Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(3)(2019) pp. 89-100

Numerical Solution of Vibration Equation using Haar Wavelet

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Received: 14 September, 2018 / Accepted: 27 November, 2018 / Published online: 28 December, 2018

Abstract. In this paper, the numerical solution of widely used vibration equation with very large membrane is considered. A collocation method with Haar wavelet (HW) is applied for this purpose. The algorithm based on proposed method is very simple and easy to implement. Highly accurate approximate solutions are obtained with a reasonable number of collocation points. The method is tested upon several different exacts solutions of the vibration equation including polynomials and infinite series solutions. The numerical results confirm the accuracy, efficiency and robustness of the proposed method.

AMS (MOS) Subject Classification Codes: 65M22; 35Q99
Key Words: Vibration equation, Haar wavelet, Collocation method.

1. Introduction

Membranes are used as components in speakers, microphones and other devices. Vibration of large membranes performs a key role in evaluating the two-dimensional wave mechanism and its broadcast [19]. The vibration of large membranes can be modelled using vibration equation. The standard form of vibration equation is given by [5]:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad r \in \Omega, \ t \geq \ 0. \tag{1. 1}$$

The above equation is subject to two initial conditions (ICs):

$$u(r,0) = \xi(r), \ \frac{\partial u}{\partial t}(r,0) = c\zeta(r), \tag{1. 2}$$

and the following boundary condition (BCs):

$$u(r,t) = g(t), \ t \in \partial\Omega, \tag{1. 3}$$

where $u(r,t)$ denotes the distance of position of a particle which is placed at the position denoted by r at time t. The constant c denotes the wave velocity, Ω is the domain and $\partial\Omega$ is the boundary.

The vibration of large membranes is significantly important in science and engineering as vibrations occur in almost all branches of science [22, 20, 5]. Some of the applications of vibration phenomenon are following. The vibration characteristics of a membrane is useful to understand earshot. The knowledge of vibration behaviour of membranes is required by the designs of hearing aid devices. The way material vibrates under given condition is an important aspect of designing of civil construction or mechanical material. Historically, in the absence of vibration effect unexpected failures of bridges and columns have occurred.

Mathematical modelling of several phenomena occurring in different fields of science and engineering results in equations which do not have exact solutions. Due to this reason, several researchers have worked on finding efficient and robust numerical schemes for handling such problems. In past, Finite Element Method (FEM) and Finite Difference Method (FDM) were commonly used for numerical approximations. These methods are very effective but every numerical method has its own disadvantages too. Therefore, researchers are always working for finding new and more efficient numerical methods. In the recent past, due to rapid advancement in the field of computer technology, a variety of numerical techniques [3, 18, 17, 1] have been developed and applied successfully to several different types of problems arising in engineering, physics, chemistry and other disciplines.

Now a days, among various numerical methods, methods based on wavelet theory are also common for numerical approximations of several types of problems including numerical integration, interpolation as well as numerical solution of differential equations. In certain situations, wavelet methods are preferred over classical methods including FEM and FDM. A mathematical function that has the property of dividing a function into various smaller components is termed as a wavelet. The word wavelet was first used by Morlet and Grossman [7]. They used the word ondelette which means small wave in early 1980s. After some time this word was transferred to English by introducing the word wavelet. The wavelet theory got popularity partly due to the work of Grossman and Morlet [7] and Meyer [16].

While simulating singular and nonlinear equations, a high resolution is essential to obtain a good accuracy and due to this reason, the concept of Multi-Resolution Analysis (MRA) was introduced by Mallat [15]. In 1988, Daubechies discovered a method for using wavelets with compact support and scaling functions [6].

Since 1980s, wavelets have been used for the numerical solution of Partial Differential Equations (PDEs). The possibility to handle singularities, rough structures and unstable phenomena are the appealing features of this technique which is exhibited by analyzed equations. The algorithms of wavelets for numerical solution of PDEs either uses the Galerkin approach or the collocation technique. Researchers have applied different types of wavelets for numerical approximations but HW is more common due to its simple applicability and efficiency. It is broadly used in modelling and dissembling in several branches of science and engineering [21].

In 1909, Alfred Haar discovered a family of simple functions which is now called HW [8]. Most of the work on Haar functions was accomplished in 1930s. The concept of integrals of Haar family, which are calculated by Haar operational matrices, was derived by Chen and Hsiao in 1997 [4]. The HW transform is a transform that has various properties as it is compact, dyadic, orthonormal and simplest among all the wavelets. The functions in the family of HW are piecewise constant functions having a good feature of having the advantage of integrating them infinitely. These can easily handle singularities, as these may be expressed as intermediate extremities conditions. Because of their quality of compact support, they vanish out after the finite interval and hence easy to handle from mathematical point of view.

Haar functions can also be viewed as rectangular oscillatory join ups. Later, after the discovery of Daubechies, it turned out that HW is a special case of Daubechies wavelet. HW is quite simple among all the orthonormal wavelets and due to its property of having local behaviour many authors prefer HW method on other wavelet methods [2]. Haar functions are discontinuous, so their derivatives fail to exist at the discontinuity points. Due to this reason, we can not apply HW directly to solve differential equations and to overcome this difficulty the integration of wavelets has been used [21]. Several methods based on HW have been applied in number of problems including signal denoising, analysis of time frequency, nonlinear approximations as well as solving differential and integral equations [10, 11, 4, 12]. In the present research work, we investigate the performance of Haar Wavelet Collocation Method (HWCM) for solving vibration equation with large membrane.

The major contribution of this work is to apply a new method based on Haar wavelet proposed in [2] for numerical solution of vibration equation with large membrane. To the best of authors' knowledge, the method proposed in [2] has not been applied to such equation before. This method has several advantages over other numerical methods for solving PDEs which include simplicity of implementation, direct implementation of the method without converting it into ODEs and incorporation of boundary conditions directly.

The rest of the article is organized in the following manner. In Section 2, definitions of HW and multi-resolution analysis are given. In Section 4, the proposed numerical method for vibration equation is elaborated. In Section 5, the convergence theorem of the HW is discussed. Numerical experiments and their results are reported in Section 6. Finally, in Section 7, few conclusions of the work are given.

2. Haar Wavelet

The functions in the HW family have compact support, that is, these functions are defined on bounded intervals. In the present work, we assume that all problems are confined to the interval $[0, 1]$. All the functions in the HW family except the scaling function defined on the interval $[0, 1]$ has a single representation which is given as follows [21]:

$$h_i(\xi) = \begin{cases} 1, & \text{for} \quad \xi \in [\delta, \eta), \\ -1, & \text{for} \quad \xi \in [\eta, \vartheta), \quad i = 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases} \tag{2. 4}$$

and the end points of the intervals in the above definition are given as

$$\delta = \frac{k}{m}, \quad \eta = \frac{k+0.5}{m}, \quad \vartheta = \frac{k+1}{m}. \tag{2. 5}$$

In this representation, the level of the wavelet is denoted by integer j, where $j = \log_2 m$. While approximating functions using HW, a largest number for the level of resolution is considered and it is denoted by the symbol J. With these notations, the integer j takes the values $j = 0, 1, \ldots, J$ and the value of m is calculated accordingly using the relation $m = 2^j, \ j = 0, 1, \ldots, J$. Moreover, the integer k in the above representation is the translation parameter and it takes the values $k = 0, 1, \ldots, m-1$. Finally, the role of integer i is to count the functions in the HW family and it satisfies the relation $i = 1 + k + m$.

For $i = 2$, we have the function $h_2(x)$ which is in fact the mother wavelet for the HW family. A worth noting property of the mother wavelet is that all the functions in the HW family can be constructed from mother wavelet using either the process of dilation or the process of translation or both.

The HW family contains one more function which is denoted by $h_1(\xi)$ and is called the scaling function. This function has the following representation:

$$h_1(\xi) = \begin{cases} 1, & \text{for} \quad \xi \in [0, 1), \\ 0, & \text{elsewhere.} \end{cases} \tag{2. 6}$$

Any member in $L_2([0, 1])$ can be represented as a summation of members of HW family in the following manner:

$$f(\xi) = \sum_{i=1}^{\infty} a_i h_i(\xi). \tag{2. 7}$$

We assume a maximum integer J which is known as maximal resolution level of the HW family for the purpose of approximation. We further identify the integer $M = 2^J$ and $N = 2M$. Thus, any function $f(\xi)$ in the space $L_2([0, 1])$ can be aproximated as a summation of HW in the following manner:

$$f(\xi) = \sum_{i=1}^{N} a_i h_i(\xi). \tag{2. 8}$$

We introduce the following notations for integrals of Haar functions:

$$p_{i,1}(\xi) = \int_0^{\xi} h_i(\xi')d\xi',$$

$$p_{i,\upsilon+1}(\xi) = \int_0^{\xi} p_{i,\upsilon}(\xi')d\xi', \quad \upsilon = 1, 2, \ldots$$

We can calculate these integrals using Eq. (2. 4) and are given below:

$$p_{i,1}(\xi) = \begin{cases} \xi - \delta, & \text{for} \quad \xi \in [\delta, \eta), \\ \vartheta - \xi, & \text{for} \quad \xi \in [\eta, \vartheta), \\ 0, & otherwise. \end{cases} \tag{2. 9}$$

$$p_{i,2}(\xi) = \begin{cases} \frac{1}{2}\left(\xi - \delta\right)^2, & \text{for} \quad \xi \in [\delta, \eta), \\ \frac{1}{2}\left(\vartheta - \xi\right)^2, & \text{for} \quad \xi \in [\eta, \vartheta), \\ 0, & otherwise. \end{cases} \tag{2. 10}$$

The general expression for the above integrals is given by

$$p_{i,n}(\xi) = \begin{cases} 0, & \text{for} \quad \xi \in [0, \delta), \\ \frac{1}{n!}\left(\xi - \delta\right)^n, & \text{for} \quad \xi \in [\delta, \eta), \\ \frac{1}{n!}\left[(\xi - \delta)^n - 2(\xi - \eta)^n\right], & \text{for} \quad \xi \in [\eta, \vartheta), \\ \frac{1}{n!}\left[(\xi - \delta)^n - 2(\xi - \eta)^n + (\xi - \vartheta)^n\right], & \text{for} \quad \xi \in [\vartheta, 1), n = 1, 2, 3, \ldots \end{cases} \tag{2. 11}$$

The index i in the above function ranges over: $i = 2, 3, \ldots$. For $i = 1$,

$$p_{i,n}(\xi) = \frac{\xi^n}{n!}, \quad \text{for } n = 1, 2, \ldots$$

3. Multi-Resolution Analysis

A Multi-Resolution Analysis (MRA) is a tool through which we can understand the concept of wavelets. Taking a function f from the class of all square integrable functions over the real line, an MRA of $L_2(R)$ generates a sequence $\varpi_j, \varpi_j + 1, \ldots$ of subspaces in such a way that the projection of f onto these spaces gives finer approximations of the function f as $j \to \infty$. An MRA of the space $L_2(R)$ is termed as a set of subspaces $\varpi_j \subset L_2(R)$, $j \in Z$, which are closed and in sequential order having the characteristics given as follows:

- Monotonicity $\cdots \subset \varpi_-1 \subset \varpi_0 \subset \varpi_1 \subset \ldots$
- The spaces ϖ_j fulfil $\bigcup_{j \in Z} \varpi_j$ is dense in $L_2(R)$ and $\bigcap_{j \in Z} \varpi_j = 0$.
- If $f(\xi) \in \varpi_0$, then $f(2^j\xi) \in \varpi_j$. In other words, this statement shows that the spaces ϖ_j can be obtained from the central space ϖ_0 by the process of scaling.
- If $f(\xi) \in \varpi_0$, then $f(2^j\xi - k) \in \varpi_j$. In other words, the spaces ϖ_j remain unchanged when the process of translation is applied to them.
- We can always find a function $\phi \in \varpi_0$ in such a way that the set $\{\phi(\xi - k) : k \in Z\}$ forms a Riesz basis in ϖ_0.

The general functions can be approximated with the help of the space ϖ_j by defining proper projection of these functions. We can approximate any square integrable function in $L_2(R)$ arbitrarily close by using these projection due to the property that the union of all the ϖ_j's is dense in $L_2(R)$. For instance, if we define the space ϖ_j as

$$\varpi_j = \varsigma_{j-1} \oplus \varpi_{j-1} = \varsigma_{j-1} \oplus \varsigma_{j-2} \oplus \varsigma_{j-2} = \bigoplus_{j=1}^{j+1} \varsigma_j \oplus \varsigma_0. \tag{3. 12}$$

then after applying translation and dilation, an MRA can be created for the sequence of spaces $\{\varpi_j, j \in Z\}$ given by Eq. (2. 4) and Eq. (2. 6) by using the scaling function $h_1(\xi)$. It can be easily seen that for all j the space ς_j is an orthogonal complement of ϖ_j in $\varpi_j + 1$. This means that with respect to any given inner product the space ς_j contains all those functions in $\varpi_j + 1$ which are orthogonal to all those in ϖ_j.The basis which is formed by the set of functions for the space ς_j is called a formation of wavelets [15].

4. NUMERICAL METHOD

We will consider, in this section, the collocation method based on HW for the vibration Eq. (1. 1). In the present work, for the sake of simplicity, we assume that $\Omega = [1, 2]$ and $t \in [0, 1]$. A similar procedure can be adopted for other intervals of space and time domains.

In the HWCM, for the intervals $r \in \Omega = [1, 2]$ and $t \in [0, 1]$, we will consider the following collocation points:

$$r_j = 1 + \frac{j - 0.5}{N}, \quad j = 1, 2, \dots N, \tag{4. 13}$$

$$t_j = \frac{j - 0.5}{N}, \quad j = 1, 2, \dots N. \tag{4. 14}$$

We assume that the highest order partial derivatives occurring in Eq. (1. 1) can be approximated as a sum of scalar multiples of functions in HW family as follows:

$$\frac{\partial^2 u}{\partial r^2} = \sum_{j=1}^{N} \sum_{i=1}^{N} a_{i,j} h_i(r) h_j(t), \tag{4. 15}$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{N} \sum_{i=1}^{N} b_{i,j} h_i(r) h_j(t), \tag{4. 16}$$

Integrating Eq. (4. 15) with respect to r and using BCs, we obtain the following equations for the unknown function $u(r, t)$ and its partial derivative with respect to r.

$$\frac{\partial u}{\partial r}(r, t) = u(2, t) - u(1, t) + \sum_{j=1}^{N} \sum_{i=1}^{N} a_{i,j} \big(p_{i,1}(r) - p_{i,2}(2)\big) h_j(t), \tag{4. 17}$$

and

$$u(r, t) = (2 - r)u(1, t) - (1 - r)u(2, t) + \sum_{j=1}^{N} \sum_{i=1}^{N} a_{i,j} \big(p_{i,2}(r) + (1 - r)p_{i,2}(2)\big) h_j(t). \tag{4. 18}$$

Similarly, integrating Eq. (4. 16) with respect to t and using ICs, the following expressions for $u(r, t)$ and its partial derivative with respect to t are obtained.

$$\frac{\partial u}{\partial t}(r, t) = \frac{\partial u}{\partial t}(r, 0) + \sum_{j=1}^{N} \sum_{i=1}^{N} b_{i,j} p_{j,1}(t) h_i(r), \tag{4. 19}$$

and

$$u(r, t) = r u(r, 0) + t \frac{\partial u}{\partial t}(r, 0) + \sum_{j=1}^{N} \sum_{i=1}^{N} b_{i,j} p_{j,2}(t) h_i(r). \tag{4. 20}$$

We substitute the above expressions in vibration Eq. (1. 1) and also compare the two expressions of the unknown function $u(r, t)$ given in Eq. (4. 18) and Eq. (4. 20). Finally, after substitution of collocation points the following system is obtained:

$$-\sum_{j=1}^{N}\sum_{i=1}^{N}a_{i,j}\Big(h_i(r_k)+\frac{1}{r_k}\big(p_{i,1}(r_k)-p_{i,2}(2)\big)\Big)h_j(t_l)+\frac{1}{c^2}\sum_{j=1}^{N}\sum_{i=1}^{N}b_{i,j}h_i(r_k)h_j(t_l)$$
$$=\frac{1}{r_k}\big(u(2,t_l)-u(1,t_l)\big),\quad k,l=1,2,\ldots,N, \tag{4. 21}$$

and

$$\sum_{j=1}^{N}\sum_{i=1}^{N}a_{i,j}\big(p_{i,2}(r_k)+(1-r_k)p_{i,2}(2)\big)h_j(t_l)-\sum_{j=1}^{N}\sum_{i=1}^{N}b_{i,j}p_{j,2}(t_l)h_i(r_k)=r_ku(r_k,0)$$
$$+t_l\frac{\partial u}{\partial t}(r_k,0)-\big((2-r_k)u(1,t_l)-(1-r_k)u(2,t_l)\big),\quad k,l=1,2,\ldots,N. \tag{4. 22}$$

Eq. (4. 21) and Eq. (4. 22) together represent a linear $N^2\times N^2$ system of equations with unknowns $a_{i,j}, i,j=1,2,\ldots,N$ and $b_{i,j}, i,j=1,2,\ldots,N$. Solving this system by any linear solver yields the unknown coefficients which can be utilized to obtain the approximate solution using either Eq. (4. 18) or Eq. (4. 20).

5. Convergence Theorem

Theorem 5.1. *Let us assume that* $f(x)=\frac{d^nu(x)}{dx^n}\in L^2(\mathbb{R})$ *is a continuous function on* $[0,1]$ *and its first derivative is bounded:*

$$\forall\quad x\in[0,1],\quad\exists\quad\eta:\left|\frac{df(x)}{dx}\right|\le\eta,\quad n\ge 2.$$

Then the HW method, based on approach proposed in [4, 9] *will be convergent, i.e.,* $|E_m|$ *vanishes as* J *goes to infinity, the convergence is of order two:*

$$\|E_m\|_2=O\left[\left(\frac{1}{2^{J+1}}\right)^2\right],$$

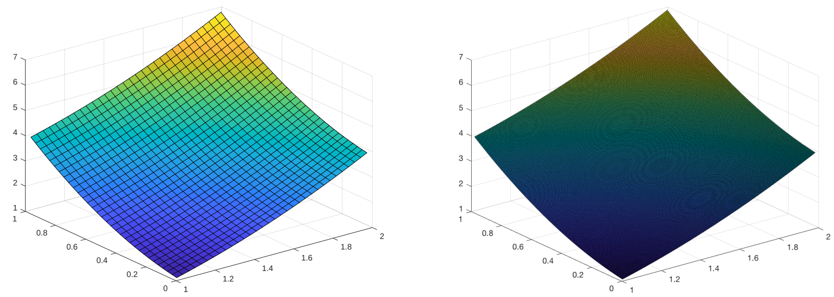
where

$$E_m=f(x)-f_M(x),\quad f_M(x)=\sum_{i=0}^{2M}a_ih_i(x).$$

Proof. For proof, see [13]. □

6. Results and Discussion

We will consider five test problems in this section in order to test the performance of the proposed numerical method. The proposed technique is applied to vibration equation with different ICs. We will denote the Maximum Absolute Errors (MAEs) using the symbol $E_c(2M)$. The reason to use the subscript $2M$ here is that the MAEs are calculated using $2M$ number of collocation points. Similarly, we will use the notation $R_c(2M)$ to denote the



Approximate solution for $N = 32 \times 32$ Analytic solution

FIGURE 1. Graphs of analytic and approximate solutions for Test Problem 6.1.

experimental rate of convergence with $2M$ number of collocation points. The experimental rate of convergence is defined as follows:

$$R_c(2M) = \frac{\log\left(E_c(2M/2)/E_c(2M)\right)}{\log 2}. \tag{6. 23}$$

Example 6.1.

Let us consider vibration Eq. (1. 1) subject to ICs:

$$u(r,0) = r^2, \quad \frac{\partial u}{\partial t}(r,0) = c, \tag{6. 24}$$

and BCs:

$$u(1,t) = 1 + ct + 2c^2t^2, \quad u(2,t) = 4 + ct + 2c^2t^2. \tag{6. 25}$$

The analytic solution for this equation is given by

$$u(r,t) = r^2 + ct + 2c^2t^2. \tag{6. 26}$$

In this case, the analytic solution is a quadratic polynomial. The proposed HW method is used for the numerical solution of this problem with $c = 1$ and a comparison of numerical solution versus analytic solution is shown in Fig. 1. In this figure, the approximate solution is calculated using 32×32 number of collocation points and the figure demonstrates the good performance of the proposed method for this test problem. From the figure, it can be easily observed that numerical solution matches closely with the analytic solution.

Example 6.2.

Considering vibration Eq. (1. 1) subject to ICs:

$$u(r,0) = r^2, \quad \frac{\partial u}{\partial t}(r,0) = cr^2 \tag{6. 27}$$

and BCs:

$$u(1,t) = 1 + ct + 2c^2t^2 + \frac{2}{3}c^3t^3, \quad u(2,t) = 4 + 4ct + 2c^2t^2 + \frac{2}{3}c^3t^3. \tag{6. 28}$$

TABLE 1. $E_c(2M)$ and $R_c(2M)$ for Test Problem 6.2.

J	$2M$	$E_c(2M)$	$R_c(2M)$
0	2×2	1.8684×10^{-02}	—
1	4×4	7.7593×10^{-03}	1.2678
2	8×8	2.2844×10^{-03}	1.7641
3	16×16	6.1180×10^{-04}	1.9010
4	32×32	1.5790×10^{-04}	1.9540
5	64×64	4.0085×10^{-05}	1.9779

The analytic solution is

$$u(r,t) = r^2 + ctr^2 + 2c^2t^2 + \frac{2}{3}c^3t^3. \tag{6. 29}$$

The analytic solution in this test problem is a cubic polynomial. The proposed technique is used for the approximate solution of this test problem with $c = 1$. In Table 1, we have reported the numerical outcomes for this test problem. It is evident from this table that the numerical results obtained by applying the proposed method are very accurate. The MAEs are decreased up to order of 10^{-05} for 64×64 number of collocation points. We can obtain even better results if we increase the number of nodes or collocation points. An important characteristic of the HWCM, which can also be observed from this table, is that the method always gets more accurate as the grid gets finer.

The last column of Table 1 exhibits the numerical convergence rates for this test problem using HWCM. We observe that as we increase the number N, the convergence rate is approaching to 2. The numerical results for this test problem are in accordance with the theoretical results already proved in [12, 14].

Example 6.3.

Considering vibration Eq. (1. 1) subject to ICs:

$$u(r,0) = r, \quad \frac{\partial u}{\partial t}(r,0) = c. \tag{6. 30}$$

The analytic solution is

$$u(r,t) = r + ct + \frac{1}{2r}c^2t^2 + \frac{1}{24r^3}c^4t^4 + \frac{1}{80r^5}c^6t^6 + \dots. \tag{6. 31}$$

In this case, the BCs are obtained from the analytic solution.

Here, we consider more complicated analytic solution which is in the form of an infinite series. Many numerical methods produce incorrect results for such type of analytic solutions. However, the performance of HWCM is equally good for this type of analytic solution as well.

In Table 2, we have shown, for $c = 1$, the MAEs at the collocation points and the corresponding experimental convergence rates using various number of collocation points. We observe that for this test problem too the MAEs are improved with the increase in the size of the grid. The MAE for this test problem using 64×64 number of nodes is

TABLE 2. $E_c(2M)$ and $R_c(2M)$ for Test Problem 6.3.

J	$2M$	$E_c(2M)$	$R_c(2M)$
0	2×2	$8.9795{\times}10^{-04}$	—
1	4×4	$2.3467{\times}10^{-04}$	1.9360
2	8×8	$7.9163{\times}10^{-05}$	1.5677
3	16×16	$3.1772{\times}10^{-05}$	1.3171
4	32×32	$9.3333{\times}10^{-06}$	1.7673
5	64×64	$2.4165{\times}10^{-06}$	1.9495

TABLE 3. $E_c(2M)$ and $R_c(2M)$ for Test Problem 6.4.

J	$2M$	$E_c(2M)$	$R_c(2M)$
0	2×2	$3.2007{\times}10^{-03}$	—
1	4×4	$1.3259{\times}10^{-03}$	1.2714
2	8×8	$3.9098{\times}10^{-04}$	1.7618
3	16×16	$1.0476{\times}10^{-04}$	1.9000
4	32×32	$2.7046{\times}10^{-05}$	1.9536
5	64×64	6.8672×10^{-06}	1.9776

order of 10^{-6} which shows an excellent performance of the HW method. Once again, the experimental convergence rates are converging to 2 validating the results in [12, 14].

Example 6.4.

Considering vibration Eq. (1. 1) subject to ICs:

$$u(r,0) = r^2, \quad \frac{\partial u}{\partial t}(r,0) = cr. \tag{6. 32}$$

The analytic solution is

$$u(r,t) = r^2 + ctr + 2c^2t^2 + \frac{1}{6}\frac{1}{r}c^3t^3 + \frac{1}{120}\frac{1}{r^3}c^5t^5 + \dots \tag{6. 33}$$

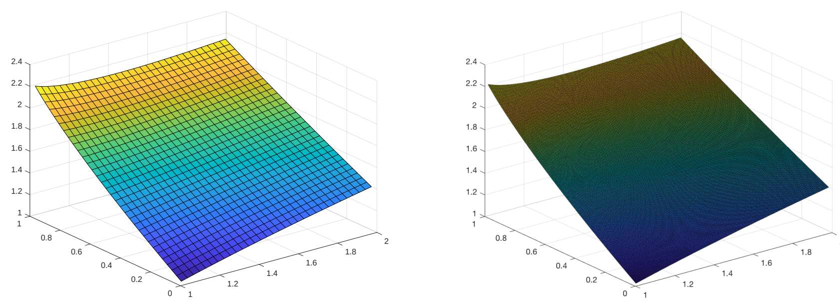
whereas the BCs are obtained from the analytic solution.

Like the previous test problem, in this case also the analytic solution is expressed in terms of an infinite series but an equally good accuracy of the proposed HW method can be seen from the numerical outcomes given in Table 3. The MAEs for $c = 1$ are reduced up to order of 10^{-6} for $2M = 64$ number of collocation points. It is worth mentioning that the accuracy of the HW method may be improved further but it will require more computational cost. As in the last two cases, the convergence rates in this test problem too approach to 2.

Example 6.5.

Considering vibration Eq. (1. 1) subject to ICs:

$$u(r,0) = \sqrt{r}, \quad \frac{\partial u}{\partial r}(r,0) = \frac{c}{\sqrt{r}}. \tag{6. 34}$$



Approximate solution for $2M = 32 \times 32$ Analytic solution

FIGURE 2. Graphs of analytic and approximate solutions for Test Problem 6.5.

The analytic solution is

$$u(r,t) = \sqrt{r} + \frac{1}{\sqrt{r}}ct + \frac{1}{8}\frac{1}{r^{3/2}}c^2t^2 + \frac{1}{24}\frac{1}{r^{5/2}}c^3t^3 + \frac{3}{128}\frac{1}{r^{7/2}}c^4t^4 + \frac{5}{384}\frac{1}{r^{9/2}}c^5t^5 + \frac{49}{5120}\frac{1}{r^{11/2}}c^6t^6 + \dots \tag{6. 35}$$

whereas the BCs are taken from the analytic solution.

Finally, we consider this test problem with a more complicated analytic solution involving an infinite series with terms having non-integer powers of the space variable r. In spite of complicated analytic solution, the performance of the technique does not deteriorate in this case as well. The comparison of analytic and approximate solutions for $c = 1$ is shown in Fig. 2 which a close resemblance between the two solutions can be observed easily.

7. CONCLUSION

The numerical solution of vibration equation is considered and HW is used for approximation of the analytic solution. The vibration equation is discretized using the collocation procedure. The method is validated upon five test problems and a very good performance of the method is observed from the numerical results. In all the problems, the accuracy is increased as we increase the number of nodes. It is also observed that the convergence rates are approaching to 2 numerically which confirms the theoretical results.

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