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On the Critical Group of Certain Subdivided Wheel Graphs

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Abstract. In this paper, we study the family of graphs $\widehat{W}_{2n}$ for $n \geq 2$, defined by removing the alternate spokes of a wheel graph with $2n$ rim vertices. We then determine the abstract structure of the critical group of the graph $\widehat{W}_{2n}$ and show that the critical group of this whole family of graphs $\widehat{W}_{2n}$ is the product of two cyclic groups.

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Key Words: Laplacian matrix, critical group, Smith normal form, tree number.

1. Introduction and main results

2. Introduction

Let G be graph of order n and multiple edges may be allowed. Let $A(G)$ be the adjacency matrix of G and $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ the degree matrix. Then the Laplacian is given as $L(G) = D(G) - A(G)$. The critical group of G is closely connected with $L(G)$ as follows: thinking of $L(G)$ as a linear map $\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, its *cokernel* has the form $\text{coker}L(G) = \frac{\mathbb{Z}^n}{L(G)\mathbb{Z}^n} \cong \mathbb{Z} \oplus S(G)$, where $S(G)$ is called the *critical group* of G. [2], [9].

Let v_r be a vertex (called root) of a graph G and consider $\Delta_i = d_i x_i - \sum a_{ij} x_j$ and where a_{ij} is the number of edges between vertices v_i and v_j, and $x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ whose unique non-zero entry is in the position $i \ \forall \ i = 1, 2, \ldots, n$. Then $S(G) = \frac{\mathbb{Z}^n}{\text{span}(\Delta_1, \ldots, \Delta_{r-1}, x_r, \Delta_{r+1}, \ldots, \Delta_n)}$. The critical group $S(G)$ is independent of the choice of v_r; for more details see [5].

The explicit structure of $S(G)$ for a given family of graphs is not always easy. In the last ten years, a series of papers has been written in this regards: wheel graphs [2], the Möbius ladder graphs [4], the Cayley graph $\mathcal{D}_n$ of dihedral group [6], the squared cycle C_n^2 [7], the graph $K_3 \times C_n$ [8], complete graphs [9], the graphs $K_m \times P_n$ [10], the graphs $3 \times n$ twisted bracelets [12] and for the cone of the hypercube [1].

It is interesting to note that almost all such families of graphs studied are regular. There is very little known about the critical group of families of irregular graphs. In this paper, we consider one such family of irregular graphs and calculate its critical group.

Consider the following sequence a_i of integers with initial conditions $a_1 = 0, a_2 = 1, a_3 = 2, a_4 = 5$,

$$a_i = 4a_{i-2} - a_{i-4}.$$

The aim of this paper is to compute the abstract structure of $S(\widehat{W}_{2n})$. We construct $\widehat{W}_{2n}$ by considering a cycle $C_{2n} : v_1, v_2, v_3, \ldots, v_{2n}, v_1$ and a new vertex v_0 adjacent to n vertices $v_1, v_3, v_5, v_7, \ldots, v_{2n-1}$ of C_{2n} . This graph has order $2n+1$ and size $3n$. Equivalently, it can be obtained from wheel W_{n+1} by adding a vertex on each edge of the cycle.

The main result of this paper is given as:

$$S(\widehat{W}_{2n}) = \begin{cases} \mathbb{Z}_{a_{n+1}} \oplus \mathbb{Z}_{2a_{n+1}}, & \text{if } n \text{ is odd;} \\ \mathbb{Z}_{a_{n+1}} \oplus \mathbb{Z}_{3a_{n+1}}, & \text{if } n \text{ is even.} \end{cases}$$

In section 1, we state the main result of this paper and give the reader a road map to its proof in sections 2 and 3. It suffices to analyze the Smith normal form(SNF) of a certain 2×2 matrix. We do this by looking at gcd's of certain auxiliary sequences of integers and prove certain facts about their modular residues.

3. System of relations for the cokernel of the Laplacian of $\widehat{W}_{2n}$

In this section, we shall first show that there are at most two generators for the critical group $S(\widehat{W}_{2n})$ of the graph $\widehat{W}_{2n}$ and reduce the relation matrix to the special matrix A_{2n}. Then, we shall give some properties of the entries of the matrix A_{2n}.

Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{2n}$, whose unique non-zero entry is in the position corresponding to vertex v_i. Let x_i be the image of e_i in $S(\widehat{W}_{2n})$. The vertex v_0 is chosen to be the root vertex, hence $x_0 = 0$ in $S(\widehat{W}_{2n})$. The relations of coker$\overline{L}(\widehat{W}_{2n})$ give rise to the following system of equations:

$$3x_{i-1} - x_i - x_{i-2} = 0; \quad i = \text{even} \tag{3. 1}$$

$$2x_{i-1} - x_i - x_{i-2} = 0; \quad i = \text{odd}. \tag{3. 2}$$

LEMMA 3.1. *There are two sequences (a_i) and (b_i) of integers such that*

$$x_i = a_i x_2 - b_i x_1, \quad 3 \leq i \leq 2n. \tag{3. 3}$$

Moreover, the sequences have the following recurrence relations with initial conditions $a_1 = 0, b_1 = -1, a_2 = 1, b_2 = 0, a_3 = 2, b_3 = 1, a_4 = 5, b_4 = 2$,

$$\begin{cases} a_i = 4a_{i-2} - a_{i-4}, \\ b_i = 4b_{i-2} - b_{i-4}, \\ b_i = a_{i-1} \quad \textit{i=odd}. \end{cases}$$

Proof. We know that there are $2n$ rim vertices of the graph $\widehat{W}_{2n}$. So we have $2n$ equations in $2n$ variables. We keep two equations $x_{2n} = 3x_1 - x_2$, and $x_1 = 2x_{2n} - x_{2n-1}$, corresponding to the vertices v_1 and v_{2n}. Each of the remaining $2n-2$ equations, represent x_i variable in terms of x_1 and x_2 as follows:

$x_3 = 2x_2 - x_1$ and $x_4 = 3x_3 - x_2$ are the equations corresponding to the vertices v_2 and v_3. One can easily express x_4 in terms of x_1 and x_2 by substituting x_3 in the equation $x_4 = 3x_3 - x_2$ as $x_4 = 5x_2 - 3x_1$. In a similar manner one can write $x_5 = 8x_2 - 5x_1$, $x_6 = 19x_2 - 12x_1$, $x_7 = 30x_2 - 19x_1$, $x_8 = 71x_2 - 45x_1$, $x_9 = 112x_2 - 71x_1$, $x_{10} = 265x_2 - 168x_1$, and so on. It is easy to see that each x_i can be written in terms of x_1 and x_2 such that

$$x_i = a_i x_2 - b_i x_1, \quad 3 \leq i \leq 2n,$$

where the two sequences a_i and b_i have the following recurrence relations;

$$\begin{aligned} a_i &= 3a_{i-1} - a_{i-2}; \quad i = \text{even} && (3.\ 4)\\ a_i &= 2a_{i-1} - a_{i-2}; \quad i = \text{odd} && (3.\ 5)\\ b_i &= 3b_{i-1} - b_{i-2}; \quad i = \text{even} && (3.\ 6)\\ a_{i-1} = b_i &= 2b_{i-1} - b_{i-2}; \quad i = \text{odd.} && (3.\ 7) \end{aligned}$$

From the above equations, it is easy to get the required recurrent relations between the two sequences a_i and b_i, which completes the proof. □.

We know from lemma 3.1 and the system of equations (2.1 & 2.2) that the group $S(\widehat{W}_{2n})$ has at most 2 generators. So, there are at least $2n-2$ diagonal entries of the Smith normal form of $\overline{L}(\widehat{W}_{2n})$ equal 1. However, the remaining invariant factors of $\overline{L}(\widehat{W}_{2n})$ hide inside the relation matrix induced by x_2 and x_1. From the structure of the graph $\widehat{W}_{2n}$, we have $x_{2n} = 3x_1 - x_2$, and $x_1 = 2x_{2n} - x_{2n-1}$, corresponding to the vertices v_1 and v_{2n}. Then by equation (3. 3), we have following system of two equations:

$a_{2n}x_2 - b_{2n}x_1 = x_{2n} = 3x_1 - x_2$, and $x_1 = 2x_{2n} - x_{2n-1} = 2[a_{2n}x_2 - b_{2n}] - [a_{2n-1}x_2 - b_{2n-1}x_1] = a_{2n+1}x_2 - b_{2n+1}x_1$. Thus, we have the required form of the matrix A_{2n}.

$$A_{2n} = \begin{pmatrix} a_{2n+1} & a_{2n}+1 \\ b_{2n+1}+1 & b_{2n}+3 \end{pmatrix}. \qquad (3.\ 8)$$

From the above argument, one can reduce $\overline{L}(\widehat{W}_{2n})$ up to equivalence to $I_{2n-2} \oplus (A_{2n})$ by performing some row and column operations. Now, we only need to evaluate the SNF of the matrix A_{2n}.

4. Analysis of the coefficients of the Smith normal form of A_{2n}

In this section, we shall try to find the SNF of A_{2n} by calculating its diagonal entries. Let us define the following sequences of positive integers with initial conditions, $K_0 = 1$ and $K_1 = 2$

$$\begin{aligned} K_m &= 4K_{m-1} - K_{m-2},\\ L_m &= K_m + K_{m-1}. \end{aligned}$$

Some useful properties of the above sequences are given in the following proposition:

PROPOSITION 4.1. *Let m be any positive integer. Then*

- $3 \nmid K_m$,
- $2 \nmid K_m$ *if* $m \equiv 0 (mod\ 2)$,
- $2 \mid K_m$ *if* $m \equiv 1 (mod\ 2)$,
- $2 \nmid K_m + K_{m-1}$,
- $3 \mid L_m$,
- $2 \nmid L_m$.

LEMMA 4.1.

- $a_{2m} = K_m - K_{m-1}$ *if* $m \equiv 0 (mod\ 2)$,
- $a_{2m+1} = \frac{1}{3}(K_{m+1} - K_{m-1})$ *if* $m \equiv 1 (mod\ 2)$.

Proof. It is easy to prove by induction.

PROPOSITION 4.2. *The sequences K_m and L_m are relatively prime for each m.*

Proof. To the contrary, suppose that there exists a prime p such that $p \mid K_m$ and $p \mid L_m$, then $p \neq 2$. Since $K_m = 4K_{m-1} - K_{m-2}$ this implies that $p \mid K_{m-2}$, hence, we get $p \mid K_{m-1}$ and $p \mid K_{m-2}$. But we know that, $K_{m-1} = 4K_{m-2} - K_{m-3}$ this implies that $p \mid K_{m-3} \Rightarrow \cdots p \mid K_{m-j} \cdots p \mid K_1 = 2$ thus $p = 2$, a contradiction to our supposition. □

LEMMA 4.2. *If $n = 2m+1$, then the following relation between the entries of the matrix A_{2n} which is defined in equation (3. 8) holds,*

$$\begin{aligned} a_{2n+1} = a_{4m+3} &= \frac{2}{3}L_{m+1}a_{n+1}, \\ a_{2n}+1 = a_{4m+2}+1 &= 2K_m a_{n+1}, \\ b_{2n+1}+1 = b_{4m+3}+1 &= 2K_m a_{n+1}, \\ b_{2n}+3 = b_{4m+2}+3 &= L_m a_{n+1}. \end{aligned}$$

If $n = 2m$, then the following relations holds,

$$\begin{aligned} a_{2n+1} = a_{4m+1} &= 2K_m a_{n+1}, \\ a_{2n}+1 = a_{4m}+1 &= L_m a_{n+1}, \\ b_{2n+1}+1 = b_{4m+1}+1 &= L_m a_{n+1}, \\ b_{2n}+3 = b_{4m}+3 &= 3K_{m-1}a_{n+1}. \end{aligned}$$

Proof. We prove the first equation in each case; the remaining equations can be proved in a similar fashion. The proof is done by simultaneously taking induction on n. The equations are true for $n = 2, 3$.

The inductive hypothesis states that the equations are true for all integers less than or equal to n, i.e. $a_{4m+3} = \frac{2}{3}L_{m+1}a_{n+1}$ and $a_{4m+1} = 2K_m a_{n+1}$. We have to show that $a_{4m+7} = \frac{2}{3}L_{m+2}a_{2(m+2)}$ and $a_{4m+5} = 2K_{m+1}a_{2m+3}$. Since $a_{4m+7} = 15a_{4m+3} - 4a_{4m+1}$, by the inductive step we have, $a_{4m+7} = 10L_{m+1}a_{2m+2} - 8K_m a_{2m+1}$. Using equation (2.5) we get, $a_{4m+7} = (10K_{m+1} - 6K_m)a_{2m+2} + 8K_m a_{2m+3}$. Using equation (2.4) and lemma 3.2 we have $a_{4m+7} = 10a_{2m+3}(3K_{m+1} - K_m) - a_{2m+4}(10K_{m+1} - 6K_m) = \frac{2}{3}L_{m+2}a_{2(m+2)}$. Now we shall prove that $a_{4m+5} = 2K_{m+1}a_{2m+3}$. Since $a_{4m+5} = 4a_{4m+3} - a_{4m+1}$, then by inductive hypothesis, we get the identity $a_{4m+5} = 4(\frac{2}{3}L_{m+1}a_{2m+2}) - 2K_m a_{2m+1}$, by equation (2.5) we get $a_{4m+5} = [\frac{8}{3}L_{m+1} - 4K_m]a_{2m+2} + 2K_m a_{2m+3}$ and applying lemma 3.2 leads to the desired identity. □

PROPOSITION 4.3.

$$\gcd(a_{2n+1},\ a_{2n}+1,\ b_{2n+1}+1,\ b_{2n}+3) = a_{n+1} \quad \forall\ n \geq 2. \tag{4. 9}$$

Proof. By lemma 4.2 and proposition 4.2, we have the desired result. □

PROPOSITION 4.4. *If n is odd, then*

$$\det A_{2n} = 2a_{n+1}^2,$$

where A_{2n} is defined in (3. 8).

Proof. Since $n = 2m + 1$, then by lemma 4.2, we have

$$\begin{aligned} \det A_{2n} &= a_{4m+3}(b_{4m+2} + 3) - (a_{4m+2} + 1)^2 \\ &= -\frac{2}{3}a_{n+1}^2\Big[(K_m - K_{m-1})^2 - 2K_mK_{m-1}\Big] \\ &= -\frac{2}{3}a_{n+1}^2\Big[(K_{m-1} - K_{m-2})^2 - 2K_{m-1}K_{m-2}\Big] \\ &\vdots \\ &= -\frac{2}{3}a_{n+1}^2\Big[(K_2 - K_1)^2 - 2K_2K_1\Big] \\ &= 2a_{n+1}^2. \end{aligned}$$

□

THEOREM 4.1. *If n is odd, then the critical group of $\widehat{W}_{2n}$ is the direct product of two cyclic groups. In particular*

$$S(\widehat{W}_{2n}) = \mathbb{Z}_{a_{n+1}} \oplus \mathbb{Z}_{2a_{n+1}}.$$

Proof. Since the matrix A_{2n} has Smith normal form as $\mathrm{diag}(s_{11}, s_{22})$ and s_{11} is equal to the gcd of all the entries of A_{2n}, by proposition 4.3, we have

$$s_{11} = a_{n+1}. \tag{4. 10}$$

Also $s_{11}s_{22}$ is equal to the determinant of the matrix A_{2n} and by proposition 4.4, we have

$$s_{11}s_{22} = 2a_{n+1}^2. \tag{4. 11}$$

Combining (4. 10) and (4. 11), we obtain

$$s_{22} = 2a_{n+1}, \tag{4. 12}$$

which competes the proof. □

PROPOSITION 4.5. *If n is even, then*

$$\det A_{2n} = 3a_{n+1}^2,$$

where A_{2n} is defined in (3. 8).

Proof. Since $n = 2m$, then by lemma 4.2, we have

$$\begin{aligned} \det A_{2n} &= a_{4m+1}(b_{4m} + 3) - (a_{4m} + 1)^2 \\ &= -a_{n+1}^2\Big[(K_m - K_{m-1})^2 - 2K_mK_{m-1}\Big] \\ &= -a_{n+1}^2\Big[(K_{m-1} - K_{m-2})^2 - 2K_{m-1}K_{m-2}\Big] \\ &\vdots \\ &= -a_{n+1}^2\Big[(K_2 - K_1)^2 - 2K_2K_1\Big] \\ &= 3a_{n+1}^2. \end{aligned}$$

□

THEOREM 4.2. *If n is even, then the critical group of $\widehat{W}_{2n}$ is the direct product of two cyclic groups. In particular*

$$S(\widehat{W}_{2n}) = \mathbb{Z}_{a_{n+1}} \oplus \mathbb{Z}_{3a_{n+1}}.$$

Proof. Since s_{11} is equal to the gcd of all the entries of A_{2n}, by proposition 4.2, we have

$$s_{11} = a_{n+1}. \tag{4. 13}$$

Also $s_{11}s_{22} = \det A_{2n}$ and by proposition 4.5, we have

$$s_{11}s_{22} = 3a_{n+1}^2. \tag{4. 14}$$

Combining (4. 13) and (4. 14), we obtain

$$s_{22} = 3a_{n+1}, \tag{4. 15}$$

which competes the proof. □

THEOREM 4.3. *If $n_1 \mid n_2$, then the critical group of $\widehat{W}_{2n_1}$ is isomorphic to a subgroup of the critical group of $\widehat{W}_{2n_2}$.*

Proof. It is sufficient to prove that every invariant factor of $S(\widehat{W}_{2n_1})$ is a divisor of the corresponding one of $S(\widehat{W}_{2n_2})$ and if $n_1 \mid n_2$ then $a_{n_1+1} \mid a_{n_2+1}$. There are four cases to be considered, depending on the parties of n_1 and n_2. Since the proofs are all very similar, we verify only the case in which both are odd. We know that

$$a_i = 4a_{i-2} - a_{i-4}$$

and its characteristic equation is

$$t^4 - 4t^2 + 1 = 0. \tag{4. 16}$$

$\Rightarrow \pm\frac{\sqrt{3}\pm 1}{\sqrt{2}}$ are the roots of this polynomial.
Put $p_k = a_{2k}$ and $q_k = a_{2k-1}$, then $t^2 - 4t + 1$ is the characteristic polynomial for $\{p_k\}$ and $\{q_k\}$, hence

$$\begin{aligned} p_k &= \frac{\sqrt{3}-1}{2}\Big(2+\sqrt{3}\Big)^k - \frac{\sqrt{3}+1}{2}\Big(2-\sqrt{3}\Big)^k, \\ q_k &= \frac{2\sqrt{3}-3}{3}\Big(2+\sqrt{3}\Big)^k - \frac{2\sqrt{3}+3}{3}\Big(2-\sqrt{3}\Big)^k. \end{aligned}$$

Suppose that n_1 and n_2 are odd integers such that $n_1 \mid n_2$, then $n_2 = sn_1$, so

$$\begin{aligned} a_{n_2+1} &= \frac{1}{\sqrt{2}}\Big[\Big(\frac{\sqrt{3}+1}{\sqrt{2}}\Big)^{n_2} - \Big(\frac{\sqrt{3}-1}{\sqrt{2}}\Big)^{n_2}\Big], \\ a_{n_1+1} &= \frac{1}{\sqrt{2}}\Big[\Big(\frac{\sqrt{3}+1}{\sqrt{2}}\Big)^{n_1} - \Big(\frac{\sqrt{3}-1}{\sqrt{2}}\Big)^{n_1}\Big], \end{aligned}$$

we get

$$\frac{a_{n_2+1}}{a_{n_1+1}} = \sum_{k=0}^{s-1}\Big(\frac{\sqrt{3}+1}{\sqrt{2}}\Big)^{n_1k}\Big(\frac{\sqrt{3}-1}{\sqrt{2}}\Big)^{n_1(s-k-1)}.$$

$\frac{\sqrt{3}\pm 1}{\sqrt{2}}$ are the roots of the polynomial defined in (4. 16), hence $\frac{a_{n_2+1}}{a_{n_1+1}}$ is an algebraic integer. But $\frac{a_{n_2+1}}{a_{n_1+1}} \in \mathbb{Q}$ and the only algebraic integers over the set of rationals $\mathbb{Q}$ are the set of integers, therefore $\frac{a_{n_2+1}}{a_{n_1+1}} \in \mathbb{Z}$, this implies that $a_{n_1+1} \mid a_{n_2+1}$. As noted above the proofs of the other cases are similar and so omit them. □

5. THE TREE NUMBER

Let G be a graph, then the tree number $k(G)$ is equal to the number of spanning trees of the graph G. In this section, we shall give the closed formula for the number of spanning trees for the graph $\widehat{W}_{2n}$, for details see [3].

A *regular graph* is a graph in which each vertex has the same number of neighbors; i.e., every vertex has the same degree. A regular graph with vertices of degree r is called an r-regular graph or a regular graph of degree r.

A *nearly regular graph* is a graph in which all the vertices except one have a fixed degree r, while the vertex not of this degree is called an *exceptional vertex*, for example the wheel graph is a nearly regular graph of degree 3.

The *characteristic polynomial* of a graph is the characteristic polynomial of its adjacency matrix.

PROPOSITION 5.1. *[3]*
Let G be a nearly regular graph of degree r and H be its subgraph obtained by removing the exceptional vertex, then

$$k(G) = P_H(r),$$

where $P_H(t)$ is the characteristic polynomial of the graph H.

REMARK 1. *Since the wheel graph W_n is a nearly regular graph of degree 3, so by proposition 5.1, we get*

$$k(W_n) = P_{C_n}(3).$$

The characteristic polynomial of a cycle C_n is given as

$$P_{C_n}(t) = 2T_n\left(\frac{t}{2}\right) - 2, \tag{5. 17}$$

where

$$T_n(t) = \frac{n}{2}\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{n-m}\begin{pmatrix} n-m \\ m \end{pmatrix}(2t)^{n-2m}$$

is the Chebyshev polynomial of the first kind. It is easy to see that, it gives the same number of spanning trees of wheel graph given by N.Biggs in [2].

A very interesting application of the proposition 5.1 is given as; the inner dual planar graph G^{**} is the subgraph of the usual dual G^* obtained by deleting the vertex corresponding to the infinite region of the original planar graph.

Let G be a planar graph in which any finite region is bounded by a cycle of fixed length r. Then G^* is a nearly regular graph, so we have the following result.

PROPOSITION 5.2. *[3]*
Let G be a planar graph in which any bounded region is a cycle of length r, then

$$k(G) = P_G^{**}(r),$$

where $P_G^{**}(t)$ is the characteristic polynomial of the graph G^{**}.

THEOREM 5.1. *The tree number for the graph $\widehat{W}_{2n}$ is*

$$k(\widehat{W}_{2n}) = P_{C_n}(4) = 2T_n(2) - 2, \tag{5. 18}$$

where $T_n(t)$ is the Chebyshev polynomial of the first kind.

Proof. The graph $\widehat{W}_{2n}$ is a planar graph in which bounded regions are bounded by a cycle of length 4 and the total number of bounded regions are n. Hence, the inner dual is a cycle of length n and its characteristic polynomial is defined in (5. 17). The result follows accordingly. □

6. ACKNOWLEDGMENTS

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