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Generalized Extension of Morphisms in Geometry of Configuration and Infinitesimal Polylogarithmic Groups Complexes

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Abstract. In this research work, a generalized extension of morphisms has been proposed to define generalized geometry between the two generalized chain complexes: Grassmannian configuration, and a variant of Cathelineau infinitesimal. Initially, the new morphisms are introduced to extend the geometry of Grassmannian and variant of Cathelineau infinitesimal chain complexes for weight $n = 6$. Finally, generalizations of this extension are proposed through generalized morphisms, and the commutativity of associated diagrams is also shown.

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1. Introduction

Configuration chain complex is used as an important tool in differential manifold and algebraic K-theory, and is naturally related to polylogarithmic groups and its chain complexes. Consider $G_{k+1}(n)$ to be a group formed by configuration of $(k + 1)$ points of n-dimensional vector space over a field F at a generic position, it is also a free abelian group. These free abelian groups are connected through two types of differential morphisms, d and p, to form Grassmannian chain and co-chain complexes, introduced by Suslin [19]. Khalid et al. [14] have introduced higher order mixed partial differential morphisms to define new form of Grassmannian complex called 2^{nd} and 3^{rd} order Grassmannian complex. The same author [15] also generalized differential maps between free abelian groups to define N^{th}

order Grassmannian chain complex. Leibniz introduced p-logarithms series as

$$Li_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}, \quad | z |< 1$$

which is an absolutely convergent series over a unit disc. Dilogarithm functions [2, 3] for $p = 2$ were studied by many authors but the most important were the functional equations called Abel's [1] five term relation. Trilogarithm for $p = 3$ and its group was defined by Goncharov [8], in which the most significant was the generalized triple cross ratio property of six points. Goncharov generalized classical polylog groups complex and called it Goncharov Complex. Goncharov also defined geometry between configuration and his own chain complexes through morphisms only for weight $N = 2$ and 3 and showed commutativity of all diagrams. Khalid et al. [17] generalized geometry between Goncharov and Grassmannian configuration chain complexes for weight N. Cathelineau [4–6] defined variant of Goncharov complex by using derivation maps in two ways; one was infinitesimal while other was tangential setting. The same author also generalized infinitesimal chain complex called Cathelineau complex.
Hussain [11] used Kähler differential to introduce a variant of Cathelineau complex and connect Grassmannain complex with variant of Cathelineau complex (both infinitesimal and tangential) for weight N = two and three and proved the commutativity of all the diagrams. Khalid et al. [12] introduced new geometry between variant of Cathelineau infinitesimal polylog group complex with Grassmannain configuration chain complex up to weight 4 and also generalized two morphism in generalized geometry of configuration with infinitesimal complexes for generalized weight n [13]. Further, the same author [16], also introduced the extension of morphisms to extend the geometry of configuration and infinitesimal chain complexes both for weight $n = 4$ and 5 and proved the commutativity of corresponding diagrams.
In this article, the work of [16] is extended to include and introduce generalized morphisms for generalized geometry of configuration and a variant of Cathelineau infinitesimal polylogarithmic groups chain complexes. Initially, four new morphisms are introduced in weight 6 and then generalized $(N-2)$ homomorphisms are defined in generalized geometry between these chain complexes. At the end, a generalized diagram will be constructed and shown to be commutative.

2. Literature Review

This section includes the introduction and explanation of configuration chain complexes, cross ratio, classical polylog groups chains, Cathelineau and its analogy infinitesimal polylogarithmic groups chain complexes.

2.1. **Grassmannian Configuration Chain Complex.** Let $G_k(V_n)$ be a group generated by elements $(v_1, ..., v_k)$ of some vector space V_n, defined over field F. There exist two simplicial complexes with differential maps $(G_m(V_n), d)$ and $(G_m(V_n), p)$

$$d : G_k(V_n) \to G_{k-1}(V_n),$$

defined as $d(q_0, ..., q_n) = \sum_0^n (-1)^i (q_0, ...\hat{q}_i, ..., q_n)$, where $\hat{q}_i$ means leaving out.

$$p : G_k(V_n) \to G_{k-1}(V_{n-1})$$

defined as $p(q_0, ..., q_n) = \sum_0^n (-1)^i (q_i | q_0, ...\hat{q}_i, ..., q_n)$, where q_i is both leaving and projection point. Using these two differential morphisms, the Grassmannian configuration is formed, given by

$$\begin{array}{ccccc} G_{k+5}(k+2) & \xrightarrow{d} & G_{k+4}(k+2) & \xrightarrow{d} & G_{k+3}(k+2) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{k+4}(k+1) & \xrightarrow{d} & G_{k+3}(k+1) & \xrightarrow{d} & G_{k+2}(k+1) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{k+3}(k) & \xrightarrow{d} & G_{k+2}(k) & \xrightarrow{d} & G_{k+1}(k) \end{array} \qquad (A)$$

Diagram (A) is bi-complex and commutative [8, 10].

2.2. **Cross Ratio Property.** In the field of algebraic and differential geometry, cross ratio property of four collinear points plays a vital rule. Siegel [18] was the first to introduce this important cross ratio property of four points given as

$$1 = r(q_0, q_1, q_2, q_3) + r(q_0, q_2, q_1, q_3),$$

where $r(q_0, q_1, q_2, q_3)$ is cross ration of four points defined as

$$r(q_0, q_1, q_2, q_3) = \frac{\triangle(q_0, q_3)\triangle(q_1, q_2)}{\triangle(q_0, q_2)\triangle(q_1, q_3)}$$

This property will be used in calculation and simplification of all results.

2.3. **Classical Polylogarithmic Groups and its Chain Complexes.** For weight 1, let $Z[\mathbf{P}^1_F/\{0, 1, \infty\}]$ be a free module generated by an element $[q] \in \mathbf{P}^1_F$ [7, 8],where symbol $[q]$ shows classical log(q). let F be a field which will be used in this research while $F^{\cdot\cdot}$ is a set without 0 and 1. The group $\mathcal{B}_1(F) = Z[F^{\cdot\cdot}]/\left\langle [q] - [r] + [\frac{r}{q}] - [\frac{1-r}{1-q}] + [\frac{1-r^{-1}}{1-q^{-1}}] \right\rangle$, where $x \neq r$,$q, r \neq 0, 1$ and $F^{\cdot\cdot} = F - (0, 1)$ [9]. This group is isomorphic to multiplicative group $F^\times$. For dilogarithm weight 2 Goncharov [8] defined a factor subgroup $\mathcal{B}_2(F) = Z[F^{\cdot\cdot}]/\left\langle \sum_{i=0}^{4} (-1)^i [r(q_0, ..., \hat{q}_i, ..., q_4)], q_i \in \mathbf{P}^1_F \right\rangle$. Using similar way for weight n Goncharov [8] generalized group $\mathcal{B}_n(F) = Z[F^{\cdot\cdot}]/R_n(F)$., where $R_n(F)$ is kernel of map $\delta_n : Z[\mathbf{P}^1_F] \to \mathcal{B}_{n-1}(F) \otimes F^\times$. Goncharov [8] introduced generalized chain complex given as

$$\mathcal{B}_n(F) \xrightarrow{d_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \mathcal{B}_2(F) \otimes \wedge^{n-2} F^\times \xrightarrow{d_1} \wedge^n F^\times$$

2.4. **Cathelineau Chain.** Cathelineau [4, 5] introduced F-vector space and also define variant of Goncharov generalized group $\mathcal{B}_n(F)$, For weight 1 let $\beta_1(F) = F$, For weight 2 let $\beta_2(F) = F[F^{\cdot\cdot}]/\left\langle [q] - [r] + q[\frac{r}{q}] + (1-q)[\frac{1-r}{1-q}] \right\rangle, q, r \in F^{\cdot\cdot}, q \neq r$ [5] and for weight N Cathelineau generalized a group as $\beta_n(F) = Z[F^{\cdot\cdot}]/R_n(F)$ [5]. The generalized infinitesimal chain complex introduced by Cathelineau in [5] for these generalized groups and higher Bloch groups $\mathcal{B}_n(F)$ is given as

$$\beta_n(F) \xrightarrow{\partial_n} \begin{matrix} \beta_{n-1}(F)\otimes F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F) \end{matrix} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \begin{matrix} \beta_2(F)\otimes\wedge^{n-2}F^\times \\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times \end{matrix} \xrightarrow{\partial_0} F\otimes\wedge^{n-1}F^\times$$

2.4.1. *Analogy of Cathelineau Infinitesimal Polylog Chain Complex.* Let

$$R_2 = \sum_{i=0}^{4} [\![r(q_0.,,,q_3)]\!]_2^D$$

is five term relation and sub group of $F[F^{\cdot\cdot}]$. Define a group $\beta_2^D(F) = F[F^{\cdot\cdot}]/\langle R_2 \rangle$ [11]. For weight 3, define a factor group $\beta_3^D(F) = F[F^{\cdot\cdot}]/\left\langle \sum_{i=0}^{6}(-1)^i [\![r_3(q_0,...,\hat{q}_i,...,q_6)]\!]_3^D \right\rangle$ Using the same steps, $\beta_n^D(F) = F[F^{\cdot\cdot}]/\langle R_n \rangle$. Finally for weight N, a chain complex for these groups was generalized $\beta_n^D(F)$ and $\mathcal{B}_n(F)$ called variant of infinitesimal Cathelineau polylogarithmic chain complex given by

$$\beta_n^D(F) \xrightarrow{\partial_n^D} \begin{matrix} \beta_{n-1}^D(F)\otimes F^\times \\ \oplus \\ nF\otimes\mathcal{B}_{n-1}(F) \end{matrix} \xrightarrow{\partial_{n-1}^D} \cdots \xrightarrow{\partial_1^D} \begin{matrix} \beta_2^D(F)\otimes\wedge^{n-2}F^\times \\ \oplus \\ (F)\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times \end{matrix} \xrightarrow{\partial_0^D} F\otimes\wedge^{n-1}F^\times$$

3. Geometry of Variant of Cathelineau and Grassmannian Complexes

For geometry of Grassmannian configuration and a variant of Cathelineau infinitesimal complexes, Khalid et al. [12, 13] first introduced morphisms to connect these two chain complexes, then generalized only two morphism up to weight n for projective spaces. Khalid et al. [16] also extend this geometry, using extension of morphisms, first for special case weight $N = 4$ and then for $N = 5$, while proving all polygons to be commutative. Therefore this section is following the work of Khalid et al. [16] to introduce generalized commutative polygons through generalized extension of morphism.

3.1. **Geometry for Weight 6.** For this geometry, Khalid et al. [13] connect the sub complexes of variant of Cathelineau infinitesimal and configuration chain complexes given by

$$\begin{array}{ccccc} G_{10}(3) & \xrightarrow{d} & G_9(3) & & \\ \downarrow p & & \downarrow p & & \\ G_9(2) & \xrightarrow{d} & G_8(2) & \xrightarrow{h_1^6} & \beta_2^D(F)\otimes\wedge^4F^\times \oplus F\otimes\mathcal{B}_2(F)\otimes\wedge^3F^\times \\ \downarrow p & & \downarrow p & & \downarrow \partial^D \\ G_8(1) & \xrightarrow{d} & G_7(1) & \xrightarrow{h_0^6} & F\otimes\wedge^5F^\times \end{array} \qquad \text{(B)}$$

where

$$h_0^6 : (q_0,...,q_6) = \sum_{i=0}^{6}(-1)^i D\log(q_i)\otimes\frac{(q_{i+1})}{(q_{i+2})}\wedge...\wedge\frac{(q_{i+5})}{(q_{i+6})} \quad (mod\ 7)$$

where $q_i = \triangle(q_i)$ is determinant of an element in single dimensional vector space.

$$h_1^6(q_0,...,q_7) = \frac{1}{15}\Big[\sum_{i\neq j\neq k\neq l}^{7}(-1)^i([\![r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_7)]\!]_2^D \otimes \prod_{l\neq r}^{7}(q_l,q_r) \wedge \prod_{k\neq r}^{7}(q_k,q_r)$$
$$\wedge \prod_{j\neq r}^{7}(q_j,q_r) \wedge \prod_{i\neq r}^{7}(q_i,q_r) - D\log(\prod_{i\neq r}^{7}(q_i,q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_7)]_2$$
$$\otimes \prod_{j\neq r}^{7}(q_j,q_r) \wedge \prod_{k\neq r}^{7}(q_k,q_r) \wedge \prod_{l\neq r}^{7}(q_l,q_r) + D\log(\prod_{j\neq r}^{7}(q_j,q_r))\otimes$$
$$[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_7)]_2 \otimes \prod_{k\neq r}^{7}(q_k,q_r) \wedge \prod_{l\neq r}^{7}(q_l,q_r) \wedge \prod_{i\neq r}^{7}(q_i,q_r)$$
$$- D\log(\prod_{k\neq r}^{7}(q_k,q_r)) \otimes [q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_7]_2 \otimes \prod_{i\neq r}^{7}(q_i,q_r)\wedge$$
$$\prod_{j\neq r}^{7}(q_j,q_r)\Big] \wedge \prod_{l\neq r}^{7}(q_l,q_r) + D\log(\prod_{l\neq r}^{7}(q_l,q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_7)]_2$$
$$\otimes \prod_{i\neq r}^{7}(q_i,q_r) \wedge \prod_{j\neq r}^{7}(q_j,q_r) \wedge \prod_{k\neq r}^{7}(q_k,q_r)\Big] \quad (mod\ 8)$$

where $(q_k,q_r) = \triangle(q_k,q_r)$ shows determinant of two different points in 2-dimensional vector space.

Lemma 3.2. $h_0^6 \circ d = 0$ *and* $h_1^6 \circ d = 0$.
it show that diagram (B) *is bi-complex. For proof, see [13].*

Lemma 3.3. $h_0^6 \circ p = \partial^D \circ h_1^6$.
This lemma show that diagram (B) *is commutative for weight* 6*. For proof, see [13].*

3.3.1. *Extension of Morphisms for Weight 6.* Here, four new morphisms h_2^6, h_3^6, h_4^6 and h_5^6 will be introduced to extend the geometry of above complexes given by

$$\begin{array}{ccccc}
G_{13}(6) & \xrightarrow{d} & G_{12}(6) & \xrightarrow{h_5^6} & \beta_6^D(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{12}(5) & \xrightarrow{d} & G_{11}(5) & \xrightarrow{h_4^6} & \beta_5^D(F)\otimes F^\times \oplus F\otimes \mathcal{B}_5(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{11}(4) & \xrightarrow{d} & G_{10}(4) & \xrightarrow{h_3^6} & \beta_4^D(F)\otimes \wedge^2 F^\times \oplus F\otimes \mathcal{B}_4(F)\otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{10}(3) & \xrightarrow{d} & G_9(3) & \xrightarrow{h_2^6} & \beta_3^D(F)\otimes \wedge^3 F^\times \oplus F\otimes \mathcal{B}_3(F)\otimes \wedge^2 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_9(2) & \xrightarrow{d} & G_8(2) & \xrightarrow{h_1^6} & \beta_2^D(F)\otimes \wedge^4 F^\times \oplus F\otimes \mathcal{B}_2(F)\otimes \wedge^3 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_8(1) & \xrightarrow{d} & G_7(1) & \xrightarrow{h_0^6} & F\otimes \wedge^5 F^\times
\end{array} \tag{C}$$

where

$$h_2^6(q_0,...,q_8) = -\frac{1}{66}\sum_{i\neq j\neq k}^{8}(-1)^i Alt_6\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]\!]_3^D \otimes \prod_{k\neq r}^{8}(q_k,q_r)\wedge\prod_{j\neq r}^{8}(q_j,q_r)$$
$$\wedge\prod_{i\neq r}^{8}(q_i,q_r) -$$
$$D\log\Big(\prod_{i\neq r}^{8}(q_i,q_r)\Big)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\otimes\prod_{j\neq r}^{8}(q_j,q_r)$$
$$\wedge\prod_{k\neq r}^{8}(q_k,q_r) +$$
$$D\log\Big(\prod_{j\neq r}^{8}(q_j,q_r)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\otimes\prod_{k\neq r}^{8}(q_k,q_r)$$
$$\wedge\prod_{i\neq r}^{8}(q_i,q_r) -$$
$$D\log\Big(\prod_{k\neq r}^{8}(q_k,q_r)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\otimes\prod_{i\neq r}^{8}(q_i,q_r)$$

$$\wedge \prod_{j\neq r}^{8}(q_j,q_r)\Big] \quad (mod\ 9)$$

In Alt_n there are $n!$ relation. For example Alt_6 show alternating group of six elements and there are $6! = 720$ relation in this group but due to symmetries it reduced to 120 relation.

$$h_3^6(q_0,...,q_9) = \frac{1}{153}\sum_{i\neq j}^{9}(-1)^i Alt_8\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_9)]\!]_4^D \otimes \prod_{i\neq r}^{9}(q_i,q_r)\wedge \prod_{j\neq r}^{9}(q_j,q_r)$$

$$- D\log(\prod_{i\neq r}^{9}(q_i,q_r))\otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_9)]_4 \otimes \prod_{j\neq r}^{9}(q_j,q_r)$$

$$+ D\log(\prod_{j\neq r}^{9}(q_j,q_r))\otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_9)]_4 \otimes \prod_{i\neq r}^{9}(q_i,q_r)\Big] \quad (mod\ 10)$$

$$h_4^6(q_0,...,q_{10}) = -\frac{1}{276}\sum_{i=0}^{10}(-1)^i Alt_{10}\Big[[\![r(q_0,...,\hat{q}_i,...,q_{10})]\!]_5^D \otimes \prod_{i\neq r}^{10}(q_i,q_r)$$

$$- D\log(\prod_{i\neq r}^{10}(q_i,q_r))\otimes [r(q_0,...,\hat{q}_i,...,q_{10})]_5\Big] \quad (mod\ 11)$$

$$h_5^6(q_0,...,q_{11}) = \frac{1}{435}Alt_{12}[\![r(q_0,...,q_{11})]\!]_6^D \quad (mod\ 12)$$

Now, proving that each square of extended diagram (C) is commutative.

Lemma 3.4. $h_1^6 \circ p = \partial^D \circ h_2^6$

Proof. : Let $(q_0,...,q_8) \in G_9(4)$, and apply morphism p

$$p(q_0,.....,q_8) = \sum_{i=0}^{8}(-1)^i(q_i|q_0,...,\hat{q}_i,...,q_8)$$

after applying morphism h_1^6, then

$$h_1^6 \circ p(q_0,...,q_8) = \frac{1}{15}\Big[\sum_{i\neq j\neq k\neq l}^{8}(-1)^i([\![r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]\!]_2^D \otimes \prod_{l\neq r}^{8}(q_l,q_r)\wedge \prod_{k\neq r}^{8}(q_k,q_r)$$

$$\wedge \prod_{j\neq r}^{8}(q_j,q_r)\wedge \prod_{i\neq r}^{8}(q_i,q_r) - D\log(\prod_{i\neq r}^{8}(q_i,q_r))\otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2$$

$$\otimes \prod_{j\neq r}^{8}(q_j,q_r)\wedge \prod_{k\neq r}^{8}(q_k,q_r)\wedge \prod_{l\neq r}^{8}(q_l,q_r) + D\log(\prod_{j\neq r}^{8}(q_j,q_r))\otimes$$

$$[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2 \otimes \prod_{k\neq r}^{8}(q_k,q_r)\wedge \prod_{l\neq r}^{8}(q_l,q_r)\wedge \prod_{i\neq r}^{8}(q_i,q_r)$$

$$- D\log(\prod_{k\neq r}^{8}(q_k,q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2\otimes\prod_{i\neq r}^{8}(q_i,q_r)\wedge$$

$$\prod_{j\neq r}^{8}(q_j,q_r)\Big]\wedge\prod_{l\neq r}^{8}(q_l,q_r)+D\log(\prod_{l\neq r}^{8}(q_l,q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2$$

$$\otimes\prod_{i\neq r}^{8}(q_i,q_r)\wedge\prod_{j\neq r}^{8}(q_j,q_r)\wedge\prod_{k\neq r}^{8}(q_k,q_r)\Big] \qquad (3.\ 1)$$

let's take $(q_0,...,q_8)\in G_9(4)$, and apply morphism h_2^6

$$h_2^6(q_0,...,q_8)=-\frac{1}{66}\sum_{i=0}^{8}(-1)^i Alt_6\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]\!]_3^D\otimes\prod_{k\neq r}^{8}(q_k,q_r)\wedge\prod_{j\neq r}^{8}(q_j,q_r)$$

$$\wedge\prod_{i\neq r}^{8}(q_i,q_r)-$$

$$D\log(\prod_{i\neq r}^{8}(q_i,q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\otimes\prod_{j\neq r}^{8}(q_j,q_r)$$

$$\wedge\prod_{k\neq r}^{8}(q_k,q_r)+$$

$$D\log(\prod_{j\neq r}^{8}(q_j,q_r)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\otimes\prod_{i\neq r}^{8}(q_i,q_r)$$

$$\wedge\prod_{k\neq r}^{8}(q_k,q_r)-$$

$$D\log(\prod_{k\neq r}^{8}(q_k,q_r)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\otimes\prod_{i\neq r}^{8}(q_i,q_r)$$

$$\wedge\prod_{j\neq r}^{8}(q_j,q_r)\Big]$$

after applying morphism ∂^D and all properties (tensor, wedge and Siegel cross ratio) the equation becomes

$$\partial^D\circ h_2^6(q_0,...,q_8)=\frac{1}{15}\Big[\sum_{i\neq j\neq k\neq l}^{8}(-1)^i([\![r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]\!]_2^D\otimes\prod_{l\neq r}^{8}(q_l,q_r)\wedge\prod_{k\neq r}^{8}(q_k,q_r)$$

$$\wedge\prod_{j\neq r}^{8}(q_j,q_r)\wedge\prod_{i\neq r}^{8}(q_i,q_r)-D\log(\prod_{i\neq r}^{8}(q_i,q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2$$

$$\otimes\prod_{j\neq r}^{8}(q_j,q_r)\wedge\prod_{k\neq r}^{8}(q_k,q_r)\wedge\prod_{l\neq r}^{8}(q_l,q_r)+D\log(\prod_{j\neq r}^{8}(q_j,q_r))\otimes$$

$$[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2 \otimes \prod_{k\neq r}^{8}(q_k,q_r) \wedge \prod_{l\neq r}^{8}(q_l,q_r) \wedge \prod_{i\neq r}^{8}(q_i,q_r)$$

$$- D\log(\prod_{k\neq r}^{8}(q_k,q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2 \otimes \prod_{i\neq r}^{8}(q_i,q_r)\wedge$$

$$\prod_{j\neq r}^{8}(q_j,q_r)\Big] \wedge \prod_{l\neq r}^{8}(q_l,q_r) + D\log(\prod_{l\neq r}^{8}(q_l,q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_8)]_2$$

$$\otimes \prod_{i\neq r}^{8}(q_i,q_r) \wedge \prod_{j\neq r}^{8}(q_j,q_r) \wedge \prod_{k\neq r}^{8}(q_k,q_r)\Big] \quad (3.\ 2)$$

by using Eq. (3. 1) and (3. 2), it is observed that $h_1^6 \circ p = \partial^D \circ h_2^6$

Lemma 3.5. $h_2^6 \circ p = \partial^D \circ h_3^6$

Proof. : Let $(q_0,...,q_9) \in G_{10}(5)$, applying differential morphism p

$$p(q_0,.....,q_9) = \sum_{i=0}^{9}(-1)^i(q_i|q_0,...,\hat{q}_i,...,q_9)$$

by applying map h_2^6, it becomes

$$h_2^6 \circ p(q_0,...,q_9) = -\frac{1}{66}\sum_{i\neq j\neq k\neq l}^{9}(-1)^i Alt_6\Big[[\![r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]\!]_3^D \otimes \prod_{l\neq r}^{9}(q_i|q_l,q_r)\wedge$$

$$\prod_{k\neq r}^{9}(q_i|q_k,q_r) \wedge \prod_{j\neq r}^{9}(q_i|q_j,q_r)-$$

$$D\log(\prod_{j\neq r}^{9}(q_i|q_j,q_r)) \otimes [r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]_3 \otimes \prod_{k\neq r}^{9}(q_i|q_k,q_r)$$

$$\wedge \prod_{l\neq r}^{9}(q_i|q_l,q_r)+$$

$$D\log(\prod_{k\neq r}^{9}(q_i|q_k,q_r) \otimes [r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]_3 \otimes \prod_{l\neq r}^{9}(q_i|q_l,q_r)$$

$$\wedge \prod_{j\neq r}^{9}(q_i|q_j,q_r)-$$

$$D\log(\prod_{l\neq r}^{9}(q_i|q_l,q_r) \otimes [r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]_3 \otimes \prod_{j\neq r}^{9}(q_i|q_j,q_r)$$

$$\wedge \prod_{k\neq r}^{9}(q_i|q_k,q_r)\Big] \quad (3.\ 3)$$

let take $(q_0, ..., q_9) \in G_{10}(5)$ again and applying morphism h_3^6

$$h_3^6(q_0,...,q_9) = \frac{1}{153}\sum_{i\neq j}^{9}(-1)^i Alt_8\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_9)]\!]_4^D \otimes \prod_{i\neq r}^{9}(q_i,q_r) \wedge \prod_{j\neq r}^{9}(q_j,q_r)$$
$$- D\log(\prod_{i\neq r}^{9}(q_i,q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_9)]_4 \otimes \prod_{j\neq r}^{9}(q_j,q_r)$$
$$+ D\log(\prod_{j\neq r}^{9}(q_j,q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_9)]_4 \otimes \prod_{i\neq r}^{9}(q_i,q_r)\Big]$$

now by applying morphism ∂^D and all properties: tensor, wedge and Siegel cross ratio,

$$\partial^D \circ h_3^6(q_0,...,q_9) = -\frac{1}{66}\sum_{i\neq j\neq k\neq l}^{9}(-1)^i Alt_6\Big[[\![r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]\!]_3^D \otimes \prod_{l\neq r}^{9}(q_i|q_l,q_r)\wedge$$
$$\prod_{k\neq r}^{9}(q_i|q_k,q_r) \wedge \prod_{j\neq r}^{9}(q_i|q_j,q_r) -$$
$$D\log(\prod_{j\neq r}^{9}(q_i|q_j,q_r)) \otimes [r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]_3 \otimes \prod_{k\neq r}^{9}(q_i|q_k,q_r)$$
$$\wedge \prod_{l\neq r}^{9}(q_i|q_l,q_r) +$$
$$D\log(\prod_{k\neq r}^{9}(q_i|q_k,q_r) \otimes [r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]_3 \otimes \prod_{l\neq r}^{9}(q_i|q_l,q_r)$$
$$\wedge \prod_{j\neq r}^{9}(q_i|q_j,q_r) -$$
$$D\log(\prod_{l\neq r}^{9}(q_i|q_l,q_r)) \otimes [r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_9)]_3 \otimes \prod_{j\neq r}^{9}(q_i|q_j,q_r)$$
$$\wedge \prod_{k\neq r}^{9}(q_i|q_k,q_r)\Big] \qquad (3.\ 4)$$

From Eq. (3. 3) and (3. 4), it is observed $h_2^6 \circ p = \partial^D \circ h_3^6$

Lemma 3.6. $h_3^6 \circ p = \partial^D \circ h_4^6$

Proof. : Take $(q_0, ..., q_{10}) \in G_{11}(5)$ and apply differential morphism p

$$p(q_0,.....,q_{10}) = \sum_{i=0}^{10}(-1)^i(q_i|q_0,...,\hat{q}_i,...,q_{10})$$

by applying differential map h_3^6, to get

$$h_3^6 \circ p(q_0, ..., q_{10}) = \frac{1}{153} \sum_{i \neq j \neq k}^{10} (-1)^i Alt_8 \Big[[\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_{10})]\!]_4^D \otimes \prod_{j \neq r}^{10} (q_i|q_j, q_r) \wedge \prod_{k \neq r}^{10} (q_i|q_k, q_r) - D \log(\prod_{j \neq r}^{10} (q_i|q_j, q_r)) \otimes r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_{10})]_4 \otimes \prod_{k \neq r}^{10} (q_i|q_k, q_r) + D \log(\prod_{k \neq r}^{10} (q_j, q_r)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_{10})]_4 \otimes \prod_{j \neq r}^{10} (q_i|q_j, q_r) \Big] \tag{3. 5}$$

take again $(q_0, ..., q_{10}) \in G_{11}(5)$ again and apply morphism h_4^6

$$h_4^6(q_0, ..., q_{10}) = -\frac{1}{276} \sum_{i=0}^{10} (-1)^i Alt_{10} \Big[[\![r(q_0, ..., \hat{q}_i, ..., q_{10})]\!]_5^D \otimes \prod_{i \neq r}^{10} (q_i, q_r) - D \log(\prod_{i \neq r}^{10} (q_i, q_r)) \otimes [r(q_0, ..., \hat{q}_i, ..., q_{10})]_5 \Big]$$

by applying morphism ∂^D and all tensor, wedge and Siegel cross ratio properties, to get

$$\partial^D \circ h_4^6(q_0, ...,q_{10}) = \frac{1}{153} \sum_{i \neq j \neq k}^{10} (-1)^i Alt_8 \Big[[\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_{10})]\!]_4^D \otimes \prod_{j \neq r}^{10} (q_i|q_j, q_r) \wedge \prod_{k \neq r}^{10} (q_i|q_k, q_r) - D \log(\prod_{j \neq r}^{10} (q_i|q_j, q_r)) \otimes r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_{10})]_4 \otimes \prod_{k \neq r}^{10} (q_i|q_k, q_r) + D \log(\prod_{k \neq r}^{10} (q_j, q_r)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_{10})]_4 \otimes \prod_{j \neq r}^{10} (q_i|q_j, q_r) \Big] \tag{3. 6}$$

From Eq. (3. 5) and (3. 6), $h_3^6 \circ p = \partial^D \circ h_4^6$

Lemma 3.7. $h_4^6 \circ p = \partial^D \circ h_5^6$

Proof. Take $(q_0, ..., q_{11}) \in G_{12}(6)$, and apply differential morphism p

$$p(q_0, ..., q_{11}) = \sum_{i=0}^{11} (-1)^i (q_i|q_0, ..., \hat{q}_i, ..., q_{11})$$

by applying map h_4^6, to get

$$h_4^6 \circ p(q_0, ..., q_{11}) = -\frac{1}{276}\sum_{i\neq j}^{11}(-1)^i Alt_{10}\Big[[\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_{11})]\!]_5^D \otimes \prod_{j\neq r}^{11}(q_j, q_r) - D\log(\prod_{j\neq r}^{11}(q_j, q_r)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_{10})]_5\Big] \quad (3.\ 7)$$

Let's take $(q_0, ..., q_{11}) \in G_{12}(6)$ again and apply morphism h_4^5

$$h_5^6(q_0, ..., q_{11}) = \frac{1}{435} Alt_{12}[\![r(q_0, ..., q_{11})]\!]_6^D$$

by applying morphism ∂^D and all tensor, wedge and Siegel cross ratio properties, to get

$$\partial^D \circ h_5^6(q_0, ..., q_{11}) = -\frac{1}{276}\sum_{i\neq j}^{11}(-1)^i Alt_{10}\Big[[\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_{11})]\!]_5^D \otimes \prod_{j\neq r}^{11}(q_j, q_r) - D\log(\prod_{j\neq r}^{11}(q_j, q_r)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_{10})]_5\Big] \quad (3.\ 8)$$

From Eq. (3. 7) and (3. 8), it is concluded that $h_4^6 \circ p = \partial^D \circ h_5^6$.

From Lemma (3.4) to (3.7), it have been proved that each square of extended diagram (C) is commutative.

4. Weight N

Weight N geometry of Grassmannian and variant of Cathelineau complex were constructed through two generalized morphisms h_0^N and h_1^N [13], given by

$$\begin{array}{ccccc}
G_{N+4}(3) & \xrightarrow{d} & G_{N+3}(3) & & \\
\downarrow p & & \downarrow p & & \\
G_{N+3}(2) & \xrightarrow{d} & G_{N+2}(2) & \xrightarrow{h_1^N} & \beta_2^D(F) \otimes \wedge^{N-2}F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes \wedge^{N-3}F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{N+2}(1) & \xrightarrow{d} & G_{N+1}(1) & \xrightarrow{h_0^N} & F \otimes \wedge^{N-1}F^\times
\end{array} \quad \text{(D)}$$

where

$$h_0^N(q_0, ..., q_N) = \sum_{i=0}^{N}(-1)^i D\log(q_i) \otimes \frac{(q_{i+1})}{(q_{i+2})} \wedge ... \wedge \frac{(q_N)}{(q_{N+1})} \quad (mod\ (N+1))$$

and

$$h_1^N(q_0, ..., q_{N+1}) = (-1)^N \frac{1}{{}^N C_2} \sum_{i=0}^{N+1}(-1)^i \Big([\![r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N, ...q_{N+1})]\!]_2^D \otimes$$

$$\prod_{N+1\neq r}^{N+1}(q_{N+1},q_r)\wedge ...\wedge\prod_{i\neq r}^{N+1}(q_i,q_r)-$$

$$D\log(\prod_{i\neq r}^{N+1}(q_i,q_r))\otimes[q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,q_{N+1}]_2\otimes\prod_{j\neq r}^{N+1}(q_{i+1},q_r)\wedge$$

$$...\wedge\prod_{N+1\neq r}^{N+1}(q_{N+1},q_r)+$$

$$D\log(\prod_{i+1\neq r}^{N+1}(q_{i+1},q_r))\otimes[q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,q_{N+1}]_2\otimes\prod_{i\neq r}^{N+1}(q_i,q_r)\wedge$$

$$...\wedge\prod_{N+1\neq r}^{N+1}(q_{N+1},q_r)-$$

$$D\log(\prod_{i+2\neq r}^{N+1}(q_{i+2},q_r))\otimes[q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,q_{N+1}]_2\otimes\prod_{i\neq r}^{N+1}(q_i+2,q_r)\wedge$$

$$...\wedge\prod_{N+1\neq r}^{N+1}(q_{N+1},q_r)+$$

.

.

.

$$(-1)^{N+1}D\log(\prod_{N+1\neq r}^{N+1}(q_{N+1},q_r))\otimes[q_0,...,,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...,q_{N+1}]_2\otimes$$

$$\prod_{i\neq r}^{N+1}(q_i,q_r)\wedge...\wedge\prod_{N\neq r}^{N+1}(q_N,q_r)\Big)\quad (mod\ N+2)$$

Theorem 4.1. $$h_0^N\circ p=\partial^D\circ h_1^N.$$

Above theorem showed that generalized diagram (D) with two generalized morphisms is commutative. For proof, see [13].

4.2. **Generalized Extension of Morphisms for Weight N.** For generalized extension, initially, $(N-2)$ new morphisms $h_2^N, h_3^N, ..., h_{N-2}^N, h_{N-1}^N$, $(N\geq 3)$ are introduced to produce generalized geometry between configuration and variant of infinitesimal Cathelineau chain complexes, then generalized diagrams of Grassmannian and variant of Cathelineau Infinitesimal chain complexes are constructed as follows

$$
\begin{array}{ccccc}
G_{2N+1}(N) & \xrightarrow{d} & G_{2N}(N) & \xrightarrow{h^N_{N-1}} & \beta^D_n(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{2N}(N-1) & \xrightarrow{d} & G_{2N-1}(N-1) & \xrightarrow{h^N_{N-2}} & \beta^D_{n-1}(F)\otimes F^\times \oplus F\times \mathcal{B}_{n-1}(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
\vdots & & \vdots & & \vdots \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{N+3}(2) & \xrightarrow{d} & G_{N+2}(2) & \xrightarrow{h^N_1} & \beta^D_2(F)\otimes \wedge^{N-2}F^\times \oplus F\otimes \mathcal{B}_2(F)\otimes \wedge^{N-3}F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{N+2}(1) & \xrightarrow{d} & G_{N+1}(1) & \xrightarrow{h^N_0} & F\otimes \wedge^{N-1}F^\times
\end{array}
\tag{E}
$$

where

$$
h^N_{N-2}(q_0,...,q_{2N-2}) = (-1)^N \frac{1}{{}^{(N-2)N}C_2} \sum_{i=0}^{2N-2} (-1)^i Alt_{2n-2}\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...,
$$

$$
q_{2N-2})]\!]^D_{N-1} \otimes \prod_{2(N-1)\neq r}^{2N-2} (q_{2(N-1)},q_r) \wedge ... \wedge \prod_{i\neq r}^{2N-2}(q_i,q_r) - D\log(\prod_{i\neq r}^{2N-2}(q_i,q_r))\otimes
$$

$$
[r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...q_{2N-2})]_{N-1} \otimes \prod_{j\neq r}^{2N-2}(q_{i+1},q_r) \wedge ... \wedge \prod_{N+4\neq r}^{2N-1}(q_{2N-2},q_r)
$$

$$
+ D\log(\prod_{i+1\neq r}^{2N-2}(q_{i+1},q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...q_{2N-2})]_{N-1} \otimes \prod_{i\neq r}^{2N-2}(q_i,q_r)
$$

$$
\wedge ... \wedge \prod_{N+1\neq r}^{2N-2}(q_{N+1},q_r) - D\log(\prod_{i+2\neq r}^{2N-2}(q_{i+2},q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...,
$$

$$
q_{2N-2})]_{N-1} \otimes \prod_{i\neq r}^{2N-2}(q_i+2,q_r) \wedge ... \wedge \prod_{N+1\neq r}^{2N-2}(q_{2(N-1)-1},q_r)
$$

$+$

.

.

.

$$
(-1)^{2N-2} D\log(\prod_{2N-2\neq r}^{2N-2}(q_{2(N-1)},q_r)) \otimes [r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...q_{2N-2})]_{N-1}\otimes
$$

$$\prod_{i\neq r}^{2N-1}(q_i,q_r)\wedge ...\wedge \prod_{N\neq r}^{2N+1}(q_N,q_r)\Big] \quad (mod\ 2N-1)$$

$$h^N_{N-1}(q_0,...,q_{2N-1}) = (-1)^N\frac{1}{_{N-1(N)}C_2}\sum_{i=0}^{2N-1}(-1)^i Alt_{2N}[\![r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},... ,\hat{q}_N,...q_{2N-1}]\!]^D_N \quad (mod\ 2N)$$

Now, proving that each square of extended generalized diagram (E) is commutative.

Theorem 4.3. $h^N_{N-2}\circ p = \partial^D\circ h^N_{N-1}$

Proof. Let $(q_0,...,q_{2N-1})\in G_{2N}(N)$ and apply morphism p

$$p(q_0,...,q_{2N-1}) = \sum_{i=0}^{2N-1}(q_0,...,\hat{q}_i,...,q_{2N-1})$$

$$\begin{aligned}
h^N_{N-2}\circ p(q_0,...,q_{2N-1}) = (-1)^N\frac{1}{_{(N-2)N}C_2}\sum_{i=0}^{2N-1}(-1)^i Alt_{2N-2}\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...,\\
q_{2N-1})]\!]^D_{N-1}\otimes\prod_{2N-1\neq r}^{2N-1}(q_{2N-1},q_r)\wedge...\wedge\prod_{i\neq r}^{2N-1}(q_i,q_r) - D\log(\prod_{i\neq r}^{2N-1}(q_i,q_r))\otimes\\
[r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...q_{2N-1})]_{N-1}\otimes\prod_{j\neq r}^{2N-1}(q_{i+1},q_r)\wedge...\wedge\prod_{N+4\neq r}^{2N-1}(q_{2N-1},q_r)\\
+D\log(\prod_{i+1\neq r}^{2N-1}(q_{i+1},q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...q_{2N-1})]_{N-1}\otimes\\
\prod_{i\neq r}^{2N-1}(q_i,q_r)\wedge...\wedge\prod_{N+1\neq r}^{2N-1}(q_{N+1},q_r) - D\log(\prod_{i+2\neq r}^{2N-1}(q_{i+2},q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},\\
...,\hat{q}_N,...,q_{2N-1})]_{N-1}\otimes\prod_{i\neq r}^{2N-1}(q_{i+2},q_r)\wedge...\wedge\prod_{N+1\neq r}^{2N-1}(q_{2N-1},q_r)\\
+\\
.\\
.\\
.\\
(-1)^{2N-1}D\log(\prod_{2N-1\neq r}^{2N-1}(q_{2N-1},q_r))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_{i+1},\hat{q}_{i+2},...,\hat{q}_N,...q_{2N-1})]_{N-1}\otimes\\
\prod_{i\neq r}^{2N-1}(q_i,q_r)\wedge...\wedge\prod_{2N-2\neq r}^{2N-1}(q_{2N-2},q_r)\Big]
\end{aligned} \qquad (4.\ 9)$$

Let us consider $(q_0, ..., q_{2N-1}) \in G_{2N}(N)$ again and apply morphism h^N_{N-1}

$$h^N_{N-1}(q_0, ..., q_{2N-1}) = (-1)^N \frac{1}{{}_{(N-1)^N}C_2} \sum_{i=0}^{2N-1} (-1)^i Alt_{2N} [\![r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N, ...q_{2N-1}]\!]^D_N$$

applying map ∂^D and all properties

$$\partial^D \circ h^N_{N-1}(q_0, ..., q_{2N-1}) = (-1)^N \frac{1}{{}_{(N-2)^N}C_2} \sum_{i=0}^{2N-1} (-1)^i Alt_{2N-2} \Big[[\![r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N,$$

$$..., q_{2N-1})]\!]^D_{N-1} \otimes \prod_{2N-1 \neq r}^{2N-1} (q_{2N-1}, q_r) \wedge ... \wedge \prod_{i \neq r}^{2N-1} (q_i, q_r) - D \log(\prod_{i \neq r}^{2N-1} (q_i, q_r)) \otimes$$

$$[r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N, ...q_{2N-1})]_{N-1} \otimes \prod_{j \neq r}^{2N-1} (q_{i+1}, q_r) \wedge ... \wedge$$

$$\prod_{N+4 \neq r}^{2N-1} (q_{2N-1}, q_r) + D \log(\prod_{i+1 \neq r}^{2N-1} (q_{i+1}, q_r)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N, ...,$$

$$q_{2N-1})]_{N-1} \otimes \prod_{i \neq r}^{2N-1} (q_i, q_r) \wedge ... \wedge \prod_{N+1 \neq r}^{2N-1} (q_{N+1}, q_r) - D \log(\prod_{i+2 \neq r}^{2N-1} (q_{i+2}, q_r)) \otimes$$

$$[r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N, ..., q_{2N-1})]_{N-1} \otimes \prod_{i \neq r}^{2N-1} (q_{i+2}, q_r) \wedge ... \wedge$$

$$\prod_{N+1 \neq r}^{2N-1} (q_{2N-1}, q_r) +$$

.

.

.

$$(-1)^{2N-1} D \log(\prod_{2N-1 \neq r}^{2N-1} (q_{2N-1}, q_r)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_{i+1}, \hat{q}_{i+2}, ..., \hat{q}_N, ...q_{2N-1})]_{N-1} \otimes$$

$$\prod_{i \neq r}^{2N-1} (q_i, q_r) \wedge ... \wedge \prod_{2N-2 \neq r}^{2N-1} (q_{2N-2}, q_r) \Big] \quad (4.\ 10)$$

by using Eq. (4. 9) and (4. 10), conclude that $h^N_{N-2} \circ p = \partial^D \circ h^N_{N-1}$. It is proved that generalized diagram (E) for any weight N is commutative.

5. Conclusion

In this research work, two generalized chain complexes, Grassmannian and variant of Cathelineau infinitesimal, are connected through generalized extension of homomorphism. While other researchers have only generalized a few morphisms in the past, this research generalizes all morphisms to produce a generalized diagram between these two chain complexes, leading to the connection of configuration and Infinitesimal polylog groups chain complexes. This work will prove to be very useful for future researchers in the fields of algebraic K-theory, manifold theory, algebraic geometry, differential geometry, homological algebra, polylogarithmic group theory and chain complexes.

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