, 4\}$ *and "$*$" be a binary operation on* X_1 *defined as* $x_1 * x_2 = x_1 - \min(x_1, x_2)$ $\forall x_1, x_2 \in X_1$. *Then it can be seen from Table 1 that* $(X_1, *, 0)$ *forms a commutative BCK-algebra. Indeed, one can extend this example by taking* $X_1 = \{0, 1, \ldots, n\}$ *for any finite* $n \in N$.

Example 2.7. *Let* A *be a non-empty set and* $P(A)$ *be its power set. Then* $(P(A), \setminus, \emptyset)$ *forms a bounded commutative and implicative BCK-algebra, where the binary operation*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	2	1	0	0
4	4	3	2	1	0

TABLE 1. Cayley table for BCK-algebra $(X_1, *, 0)$

" $\setminus$" is the usual set difference. Indeed, if $A = \{a_1, a_2\}$*, then* $P(A) = \{\emptyset, S_1 = \{a_1\}, S_2 = \{a_2\}, A\}$*. It can be seen from Table 2 that* $(P(A), \setminus, \emptyset)$ *forms a* $BCK-$*algebra.*

$\setminus$	$\emptyset$	S_1	S_2	A
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
S_1	S_1	$\emptyset$	S_1	$\emptyset$
S_2	S_2	S_2	$\emptyset$	$\emptyset$
A	A	S_2	S_1	$\emptyset$

TABLE 2. Cayley table for BCK-algebra $(P(A), \setminus, \emptyset)$

The next in the sequel, is the notion of a BCK-module. It was introduced as an action of BCK-algebras on an abelian group by H.A.S. Abujabal, M. Aslam and A.B. Thaheem in [1]. Interests have been shown by several researchers in the development of its theory. Some of the developments can be seen in [5, 11, 17].

Definition 2.8. [1] **(BCK-Module)**
Let $(M, +)$ *be an abelian group and* X *be a BCK-algebra. Then* M *is said to be an* $X-$*module if there exists a mapping* $(x, m) \mapsto xm$ *from* $X \times M \to M$ *such that for all* $x_1, x_2 \in X$ *and* $m, m_1, m_2 \in M$*, following conditions are satisfied:*

(1) $(x_1 \wedge x_2)m = x_1(x_2m)$
(2) $x_1(m_1 + m_2) = x_1m_1 + x_1m_2$
(3) $0m = 0$
(4) $1m = m$, *if* X *is bounded.*

One can define a right $X-$*module in a similar way. In this paper, an* $X-$*module* M*, is use to refer a left BCK-module.*

Some examples of BCK-modules are presented here.

Example 2.9. [1] *A BCK-algebra* $(X, *, 0)$ *which is bounded and implicative forms an* $X-$*module over itself.*

Example 2.10. *Let* $(P(A), \setminus, \emptyset)$ *be the BCK-algebra defined in Example 2.7. Then the set* $M_1 = \{\emptyset, S_1\}$ *forms an abelian group w.r.t. the addition "+" defined by* $m_1 + m_2 = (m_1 \setminus m_2) \cup (m_2 \setminus m_1) \ \forall\, m_1, m_2 \in M_1$*. Define an action of* $P(A)$ *on* M_1 *by* $xm = x \cap m$

$+$	$\emptyset$	S_1
$\emptyset$	$\emptyset$	S_1
S_1	S_1	$\emptyset$

TABLE 3. Cayley table of group $(M_1, +)$

for all $x \in P(A)$ & $m \in M_1$. Then it is easy to see that M_1 forms an $P(A)-$module. Table 4 summarize the action of $P(A)$ on M_1.

$\cap$	$\emptyset$	S_1	S_2	A
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
S_1	$\emptyset$	S_1	$\emptyset$	S_1

TABLE 4. Action of $P(A)$ on M_1

Example 2.11. *Let $(X_1, *, 0)$ be the BCK-algebra discussed in Example 2.6 and $N_1 = \{0, 1\} \subset X_1$. Define an operation of addition "$+$" on N_1 as $x_1 + x_2 = \max(x_1 * x_2, x_2 * x_1)\ \ \forall\, x_1, x_2 \in N_1$. Then it is easy to see that $(N_1, +)$ forms an abelian group.*
Now if an action of X_1 is defined on N_1 by $xn = \min(x, n)\ \ \forall x \in X_1, n \in N_1$, then Table 5 shows that N_1 forms an X_1-module.

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1

TABLE 5. Action of X_1 on N_1

A subgroup N of an $X-$module M is called an $X-$submodule of M if N is also an $X-$module. Let M_1, M_2 be $X-$modules. A mapping $f : M_1 \to M_2$ is called an $X-$homomorphism if for any $x \in X$ and $m_1, m_2 \in M_1$ the following hold: 1) $f(m_1 + m_2) = f(m_1) + f(m_2)$, 2) $f(xm_1) = xf(m_1)$. An $X-$homomorphism $f : M_1 \to M_2$ which is both one to one as well as onto is called an $X-$isomorphism. The $\operatorname{Ker} f$ and $\operatorname{Im} f$, both in usual sense, are submodules of M_1 and M_2 respectively (see [17]). If N is an $X-$submodule of an $X-$module M, then quotient group M/N forms an $X-$module called the factor $X-$module w.r.t the scalar multiplication $(x, m + N) \to xm + N\ \forall x \in X, m \in M$ from $X \times (M/N) \to M/N$. (see for details and developments [1, 5, 6, 7, 11, 17]).

3. SOFT BCK-MODULES

In this section, the notion of Soft BCK-Modules is introduced. Some related examples will be discussed. The notion of soft $X-$subomdules will be introduced and discussed. The necessary conditions on summation, intersection and union of an arbitrary family of soft $X-$submodules to become a soft $X-$subomdule will also be established.

Definition 3.1. (Soft BCK-Module)
An $S-$set (F, A) over an $X-$module M is said to be a soft $X-$module ($SX-$module) over M, if for all $\varepsilon \in A$, $F(\varepsilon)$ is an $X-$submodule of M. The collection of all $SX-$modules over an $X-$module M is denoted by $SX(M)$

Some examples of soft BCK-modules are produced here.

Example 3.2. *Let M_1 be a $P(A)-$module defined in Example 2.10 and $B_1 \subset P(A) = E$. Define $G_1 : B_1 \to P(M_1)$ and $G_2 : B_1 \to P(M_1)$ by*

$$G_1(\varepsilon) = \begin{cases} \{\emptyset\} & if\ \varepsilon \not\subseteq M_1 \\ \{\emptyset, A_1\} & if\ \varepsilon \subseteq M_1 \end{cases} \quad G_2(\varepsilon) = \begin{cases} \{\emptyset\} & if\ \varepsilon \not\supseteq M_1 \\ \{\emptyset, A_1\} & if\ \varepsilon \supseteq M_1 \end{cases}$$

Then (G_1, B_1) and (G_2, B_1) are soft $P(A)-$modules over M_1.

Example 3.3. *Let N_1 be an X_1-module defined in Example 2.11 and $A_1 = \{red, green\}$ and $A_2 = \{green, blue\}$ be two subsets of the set E of all colours. Define $F_1 : A_1 \to P(N_1)$ and $F_2 : A_2 \to P(N_1)$ by*

$$F_1(\varepsilon) = \begin{cases} \{0\} & if\ \varepsilon = red \\ \{0, 1\} & if\ \varepsilon = green \end{cases} \quad F_2(\varepsilon) = \begin{cases} \{0\} & if\ \varepsilon = green \\ \{0, 1\} & if\ \varepsilon = blue \end{cases}$$

Then (F_1, A_1) and (F_2, A_2) are soft X_1-modules over N_1. Similarly, if $A_3 = X$ in Example 2.11, then $F_3 : A_3 \to P(N_1)$ defined below forms a soft X_1-module over N_1.

$$F_3(\varepsilon) = \begin{cases} \{0\} & if\ \varepsilon \notin N_1 \\ \{0, 1\} & if\ \varepsilon \in N_1 \end{cases}$$

Proposition 3.4. *Let (F_1, A_1) and (F_2, A_2) be in $SX(M)$. Then*

(1) *$(F_1, A_1) \,\tilde{\cap}\, (F_2, A_2)$ is in $SX(M)$.*
(2) *$(F_1, A_1) \,\tilde{\cup}\, (F_2, A_2)$ is in $SX(M)$, provided $A_1 \cap A_2 = \emptyset$.*

Proof. It is clear from Definition 2.3 that $(F_1, A_1) \,\tilde{\cap}\, (F_2, A_2) = (\tilde{F}, \tilde{A})$ is a $S-$set, where, $\tilde{A} = A_1 \cap A_2$ and $\forall \varepsilon \in \tilde{A}, \tilde{F}(\varepsilon) = F_1(\varepsilon) \bigcap F_2(\varepsilon)$. Also from the fact that intersection of $X-$submodules is again an $X-$submodule (see [1]), $(F_1, A_1) \,\tilde{\cap}\, (F_2, A_2)$ is a $SX-$module M.
Similarly, if $A_1 \cap A_2 = \emptyset$, then from Definition 2.3, $(F_1, A_1) \,\tilde{\cup}\, (F_2, A_2) = (\tilde{\tilde{F}}, \tilde{\tilde{A}})$ is a $S-$set such that $\tilde{\tilde{F}}(\varepsilon)$ either equals $F_1(\varepsilon)$ or $F_2(\varepsilon)$ for all $\varepsilon \in A_1 - A_2$ or $\varepsilon \in A_2 - A_1$ respectively. Since $F_1(\varepsilon)$ and $F_2(\varepsilon)$ are $X-$submodules of M, therefore $\tilde{\tilde{F}}(\varepsilon)$ is an $X-$submodules of M. □

Definition 3.5. *Let (F_1, A_1) and (F_2, A_2) be $SX-$modules over an $X-$module M. Then $(F_1, A_1) + (F_2, A_2)$ is defined as $(S, A \times B)$, where $S(\varepsilon, \delta) = F_1(\varepsilon) + F_2(\delta)\ \forall(\varepsilon, \delta) \in A_1 \times A_2$.*

Proposition 3.6. *Let (F_1, A_1) and (F_2, A_2) be in $SX(M)$. Then $(F_1, A_1) + (F_2, A_2)$ is also belongs to $SX(M)$.*

Proof. This can be easily obtained from the fact that sum of $X-$submodules is again an $X-$submodule of M (see [1]). □

Definition 3.7. *Let (F_1, A_1) and (F_2, A_2) be two $SX-$modules over $X-$modules M_1 and M_2 respectively. Then we define $(F_1, A_1) \times (F_2, A_2) = (P, A_1 \times A_2)$ as $P(\varepsilon, \delta) = F_1(\varepsilon) \times F_2(\delta)$ for all $(\varepsilon, \delta) \in A_1 \times A_2$.*

Proposition 3.8. *Let (F_1, A_1) and (F_2, A_2) be two $SX-$modules over $X-$modules M_1 and M_2 respectively. Then $(F_1, A_1) \times (F_2, A_2)$ is a $SX-$module over the $X-$module $M_1 \times M_2$.*

Proof. This is clear from the fact that the cartesian product of two $X-$modules is also an $X-$module (see[1]). □

It was shown in [1] that direct product of two $X-$moduels is isomorphic to the cartesian product. Therefore, $\oplus$ can be used instead of $\times$ in the above proposition.

Definition 3.9. *Let (F_1, A_1) and (F_2, A_2) be $SX-$modules over an $X-$module M. Then (F_2, A_2) is a soft $X-$submodule ($SX-$submodule) of (F_1, A_1) if*

(1) *$A_2 \subset A_1$ and*
(2) *$F_2(\varepsilon) < F_1(\varepsilon), \ \forall \varepsilon \in A_2$.*

This is denoted by $(F_2, A_2) \mathrel{\tilde{<}} (F_1, A_1)$.

Proposition 3.10. *Let (F_1, A_1) and (F_2, A_2) be $SX-$modules over an $X-$module M such that $A_2 \subseteq A_1$. Then $(F_2, A_2) \mathrel{\tilde{<}} (F_1, A_1)$ if $F_2(\varepsilon) \subseteq F_1(\varepsilon), \ \forall \varepsilon \in A_2$.*

Proof. It is simple to prove. □

Proposition 3.11. *Let (F, A) be a $SX-$module over an $X-$module M, and consider the nonempty family $\{(G_i, B_i)|i \in I\}$ of soft $X-$submodules of (F, A). Then*

(1) *$\sum\limits_{i \in I}(G_i, B_i) \mathrel{\tilde{<}} (F, A)$.*
(2) *$\bigcap\limits_{i \in I}(G_i, B_i) \mathrel{\tilde{<}} (F, A)$.*
(3) *$\bigcup\limits_{i \in I}(G_i, B_i) \mathrel{\tilde{<}} (F, A)$, if $B_i \cap B_j = \emptyset \ \ \forall i, j \in I$.*

Proof. It can be seen from Propositions 3.4, 3.6 and 3.10. □

Proposition 3.12. *Let (F_1, A_1) and (F_2, A_2) be two $SX-$modules over an $X-$module M such that (F_2, A_2) be soft $X-$submodule of (F_1, A_1). Let $\tilde{M}$ be an $X-$module and $f : M \to \tilde{M}$ is an $X-$homomorphism. Then $(f(F_1), A_1)$ and $(f(F_2), A_2)$ are $SX-$modules over $\tilde{M}$, and $(f(F_2), A_2) \mathrel{\tilde{<}} (f(F_1), A_1)$.*

Proof. Since $f : M \to \tilde{M}$ is an $X-$homomorphism, therefore, $f(F_1(\varepsilon))$ and $f(F_2(\delta))$ are $SX-$modules of $\tilde{M}$ $\forall \varepsilon \in A_1$ and $\forall \ \delta \in A_2$. This implies that both $(f(F_1), A_1)$ and $(f(F_2), A_2)$ are $SX-$modules over $\tilde{M}$. If $(F_2, A_2) \mathrel{\tilde{<}} (F_1, A_1)$, then indeed it follows that $F_2(\delta)$ and $f(F_2(\delta))$ are the $X-$submodules of $F_1(\delta)$ and $f(F_1(\delta))$ $\forall \delta \in A_2$, respectively. Therefore, by Definition 3.9, we immediately conclude that $(f(F_2), A_2) \mathrel{\tilde{<}} (f(F_1), A_1)$. □

4. Isomorphism Theorem of Soft BCK-Modules

In this section, we introduce the notion of soft $X-$homorphisms and soft $X-$isomorphisms of $X-$modules and establish the isomorphism theorems on $SX-$modules.

Definition 4.1. (Soft $X-$Hommorphism)
Let $(F_1, A_1) \in SX(M_1)$ and $(F_2, A_2) \in SX(M_2)$, $\phi : M_1 \to M_2,\ \&\ \theta : A_1 \to A_2$ be any two mappings. Then (ϕ, θ) is said to be a soft $X-$homomorphism ($SX-$homomorphism) if it satisfies the following conditions:

(1) $\phi : M_1 \to M_2$ *is an $X-$homomorphism;*
(2) $\theta : A_1 \to A_2$ *is surjective;*
(3) $\phi(F_1(\varepsilon)) = F_2(\theta(\varepsilon)),\ \forall \varepsilon \in A_1$.

In this case, we say (F_1, A_1) is soft $X-$homomorphic to (F_2, A_2), and it is denoted by $(F_1, A_1) \simeq (F_2, A_2)$.

In this definition, if ϕ is an $X-$isomorphism and θ is a bijection, then we say that (ϕ, θ) is a soft $X-$isomorphism and that (F_1, A_1) is soft $X-$isomorphic to (F_2, A_2), denoted by $(F_1, A_1) \cong (F_2, A_2)$.

Proposition 4.2. *Let (F_1, A_1) be a $SX-$module over an $X-$module M_1 and consider the $S-$set (F_2, A_2) over an $X-$module M_2. If $(F_1, A_1) \simeq (F_2, A_2)$ as an $S-$set, then $(F_2, A_2) \in SX(M_2)$.*

Proof. Let $(\phi, \theta) : (F_1, A_1) \to (F_2, A_2)$ be an $X-$homomorphism. Now, $\forall\ \delta \in A_2\ \ \exists\ \ \varepsilon \in A_1$ such that $g(\varepsilon) = \delta$. Therefore, $F_2(\delta) = F_2(g(\varepsilon)) = f(F_1(\varepsilon))$ is an $X-$submodule of the module M_2. This implies $(F_2, A_2) \in SX(M_2)$. □

Proposition 4.3. *Let $(F_1, A_1) \in SX(M_1)$ and $(F_2, A_2) \in SX(M_2)$. If $(\phi, \theta) : (F_1, A_1) \to (F_2, A_2)$ is an $SX-$homomorphism and $(F', A') \tilde{<} (F_1, A_1)$, then $(F_2, \theta(A')) \tilde{<} (F_2, A_2)$.*

Proof. Indeed clear, since for all $\varepsilon \in \theta(A') \subset A_2$, $F_2(\varepsilon)$ is a soft $X-$submodule of M_2. □

We conclude this section by presenting the isomorphism theorems of $SX-$modules.

Theorem 4.4. (1^{st} Iso-Theorem of $SX-$modules)
Let $(F_1, A_1) \in SX(M_1)$ and $(F_2, A_2) \in SX(M_2)$. If $(\phi, \theta) : (F_1, A_1) \to (F_2, A_2)$ is an $SX-$homomorphism and $\ker\phi \subset F(\varepsilon)$ for all $\varepsilon \in A_1$. Then the following conditions hold:

(1) *If $I(\varepsilon) = F(\varepsilon)/\ker\phi,\ J(x) = \phi(F(\varepsilon)), \varepsilon \in A_1$, then $(I, A_1) \cong (J, A_1)$.*
(2) *$(I, A_1) \cong (F_2, A_2)$, provided θ is a bijection.*

Proof. (1) From [1], $\ker\phi$ is an $X-$submodule of M_1 and therefore, $M_1/\ker\phi$ forms an $X-$module. Also, since $\ker\phi$ is an $X-$submodule of $F_1(\varepsilon)$, therefore, $F_1(\varepsilon)/\ker\phi$ is also an $X-$module, for all $\varepsilon \in A_1$. Indeed, $F_1(\varepsilon)/\ker\phi$ is an $X-$submodule of $M_1/\ker\phi$. This implies that (I, A_1) is a $SX-$module over $M_1/\ker\phi$. Now for all $\varepsilon \in A_1$, it is easy to see that $J(\varepsilon) = \phi(F_1(\varepsilon)) = F_2(\theta(\varepsilon))$ is an $X-$submodule of M_2. Therefore, (J, A_1) is a $SX-$module over M_2.
Define $\bar{\phi} : M_1/\ker\phi \to M_2$ by $\bar{\phi}(m + \ker\phi) = \phi(m)$; for all $m \in M_1$. Then

$\bar{\phi} : M_1/\ker\phi \to M_2$ is an $X-$isomorphism.
Let $i_\theta : A_1 \to A_1$ defined by $i_\theta(\varepsilon) = \varepsilon$ be an identity mapping. Then i_θ is indeed a bijection. Now, we have $\bar{\phi}(I(x)) = \bar{\phi}(F_1(\varepsilon)/\ker\phi) = \phi(F_1(\varepsilon)) = J(\varepsilon) = J(i_\theta(\varepsilon))$. Consequently, $(\bar{\phi}, i_\theta)$ is a soft $X-$isomorphism i.e. $(I, A_1) \cong (J, A_1)$.

(2) Define $\bar{\phi} : M_1/\ker\phi \to M_2$ by $\bar{\phi}(m + \ker\phi) = \phi(m)$; for all $m \in M_1$. Then $\bar{\phi}$ is an $X-$isomorphism from $M_1/\ker\phi$ to M_2. Since θ is a bijection and $\bar{\phi}(I(\varepsilon)) = \bar{\phi}(F(\varepsilon)/\ker\phi) = \phi(F(\varepsilon)) = H(\theta(\varepsilon))$, we conclude that $(I, A_1) \cong (F_2, A_2)$. □

Theorem 4.5. *Let (F, A) be in $SX(M)$. If (F_1, A_1) and (F_2, A_2) are $SX-$submodules of (F, A), then $(P_1, A_1) \simeq (Q_1, A_1)$ and $(P_2, A_2) \simeq (Q_2, A_2)$, where $P_1(\varepsilon) = F_1(\varepsilon)/(M_1 \cap M_2), Q_1(\varepsilon) = (F_1(\varepsilon) + M_2)/M_2,\ P_2(\varepsilon) = F_2(\varepsilon)/(M_1 \cap M_2), Q_2(\varepsilon) = (F_2(\varepsilon) + M_1)/M_1,\ M_1 = \bigcap_{\varepsilon\in A_1} F_1(\varepsilon)$ and $M_2 = \bigcap_{\varepsilon\in A_2} F_2(\varepsilon)$.*

Proof. Let us denote $K = \left\langle \bigcup_{\varepsilon\in A_1} F_1(\varepsilon)\right\rangle$ and $L = \left\langle \bigcup_{\varepsilon\in A_2} F_2(\varepsilon)\right\rangle$. Then, $M_1 = \bigcap_{\varepsilon\in A_1} F_1(\varepsilon)$ is an $X-$submodule of M. It is clear that M_1 is also an $X-$submodule K so that $M_1 \cap M_2$ is an $X-$submodule of K and hence, (P_1, F_1) is a $SX-$module over $K/(M_1 \cap M_2)$. It is trivial that (Q_1, F_1) is an $SX-$module over $(K + M_2)/M_2$.
Now, define a mapping $f : K/(M_1\cap M_2) \to (K+M_2)/M_2$ by $f(k+(M_1\cap M_2)) = k+M_2$ and $i_d : F_1 \to F_1$ by $i_d(\varepsilon) = \varepsilon$. Then f from $K/(M_1 \cap M_2)$ to $(K + M_2)/M_2$ is an $X-$homomorphism, where i_d is a bijection and $f(P_1(\varepsilon)) = f(F_1(\varepsilon)/(M_1 \cap M_2)) = (F_1(\varepsilon) + M_2)/M_2 = Q_1(\varepsilon) = Q_1(i_d(\varepsilon))$. This shows that $(P_1, A_1) \simeq (Q_1, A_1)$.
$(P_2, A_2) \simeq (Q_2, A_2)$ can be proved similarly. □

Theorem 4.6. (2^{nd} Iso-Theorem of $SX-$modules)
Let (F, A) be in $SX(M)$. If (F_1, A_1) and (F_2, A_2) are $SX-$submodules of (F, A) such that $F_1(\varepsilon) = M_1$ for all $\varepsilon \in A_1$, then $(P_1, A_1) \cong (Q_1, A_1)$, where $P_1(\varepsilon) = F_1(\varepsilon)/(M_1 \cap M_2),\ Q_1(\varepsilon) = (F_1(\varepsilon) + M_2)/M_2,\ M_2 = \bigcap_{\varepsilon\in A_2} F_2(\varepsilon)$.

Proof. Indeed, if we replace $K = M_1 = F_1(\varepsilon)$ for all $\varepsilon \in A_1$ in Theorem 4.5, the proof of the theorem can be similarly furnished. □

Theorem 4.7. (3^{rd} Iso-Theorem of $SX-$modules)
Let (F, A) be in $SX(M)$. If (F_1, A_1) and (F_2, A_2) are $SX-$submodules of (F, A), such that $A_1 \cap A_2 \neq \emptyset$ and $F_2(\varepsilon) \subset F_1(\varepsilon)\forall\varepsilon \in A_1 \cap A_2$, then

$$(P, A_1 \cap A_2) \cong (Q, A_1 \cap A_2),$$

where, $P(\varepsilon) = (F(\varepsilon)/M_2)/(M_1/M_2),\ Q(x) = F(\varepsilon)/M_1$ with $M_1 = \bigcap_{\varepsilon\in(A_1\cap A_2)} F_1(\varepsilon)$ and $M_2 = \bigcap_{\varepsilon\in(A_1\cap A_2)} F_2(\varepsilon)$.

Proof. It is indeed clear that M_1 and M_2 are $X-$submodules of M, and M_2 is an $X-$submodule of M_1. Therefore, $(M/M_2)/(M_1/M_2)$ forms an $X-$module (from [1]) and $(P, A_1\cap A_2)$ is a soft $X-$submodule over it. Also, $(Q, A_1 \cap A_2)$ is a $SX-$module over M/M_1. Now define the mapping $f : (M/M_2)/(M_1/M_2) \to M/M_1$ by

$$f((m + M_2) + (M_1/M_2)) = m + M_1 \quad \forall m \in M$$

and an identity mapping $i_d : A_1 \cap A_2 \to A_1 \cap A_2$ by $i_d(\varepsilon) = \varepsilon$. One can see that f is an $X-$isomorphism therefore

$$f(P(\varepsilon)) = f((F(\varepsilon)/M_2)/(M_1/M_2)) = F(\varepsilon)/M_1 = Q(i_d(\varepsilon)).$$

Hence, from Definition 4.1, $(P, A_1 \cap A_2) \cong (Q, A_1 \cap A_2)$. □

5. Exactness of Soft BCK-Modules

In the current section, all nonempty sets are considered as $X-$modules.

Definition 5.1. (Soft $X-$Exactness)
Let $(F_1, A_1), (F_2, A_2)$ and (F_3, A_3) be in $SX(M_1), SX(M_2)$ and $SX(M_3)$, respectively. Then a sequence $(F_1, A_1) \overset{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \overset{(\phi_2,\theta_2)}{\longrightarrow} (F_3, A_3)$ of $SX-$homomorphisms is said to be soft $X-$exact ($SX-$exact) at (F_2, A_2), if the following conditions are satisfied:

(1) $M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3$ *is exact.*
(2) $A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_1} A_3$ *is exact.*

Proposition 5.2. *Let(F_1, A_1), and (F_2, A_2) be in $SX(M_1)$, and $SX(M_2)$, respectively. If $(F_1, A_1) \overset{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \longrightarrow 0$. is $SX-$exact, then (ϕ_1, θ_1) is $SX-$homomorphism. In particular, if $0 \longrightarrow (F_1, A_1) \overset{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \longrightarrow 0$ is $SX-$exact, then (ϕ_1, θ_1) is a $SX-$isomorphism.*

Proof. From Definition 5.1, it follows that $M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} 0$ and $A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_1} 0$ are $X-$exact. Therefore, (ϕ_1, θ_1) are $X-$epimorphisms, which implies (ϕ_1, θ_1) is an $X-$homomorphism.

In particular, if $0 \longrightarrow (F_1, A_1) \overset{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \longrightarrow 0$ is $SX-$exact, then again from Definition 5.1, $0 \longrightarrow M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} 0$ and $0 \longrightarrow A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_1} 0$ are $X-$exact. This implies that (ϕ_1, θ_1) are $X-$isomorphisms, and hence (ϕ_1, θ_1) is a $SX-$isomorphism. □

Definition 5.3. *Let $M = 0$ and $A = 0$, then $(F_1, A_1) = 0$. We call (F_1, A_1) is a zero-$SX-$module.*

Proposition 5.4. *Let $(F_1, A_1), (F_2, A_2)$ and (F_3, A_3) be in $SX(M_1), SX(M_2)$ and $SX(M_3)$, respectively. If $(F_1, A_1) \overset{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \overset{\phi_2,\theta_2)}{\longrightarrow} (F_3, A_3)$ is $SX-$exact with (ϕ_1, θ_1) $X-$epimorphism and (ϕ_2, θ_2) $X-$monomorphism, then (F_2, A_2) is a zero-$SX-$module.*

Proof. Indeed in this case, we have the following diagram.

$$A_1 r \theta_1 d F_1 A_2 r \theta_2 d F_2 A_3 d F_3 M_1 r \phi_1 M_2 r \phi_2 M_3 \tag{5. 1}$$

The $X-$exactness of A_i's and M_i's and the fact that (ϕ_1, θ_1) are $X-$epimorphism and (ϕ_2, θ_2) are $X-$monomorphism, forces $A_2 = 0 = M_2$. This completes the proof. □

Here we recall from [1], that if N is an $X-$submodule of an $X-$module M, then M/N forms an $X-$module called quetient $X-$module.

Theorem 5.5. *Let (F_1, A_1) and (F_2, A_2) be in two $SX(M_1)$ and $SX(M_2)$, respectively. For any $M_1 \subset M_2, A_1 \subset A_2$ and $M_1 \subset F_2(\varepsilon)$ where $\varepsilon \in A_2$. If $(F_1, A_1) \stackrel{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2)$ is $SX-$homomorphism, then $0 \longrightarrow (F_1, A_1) \stackrel{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \stackrel{(\phi_2,\theta_2)}{\longrightarrow} (I, A_2/A_1) \longrightarrow 0$ is $SX-$exact, where $I(\varepsilon + A_1) = F_2(\varepsilon) = M_1$ for all $\varepsilon \in A_2$.*

Proof. Since for all $\varepsilon \in A_2$, $M_1 \leq F_2(\varepsilon) \leq M_2$, therefore, $F_2(\varepsilon)/M_1 \leq M_2/M_1 \quad \forall \varepsilon \in A_2$. Next, we define in a natural way $X-$homomorphisms $\phi_2 : M_2 \rightarrow M_2/M_1$ and $\theta_2 : A_2 \rightarrow A_2/A_1$ by;

$$\phi_2(m_2) = m_2 + M_1 \;\; and \;\; \theta_2(a_2) = a_2 + A_1 \quad \forall m_2 \in M_2 \;\&\; a_2 \in A_2.$$

It is clear that in this case,

$$0 \longrightarrow M_1 \stackrel{\phi_1}{\longrightarrow} M_2 \stackrel{\phi_2}{\longrightarrow} M_2/M_1 \longrightarrow 0$$

and

$$0 \longrightarrow A_1 \stackrel{\theta_1}{\longrightarrow} A_2 \stackrel{\theta_2}{\longrightarrow} A_2/A_1 \longrightarrow 0$$

are $X-$exact.

Finally, we see that $\phi_2(F_2(\varepsilon)) = F_2(\varepsilon) + M_1 = I(\varepsilon + A_1) = I(\theta_2(\varepsilon))$ for all $\varepsilon \in A_2$. This implies

$$0 \longrightarrow (F_1, A_1) \stackrel{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \stackrel{(\phi_2,\theta_2)}{\longrightarrow} (I, A_2/A_1) \longrightarrow 0$$

is $SX-$exact □

We conclude the paper with the following result, discussing the transitivity of two $SX-$exact sequences.

Theorem 5.6. *Let (F_i, A_i) be a $SX-$module over BCK-modules M_i for $i = 1, 2, 3, 4, 5$, respectively. If*

$$0 \longrightarrow (F_1, A_1) \stackrel{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \stackrel{(\phi_2,\theta_2)}{\longrightarrow} (F_3, A_3) \longrightarrow 0$$

and

$$0 \longrightarrow (F_3, A_3) \stackrel{(\phi_3,\theta_3)}{\longrightarrow} (F_4, A_4) \stackrel{(\phi_4,\theta_4)}{\longrightarrow} (F_5, A_5) \longrightarrow 0$$

are $SX-$exact. Then

$$0 \longrightarrow (F_1, A_1) \stackrel{(\phi_1,\theta_1)}{\longrightarrow} (F_2, A_2) \stackrel{(\phi_3\phi_2,\theta_3\theta_2)}{\longrightarrow} (F_4, A_4) \stackrel{(\phi_4,\theta_4)}{\longrightarrow} (F_5, A_5) \longrightarrow 0$$

is $SX-$exact.

Proof. We have from the hypothesis of the theorem

$$0 \longrightarrow M_1 \stackrel{\phi_1}{\longrightarrow} M_2 \stackrel{\phi_2}{\longrightarrow} M_3 \longrightarrow 0$$

and

$$0 \longrightarrow M_3 \stackrel{\phi_3}{\longrightarrow} M_4 \stackrel{\phi_4}{\longrightarrow} M_5 \longrightarrow 0$$

are $SX-$exact. This clearly implies that

$$0 \longrightarrow M_1 \stackrel{\phi_1}{\longrightarrow} M_2 \stackrel{\phi_3\phi_2}{\longrightarrow} M_4 \stackrel{\phi_4}{\longrightarrow} M_5 \longrightarrow 0$$

is $SX-$exact.
Similarly, from the hypothesis

$$0 \longrightarrow A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_3\theta_2} A_4 \xrightarrow{\theta_4} A_5 \longrightarrow 0$$

is $SX-$exact.
Since $\phi_2(F_2(\varepsilon)) = F_3\theta_2(\varepsilon))$ for all $\varepsilon \in A_2$ and $\phi_3(F_3(\varepsilon)) = F_4\theta_3(\varepsilon))$ for all $\varepsilon \in A_3$. We have $\phi_3\phi_2(F_2(\varepsilon)) = \phi_3(F_3\theta_2(\varepsilon))) = F_4\theta_3\theta_2(\varepsilon))$ for all $\varepsilon \in A_2$. This implies

$$0 \longrightarrow (F_1, A_1) \xrightarrow{(\phi_1,\theta_1)} (F_2, A_2) \xrightarrow{(\phi_3\phi_2,\theta_3\theta_2)} (F_4, A_4) \xrightarrow{(\phi_4,\theta_4)} (F_5, A_5) \longrightarrow 0$$

is $SX-$exact. □

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