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Grassmannian Complex and Second Order Tangent Complex

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Abstract. First order tangent complex is studied by Siddiqui in [11]. There he has introduced morphisms for the first order tangent complex and connected this complex to the famous Grassmannian complex. In this paper we will extend the discussion of tangent complex for the second order. To do this we will introduce second order tangent group, denoted by $T\mathcal{B}_2^2(F)$, and form a tangent complex of order 2. Then we will write morphisms in order to connect this complex with the famous Grassmannian complex.

AMS (MOS) Subject Classification Codes: 18G; 14L; 16G
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1. Introduction

In [8] Siddiui uses configurations to relate the Grassmannian complex (see ⊂ 2.2)

$$d : C_m(X) \to C_{m-1}(X)$$

where

$$d : (x_1, \dots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_1, \dots, \hat{x}_i, \dots, x_m)$$

to both infinitesimal and tangential complexes.

In one hand he connected the Grassmannian complex to the variant of Cathelineau's complexes by introducing homomorphisms for both weight 2 and weight 3. He also proved the commutativity of corresponding diagrams. On the other hand he presented the cross-ratio of four points and the famous Siegel's cross-ratio identity in tangential settings and also proved that the Goncharov's projected five term relation can also be defined for tangent group $T\mathcal{B}_2(F)$. By using these constructions he defined maps to link the Grassmannian sub complex and Cathelineau's tangential complex for both weight 2 and 3. To connect Grassmannian sub complex

$$C_5(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{d} C_4(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{d} C_3(\mathbb{A}^2_{F[\varepsilon]_2})$$

to Cathelineau's tangent complex

$$T\mathcal{B}_2(F) \xrightarrow{\partial_\varepsilon} F \otimes F^\times \oplus \bigwedge^2 F$$

for $n = 2$ he gives the morphisms $\tau^2_{1,\varepsilon}$ and $\tau^2_{0,\varepsilon}$. For $n = 3$ he uses morphisms $\tau^3_{2,\varepsilon}$,$\tau^3_{1,\varepsilon}$ and $\tau^3_{0,\varepsilon}$ to relate

$$C_6(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{d} C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{d} C_4(\mathbb{A}^3_{F[\varepsilon]_2})$$

to the Cathelineau's tangent complex of weight 3 (see chapter 4 of [8]).

He studied these complexes for the first order tangent group but in the appendix of [8] he discussed a little bit about the second order tangent group $T\mathcal{B}^2_2(F)$. There he gives the morphisms between Grassmannian complex and Cathelineau's complex by taking only pure terms occurred in the definition of maps and neglected the non pure terms. But he himself does not consider it as an authentic work because there is no logic behind ignoring the non pure terms that's why he put that work in the appendix of his dissertation. However that work gives us a guideline to move toward higher order.

In this work we have tried to study the polyloagrithmic complexes in tangential settings for second order . The second section is devoted for the essential prerequisites and basic definitions as usual. In third section we have discussed second order tangent groups. Where we have tried to define the second order tangent groups of weight 2 and 3 ,denoted by $T\mathcal{B}^2_2(F)$ and $T\mathcal{B}^2_3(F)$ respectively. We also have mentioned the relations exist in these groups . Then we moved forward to describe the geometry of configurations of $T\mathcal{B}^2_2(F)$, which includes the construction of Grassmannian complex for tangential settings in higher order, building of the cross ratio and triple- ratio in $T\mathcal{B}^2_2(F)$. Then we move to write morphisms τ^2_{0,ε^2} and τ^2_{1,ε^2} for $n = 2$ and τ^3_{0,ε^2}, τ^3_{1,ε^2} and τ^3_{2,ε^2} for $n = 3$ in order to connect the Grassmannian complex to the the Cathelineau's tangent complex. At last we write the proofs of the commutativity of resulting diagrams (see theorems (4.2),(7.1)and (7.2)).

2. Notations and Preliminaries

2.1. **Configurations.** Let $C_m(X)$ be a free abelian group generated by elements $(x_1, \dots, x_m) \in X^m$ .For any group G acting on a non empty set X the elements of the set G/X^m, consisting of the m-tuples , is called configurations of X where G is acting diagonally on X^m (see[5]) .

A configuration $(x_i|x_1, \dots, \hat{x}_i, \dots, x_m)$ of vectors in $V_{n+1}/\langle x_i\rangle$ (V_{n+1} be a vector space of dimension $n + 1$)is said to be a projective configuration and defined as a quotient space of dimension n formed by the projection of $x_j \in V_n$; $j \neq i$, projected from $C_{m+1}(n + 1)$ to $C_m(n)$

2.2. **Grassmannian Complex.** This complex is introduced by a famous German mathematician "Hermann Grassmann." The importance of this complex is because of helping in the investigation of homology of general linear groups. let X be any non empty set. Consider $C_m(X)$ be a free abelian group whose generators are configurations of m points, then we define a differential map $d : C_m(X) \to C_{m-1}(X)$ as

$$d : (x_1, \ldots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_1, \ldots, \hat{x}_i, \ldots, x_m) \tag{2. 1}$$

Also if we denote $C_m(n)$ to be a group which is free abelian and whose generators are the configurations of the elements of an n-dimensional vector space V_n and $(x_i|x_1, \ldots, \hat{x}_i, \ldots, x_m)$ be the projective configuration of the vectors x_j along the vectors x_i, where $i \neq j$; $j = 1, ..., m$, then we define a projective map $d' : C_{(m+1)}(n+1) \to C_m(n)$ as

$$d' : (x_1, \ldots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_i|x_1, \ldots, \hat{x}_i, \ldots, x_m) \tag{2. 2}$$

by using these differential maps and free abelian groups generated by configurations we have a bi- complex called Grassmannian bi-Complex (see section 2 of [8]) .
From this bi-complex we can form many sub-complexes like

$$C_{m+2}(n+2) \xrightarrow{d'} C_{m+1}(n+1) \xrightarrow{d'} C_m(n)$$

or

$$C_{m+2}(n) \xrightarrow{d} C_{m+1}(n) \xrightarrow{d} C_m(n)$$

These sub-complexes known as Grassmannian Sub-Complexes

2.3. **Tangential Configuration Space.** For any field F with zero characteristic, we define the ring of νth truncated polynomial by

$$F[\varepsilon]_\nu := F[\varepsilon]/\varepsilon^\nu$$

where $\nu \geq 1$. First we will define $\mathbb{A}^n_{F[\varepsilon]_\nu}$ as an affine space over the the ring of truncated polynomial $F[\varepsilon]_\nu$. We choose $l = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{A}^n_F \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$, $l_\varepsilon := \begin{pmatrix} a_{1,\varepsilon} \\ a_{2,\varepsilon} \\ \vdots \\ a_{n,\varepsilon} \end{pmatrix} \in \mathbb{A}^n_F$ and upto $l_{\varepsilon^n} := \begin{pmatrix} a_{1,\varepsilon^{\nu-1}} \\ a_{2,\varepsilon^{\nu-1}} \\ \vdots \\ a_{n,\varepsilon^{\nu-1}} \end{pmatrix} \in \mathbb{A}^n_F$, then one can write (see [**?**])

$$l^* = l + l_\varepsilon \varepsilon + \cdots l_{\varepsilon^{\nu-1}} \varepsilon^{\nu-1} = \begin{pmatrix} a_1 + a_{1,\varepsilon}\varepsilon + \cdots + a_{1,\varepsilon^{\nu-1}}\varepsilon^{\nu-1} \\ a_2 + a_{2,\varepsilon}\varepsilon + \cdots + a_{2,\varepsilon^{\nu-1}}\varepsilon^{\nu-1} \\ \vdots \\ a_n + a_{n,\varepsilon}\varepsilon + \cdots + a_{n,\varepsilon^{\nu-1}}\varepsilon^{\nu-1} \end{pmatrix} \in \mathbb{A}^n_{F[\varepsilon]_\nu}$$

Consider the free abelian group $C_m(\mathbb{A}^n_{F[\varepsilon]_\nu})$ whose generators are the configurations $(l_1^*, \ldots, l_m^*)$ of m-vectors in $\mathbb{A}^n_{F[\varepsilon]_\nu}$. For these configurations, we can define the differential map d as

$$d : C_{m+1}(\mathbb{A}^n_{F[\varepsilon]_\nu}) \to C_m(\mathbb{A}^n_{F[\varepsilon]_\nu})$$

where d is defined in (2.1). another differential map d' with projection can be defined as

$$d' : C_{m+1}(\mathbb{A}^n_{F[\varepsilon]_\nu}) \to C_m(\mathbb{A}^{n-1}_{F[\varepsilon]_\nu})$$

where d' is defined in (2.2)
Let $\omega \in detV_n^*$ be the volume element in a n- dimensional vector space V_n (see section 3 of [5]) , we define a $n\times n$ determinant $\Delta(x_1^*,\ldots,x_n^*) = \langle \omega, x_1\wedge x_2 \wedge \ldots \wedge x_n\rangle$; $\quad x_i \in V_n$

For simplicity, we consider the the following cases

2.3.1. *Case $n = 2$ and $\nu = 3$.* In this situation, we can write

$$\Delta(x_p^*, x_q^*) = \Delta(x_p^*, x_q^*)_{\varepsilon^0} + \Delta(x_p^*, x_q^*)_{\varepsilon^1}\varepsilon + \Delta(x_p^*, x_q^*)_{\varepsilon^2}\varepsilon^2$$

where $\Delta(x_p^*, x_q^*)_{\varepsilon^0} = \Delta(x_p, x_q)$, $\quad \Delta(x_p^*, x_q^*)_{\varepsilon^1}\varepsilon = \Delta(x_p, x_{q,\varepsilon}) + \Delta(x_{p,\varepsilon}, x_q)$ and $\Delta(x_p^*, x_q^*)_{\varepsilon^2} = \Delta(x_p, x_{q,\varepsilon^2}) + \Delta(x_{p,\varepsilon}, x_{q,\varepsilon}) + \Delta(x_{p,\varepsilon^2}, x_q)$

2.3.2. *Case $n = 3$ and $\nu = 3$.*

$$\Delta(x_p^*, x_q^*, x_r^*) = \Delta(x_p^*, x_q^*, x_r^*)_{\varepsilon^0} + \Delta(x_p^*, x_q^*, x_r^*)_{\varepsilon^1}\varepsilon + \Delta(x_p^*, x_q^*, x_r^*)_{\varepsilon^2}\varepsilon^2$$

where $\Delta(x_p^*, x_q^*, x_r^*)_{\varepsilon^0} = \Delta(x_p, x_q, x_r)$, $\quad \Delta(x_p^*, x_q^*, x_r^*)_{\varepsilon^1}\varepsilon = \Delta(x_p, x_q, x_{r,\varepsilon}) + \Delta(x_p, x_{q,\varepsilon}, x_r) + \Delta(x_{p,\varepsilon}, x_q, x_r)$ and

$$\begin{aligned}\Delta(x_p^*, x_q^*, x_r^*)_{\varepsilon^2} =& \Delta(x_p, x_q, x_r) + \Delta(x_p, x_q, x_{r,\varepsilon}) + \Delta(x_p, x_{q,\varepsilon}, x_r) + \Delta(x_{p,\varepsilon}, x_q, x_r)\\ &+\Delta(x_p, x_q, x_{r,\varepsilon^2}) + \Delta(x_p, x_{q,\varepsilon}, x_{r,\varepsilon}) + \Delta(x_p, x_{q,\varepsilon^2}, x_r) + \Delta(x_{p,\varepsilon}, x_q, x_{r,\varepsilon})\\ &+\Delta(x_{p,\varepsilon}, x_{q,\varepsilon}, x_r) + \Delta(x_{p,\varepsilon^2}, x_q, x_r)\end{aligned}$$

2.4. **Cross-ratio in $F[\varepsilon]_2$.** Let $(x_0^*, x_1^*, x_2^*, x_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_\nu})$ be the configuration of four points, then their cross ratio is

$$\mathbf{r}(x_0^*, x_1^*, x_2^*, x_3^*) = \frac{\Delta(x_0^*, x_3^*)\Delta(x_1^*, x_2^*)}{\Delta(x_0^*, x_2^*)\Delta(x_1^*, x_3^*)} \tag{2. 3}$$

But expansion of $\mathbf{r}(x_0^*, x_1^*, x_2^*, x_3^*)$ over the truncated polynomial ring $F[\varepsilon]_3$ is

$$\mathbf{r}(x_0^*, x_1^*, x_2^*, x_3^*) = \left(r_{\varepsilon^0} + r_\varepsilon\varepsilon + r_{\varepsilon^2}\varepsilon^2\right)(x_0^*, x_1^*, x_2^*, x_3^*) \tag{2. 4}$$

which gives us the following values(see [8] and [9]). Put $\nu = 1$, we get

$$\mathbf{r}(x_0^*, x_1^*, x_2^*, x_3^*) = r_{\varepsilon^0}(x_0^*, x_1^*, x_2^*, x_3^*) = r(x_0, x_1, x_2, x_3) = \frac{\Delta(x_0, x_3)\Delta(x_1, x_2)}{\Delta(x_0, x_2)\Delta(x_1, x_3)} \tag{2. 5}$$

$$r_\varepsilon(x_0^*, x_1^*, x_2^*, x_3^*) = \frac{\{\Delta(x_0^*, x_3^*)\Delta(x_1^*, x_2^*)\}_\varepsilon}{\Delta(x_0, x_2)\Delta(x_1, x_3)} - r(x_0, x_1, x_2, x_3)\frac{\{\Delta(x_0^*, x_2^*)\Delta(x_1^*, x_3^*)\}_\varepsilon}{\Delta(x_0, x_2)\Delta(x_1, x_3)} \tag{2. 6}$$

$$\begin{aligned} r_{\varepsilon^2}(x_0^*, x_1^*, x_2^*, x_3^*) =& \frac{\{(x_0^*, x_3^*)(x_1^*, x_2^*)\}_{\varepsilon^2}}{(x_0, x_2)(x_1, x_3)} - r_\varepsilon(x_0^*, x_1^*, x_2^*, x_3^*)\frac{\{(x_0^*, x_2^*)(x_1^*, x_3^*)\}_\varepsilon}{(x_0, x_2)(x_1, x_3)}\\ &- r(x_0, x_1, x_2, x_3)\frac{\{(x_0^*, x_2^*)(x_1^*, x_3^*)\}_{\varepsilon^2}}{(x_0, x_2)(x_1, x_3)}\end{aligned} \tag{2. 7}$$

where $(ab)_\varepsilon = a_\varepsilon b + ab_\varepsilon$ and $(ab)_{\varepsilon^2} = a_{\varepsilon^2}b + a_\varepsilon b_\varepsilon + ab_{\varepsilon^2}$

2.5. **First Order Tangent Group.** For any $a, a' \in F$ and $\langle a; a'] = [a+a'\varepsilon]-[a] \in \mathbb{Z}[F[\varepsilon]_2]$, we define the first order tangent Group, denoted by $T\mathcal{B}_2(F)$, is a $\mathbb{Z}$-module generated by the elements of the form $\langle a; a'] \in \mathbb{Z}[F[\varepsilon]_2]$ an quotient by the expression

$$\langle a; a'] - \langle b; b'] + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)' \right] - \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)' \right] + \left\langle \frac{a(1-b)}{b(1-a)}; \left(\frac{a(1-b)}{b(1-a)}\right)' \right], \quad a, b \neq 0, 1, a \neq b; \quad (a, a' \in F) \tag{2. 8}$$

where

$$\left(\frac{b}{a}\right)' = \frac{ab' - a'b}{a^2} \quad ; \quad \left(\frac{1-b}{1-a}\right)' = \frac{(1-b)a' - (1-a)b'}{(1-a)^2}$$
$$\left(\frac{a(1-b)}{b(1-a)}\right)' = \frac{b(1-b)a' - a(1-a)b'}{(b(1-a))^2}$$

3. Second Order Tangent Groups

First of all we will give definition of $T\mathcal{B}_2^2(F)$ and will try to construct the cross ratio, triple ratio and identities of the vectors in the configuration space $C_m(\mathbb{A}^n_{F[\varepsilon]_3})$ for $m = 4, \dots, 7$; $n = 2, 3$. We will also give the relations satisfied by $T\mathcal{B}_2^2(F)$ and will prove the five term relation.

The tangent group of order 2 $T\mathcal{B}_2^2(F)$ is a $\mathbb{Z}$-module whose generators are the elements of the form $\langle a; a', a''] \in \mathbb{Z}[F[\varepsilon]_3]$, where $\langle a; a', a''] = [a + a'\varepsilon + a''\varepsilon^2] - [a], (a, a', a'' \in F)$ and quotient by the expression (see appendix of [8])

$$\langle a; a', a''] - \langle b; b', b''] + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)', \left(\frac{b}{a}\right)'' \right] - \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)', \left(\frac{1-b}{1-a}\right)'' \right] + \left\langle \frac{a(1-b)}{b(1-a)}; \left(\frac{a(1-b)}{b(1-a)}\right)', \left(\frac{a(1-b)}{b(1-a)}\right)'' \right], \quad a, b \neq 0, 1, a \neq b \tag{3. 9}$$

where

$$\left(\frac{b}{a}\right)'' = \frac{a^2b'' - aba'' - aa'b' + b(a')^2}{a^3}; \quad \left(\frac{1-b}{1-a}\right)'' = \frac{A}{(1-a)^3};$$
$$\left(\frac{a(1-b)}{b(1-a)}\right)'' = \frac{B}{a^3(1-b)^3} \tag{3. 10}$$

where

$$A = (1-a)(1-b)a'' - (1-a)^2b'' - (1-a)a'b' + (1-b)(a')^2$$

and

$$B = (b')^2a^3 - bb''a^3 + 2bb''a^2 - 2(b')^2a^2 - bb''a + (b')^2a + bab'a' - ba'b' + b^3aa'' - b^3(a')^2 - b^3a'' - b^2aa'' + b^2(a')^2 + b^2a'' \tag{3. 11}$$

3.1. **Functional Equations of $T\mathcal{B}_2^2(F)$.** The tangent group $T\mathcal{B}_2^2(F)$ satisfies the following relations (see [7])

(1) Two term relation: $\langle a; b_1, b_2]_2^2 = -\langle 1-a; -b_1, -b_2]_2^2$
(2) Inversion relation: $\langle a; b_1, b_2]_2^2 = \left\langle \frac{1}{a}; -\frac{b_1}{a^2} - \frac{ab_2-(b_1)^2}{a^3} \right]_2^2$
(3) The five term relation of $T\mathcal{B}_2^2(F)$ is equation (3. 9) above.

Lemma 3.2. *For* $(x_0^*, x_1^*, x_2^*, x_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_3})$, *we have*

$$\{\Delta(x_0^*, x_1^*)\Delta(x_2^*, x_3^*)\}_{\varepsilon^2} = \{\Delta(x_0^*, x_2^*)\Delta(x_1^*, x_3^*)\}_{\varepsilon^2} - \{\Delta(x_0^*, x_3^*)\Delta(x_1^*, x_2^*)\}_{\varepsilon^2}$$

Proof. See Lemma (4.4) of [9] for proof. □

4. Dilogarathmic Complexes of Tangent Groups

Cathelineau formed a complex using tangents groups known as tangent complex and we have another well known complex known as Grassmannian complex. In his work Sidddiqui connected these two complexes by defining maps $\tau^2_{0,\varepsilon}$ and $\tau^2_{1,\varepsilon}$ for $n = 2; \nu = 2$. In this section we also connect the complexes mentioned above for the case $n = 2; \nu = 3$. Let us denote $C_m(\mathbb{A}^2_{F[\varepsilon]_3})$ is a free abelian group whose generators are the configuration $(x_0^*, \ldots, x_{m-1}^*) \in \mathbb{A}^2_{F[\varepsilon]_3}$ for any affine space $\mathbb{A}^2_{F[\varepsilon]_3}$ over $F[\varepsilon]_3$ then we have the following Grassmannian complex

$$\ldots \to C_5(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{d} C_4(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{d} C_3(\mathbb{A}^2_{F[\varepsilon]_3})$$

$$d : (x_0^*, \ldots, x_m^*) \mapsto \sum_{i=0}^{m} (-1)^i (x_0^*, \ldots, \hat{x}_i^*, \ldots, x_m^*)$$

Consider the following diagram

$$\begin{array}{ccccc} C_5(\mathbb{A}^2_{F[\varepsilon]_3}) & \xrightarrow{d} & C_4(\mathbb{A}^2_{F[\varepsilon]_3}) & \xrightarrow{d} & C_3(\mathbb{A}^2_{F[\varepsilon]_3}) \\ & & \downarrow{\scriptstyle \tau^2_{1,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^2_{0,\varepsilon^2}} \\ & & T\mathcal{B}_2^2(F) & \xrightarrow{\partial_{\varepsilon^2}} & F \otimes F^\times \oplus \bigwedge^2 F \end{array} \qquad \text{(D)}$$

where

$$\partial_{\epsilon^2} : \langle a; b_1, b_2] \mapsto \left(\frac{2b_2}{a} - \frac{b_1^2}{a^2}\right) \otimes (1-a) + \left(\frac{2b_2}{(1-a)} + \frac{b_1^2}{(1-a)^2}\right) \otimes a)$$
$$+ \left(\frac{2b_2}{a} - \frac{b_1^2}{a^2}\right) \wedge \left(\frac{2b_2}{(1-a)} + \frac{b_1^2}{(1-a)^2}\right); \quad a, b_1, b_2 \in F \tag{4. 12}$$

Now we come to define the vertical maps τ^2_{0,ϵ^2} and $\tau^2_{1,\epsilon^2}$of the diagram. The map τ^2_{0,ϵ^2} can be written as the sum of two maps

$$\tau^2_{0,\epsilon^2} = \tau^1 + \tau^2$$

where

$$\tau^1(x_0^*, x_1^*, x_2^*) = \sum_{i=0}^{2} (-1)^i \left(\left(2\frac{(x_i^*, x_{i+1}^*)_{\epsilon^2}}{(x_i, x_{i+1})} - \frac{(x_i^*, x_{i+1}^*)^2_{\epsilon}}{(x_i, x_{i+1})^2} \right) \otimes \frac{(x_i, x_{i+2})}{(x_{i+1}, x_{i+2})} \right), \qquad i \quad mod \quad 3 \tag{4. 13}$$

and

$$\tau^2(x_0^*, x_1^*, x_2^*)$$
$$= \sum_{i=0}^{2} (-1)^i \left[\left(2\frac{(x_i^*, x_{i+1}^*)_{\epsilon^2}}{(x_i, x_{i+1})} - \frac{(x_i^*, x_{i+1}^*)^2_{\epsilon}}{(x_i, x_{i+1})^2} \right) \wedge \left(2\frac{(x_i^*, x_{i+2}^*)_{\epsilon^2}}{(x_i, x_{i+2})} - \frac{(x_i^*, x_{i+2}^*)^2_{\epsilon}}{(x_i, x_{i+2})^2} \right) \right], \quad i \quad mod \quad 3 \tag{4. 14}$$

The map τ^1 carries the elements of $C_3(\mathbb{A}^2_{F[\varepsilon]_3})$ into $F \otimes F^\times$ and the image of the elements of $C_3(\mathbb{A}^2_{F[\varepsilon]_3})$ under the map τ^2 is in $\bigwedge^2 F$.

$$\tau^2_{1,\epsilon^2}(x_0^*, \dots, x_3^*) = \langle r(x_0, \dots, x_3); r_\epsilon(x_0^*, \dots, x_3^*), r_{\epsilon^2}(x_0^*, \dots, x_3^*)] \tag{4. 15}$$

where $r(x_0, \dots, x_3)$, $r_\epsilon(x_0^*, \dots, x_3^*)$ and $r_{\epsilon^2}(x_0^*, \dots, x_3^*)$ are the respective coefficients of the ε^0 , ε^1 and ε^2 in the cross ratio. First we have to show the maps τ^2_{0,ϵ^2} and τ^2_{1,ϵ^2} are both well defined. Since we have defined τ^2_{1,ϵ^2} as the cross ratio of four points so it is not necessary to check. Therefore we only check for τ^2_{0,ϵ^2}.

Lemma 4.1. τ^2_{0,ϵ^2} *does not depend upon the volume form ω by the vectors.*

Proof. According to the definition we can write τ^2_{0,ϵ^2} as

$$\sum_{i=0}^{2}(-1)^i\left[\left(2\frac{(x_i^*, x_{i+1}^*)_{\epsilon^2}}{(x_i, x_{i+1})} - \frac{(x_i^*, x_{i+1}^*)^2{}_\epsilon}{(x_i, x_{i+1})^2}\right)\otimes \frac{(x_i, x_{i+2})}{(x_{i+1}, x_{i+2})}\right]$$
$$+\sum_{i=0}^{2}(-1)^i\left[\left(2\frac{(x_i^*, x_{i+1}^*)_{\epsilon^2}}{(x_i, x_{i+1})} - \frac{(x_i^*, x_{i+1}^*)^2{}_\epsilon}{(x_i, x_{i+1})^2}\right)\wedge\left(2\frac{(x_i^*, x_{i+2}^*)_{\epsilon^2}}{(x_i, x_{i+2})} - \frac{(x_i^*, x_{i+2}^*)^2{}_\epsilon}{(x_i, x_{i+2})^2}\right)\right]; i \quad mod \quad 3$$

Due to the homogeneity between the factors of the terms of above expression it is not possible to get a different value by replacing the volume element ω by $\lambda\omega$. This shows that the map τ^2_{0,ϵ^2} is independent of the volume element. □

Theorem 4.2. *The diagram (D) commutes.i.e.*

$$\tau^2_{0,\epsilon^2} \circ d = \partial^2_{\epsilon^2} \circ \tau^2_{1,\epsilon^2}$$

Proof. We have already defined the map $\tau^2_{1,\epsilon^2}(x_0^*, \dots, x_3^*)$ as

$$\tau^2_{1,\epsilon^2}(x_0^*, \dots, x_3^*) = \langle r(x_0, \dots, x_3); r_\epsilon(x_0^*, \dots, x_3^*), r_{\epsilon^2}(x_0^*, \dots, x_3^*)]^2_2$$

For simplicity we put $a = r(x_0, \dots, x_3)$, $b_1 = r_\epsilon(x_0^*, \dots, x_3^*)$ and $b_2 = r_{\epsilon^2}(x_0^*, \dots, x_3^*)$ then

$$\tau^2_{1,\epsilon^2}(x_0^*, \dots, x_3^*) = \langle a; b_1, b_2]^2_2$$

Here we evaluate value of the expression $\frac{2b_2}{a} - \frac{b_1^2}{a^2}$. Since we have

$$b_1 = r_\varepsilon(l_0^*, \dots, l_3^*) = \frac{\{\Delta(x_0^*, x_3^*)\Delta(x_1^*, x_2^*)\}_\varepsilon}{\Delta(x_0, x_2)\Delta(x_1, x_3)} - r(x_0, \dots, x_3)\frac{\{\Delta(x_0^*, x_2^*)\Delta(x_1^*, x_3^*)\}_\varepsilon}{\Delta(x_0, x_2)\Delta(x_1, x_3)} \tag{4. 16}$$

Dividing by "a" and using a shorthand $\Delta(x_i^*, x_j^*)_\varepsilon = (x_i x_j)_\varepsilon$, we have

$$\begin{aligned}\frac{2b_2}{a} - \frac{b_1^2}{a^2} =& 2\left(\frac{(x_0^*x_3^*)_{\varepsilon^2}}{(x_0x_3)} + \frac{(x_1^*x_2^*)_{\varepsilon^2}}{(x_1x_2)} - \frac{(x_0^*x_2^*)_{\varepsilon^2}}{(x_0x_2)} - \frac{(x_1^*x_3^*)_{\varepsilon^2}}{(x_1x_3)}\right)\\ &-\frac{(x_0^*x_3^*)^2_\varepsilon}{(x_0x_3)^2} - \frac{(x_1^*x_2^*)^2_\varepsilon}{(x_1x_2)^2} + \frac{(x_0^*x_2^*)^2_\varepsilon}{(x_0x_2)^2} + \frac{(x_1^*x_3^*)^2_\varepsilon}{(x_1x_3)^2}\end{aligned} \tag{4. 17}$$

Similarly we can evaluate the expression as

$$\begin{aligned}\frac{2b_2}{1-a} - \frac{b_1^2}{(1-a)^2} =& 2\left(\frac{(x_0^*x_2^*)_{\varepsilon^2}}{(x_0x_2)} + \frac{(x_1^*x_3^*)_{\varepsilon^2}}{(x_1x_3)} - \frac{(x_0^*x_1^*)_{\varepsilon^2}}{(x_0x_1)} - \frac{(x_1^*x_3^*)_{\varepsilon^2}}{(x_2x_3)}\right)\\ &-\frac{(x_0^*x_2^*)^2_\varepsilon}{(x_0x_2)^2} - \frac{(x_1^*x_3^*)^2_\varepsilon}{(x_1x_3)^2} + \frac{(x_0^*x_1^*)^2_\varepsilon}{(x_0x_1)^2} + \frac{(x_2^*x_3^*)^2_\varepsilon}{(x_2x_3)^2}\end{aligned} \tag{4. 18}$$

Now by taking composition of the map τ^2_{1,ϵ^2} with the map ∂_{ϵ^2}, we get a large expression. Therefore by using homomorphic property we can split this composition into two parts to explain them separately

$$\partial_{\epsilon^2}\circ\tau^2_{1,\epsilon^2}(x_0^*,\ldots,x_3^*)=\partial^1_{\epsilon^2}\left(\langle a;b_1,b_2]_2^2\right)+\partial^2_{\epsilon^2}\left(\langle a;b_1,b_2]_2^2\right) \tag{4. 19}$$

where $\partial^1_{\epsilon^2}$ carries the elements of $T\mathcal{B}_2^2(F)$ into $F\otimes F^\times$ and $\partial^2_{\epsilon^2}$ carries the elements of $T\mathcal{B}_2^2(F)$ into $\bigwedge^2 F$. Therefore we have

$$\begin{aligned}\partial^1_{\epsilon^2}\left(\langle a;b_1,b_2]_2^2\right)=&\left(\frac{2b_2}{a}-\frac{b_1^2}{a^2}\right)\otimes(1-a)+\left(\frac{2b_2}{(1-a)}+\frac{b_1^2}{(1-a)^2}\right)\otimes a\\
=&\left\{2\frac{(x_0^*x_3^*)_{\epsilon^2}}{(x_0x_3)}+2\frac{(x_1^*x_2^*)_{\epsilon^2}}{(x_1x_2)}-2\frac{(x_0^*x_2^*)_{\epsilon^2}}{(x_0x_2)}-2\frac{(x_1^*x_3^*)_{\epsilon^2}}{(x_1x_3)}+\frac{(x_0^*,x_2^*)^2{}_\epsilon}{(x_0,x_2)^2}\right.\\
&\left.+\frac{(x_1^*,x_3^*)^2{}_\epsilon}{(x_1,x_3)^2}-\frac{(x_0^*,x_3^*)^2{}_\epsilon}{(x_0,x_3)^2}-\frac{(x_1^*,x_2^*)^2{}_\epsilon}{(x_1,x_2)^2}\right\}\otimes\frac{(x_ox_1)(x_2x_3)}{(x_0x_2)(x_1x_3)}\\
&+\left\{2\frac{(x_o^*x_2^*)_{\epsilon^2}}{(x_0x_2)}+2\frac{(x_1^*x_3^*)_{\epsilon^2}}{(x_1x_3)}-2\frac{(x_0^*x_1^*)_{\epsilon^2}}{(x_0x_1)}-2\frac{(x_2^*x_3^*)_{\epsilon^2}}{(x_2x_3)}\right.\\
&\left.+\frac{(x_0^*,x_1^*)^2{}_\epsilon}{(x_0,x_1)^2}+\frac{(x_2^*,x_3^*)^2_\epsilon}{(x_2,x_3)^2}-\frac{(x_0^*,x_2^*)^2_\epsilon}{(x_0,x_2)^2}-\frac{(x_1^*,x_3^*)^2_\epsilon}{(x_1,x_3)^2}\right\}\otimes\frac{(x_ox_3)(x_1x_2)}{(x_0x_2)(x_1x_3)}\end{aligned} \tag{4. 20}$$

$$\begin{aligned}\partial^2_{\epsilon^2}\left(\langle a;b_1,b_2]_2^2\right)=&\left(\frac{2b_2}{a}-\frac{b_1^2}{a^2}\right)\wedge\left(\frac{2b_2}{(1-a)}+\frac{b_1^2}{(1-a)^2}\right)\\
=&\left\{2\frac{(x_o^*x_3^*)_{\epsilon^2}}{(x_0x_3)}+2\frac{(x_1^*x_2^*)_{\epsilon^2}}{(x_1x_2)}-2\frac{(x_0^*x_2^*)_{\epsilon^2}}{(x_0x_2)}-2\frac{(x_1^*x_3^*)_{\epsilon^2}}{(x_1x_3)}+\frac{(x_0^*,x_2^*)^2{}_\epsilon}{(x_0,x_2)^2}\right.\\
&\left.+\frac{(x_1^*,x_3^*)^2{}_\epsilon}{(x_1,x_3)^2}-\frac{(x_0^*,x_3^*)^2{}_\epsilon}{(x_0,x_3)^2}-\frac{(x_1^*,x_2^*)^2{}_\epsilon}{(x_1,x_2)^2}\right\}\wedge\left\{2\frac{(x_o^*x_2^*)_{\epsilon^2}}{(x_0x_2)}+2\frac{(x_1^*x_3^*)_{\epsilon^2}}{(x_1x_3)}\right.\\
&\left.-2\frac{(x_0^*x_1^*)_{\epsilon^2}}{(x_0x_1)}-2\frac{(x_2^*x_3^*)_{\epsilon^2}}{(x_2x_3)}+\frac{(x_0^*,x_1^*)^2{}_\epsilon}{(x_0,x_1)^2}+\frac{(x_2^*,x_3^*)^2_\epsilon}{(x_2,x_3)^2}-\frac{(x_0^*,x_2^*)^2_\epsilon}{(x_0,x_2)^2}-\frac{(x_1^*,x_3^*)^2_\epsilon}{(x_1,x_3)^2}\right\}\end{aligned} \tag{4. 21}$$

Using (4. 13) and (4. 14), we split the second map $\tau^2_{0,\epsilon^2}\circ d(x_0^*,\ldots,x_3^*)$ into two parts. First part can be written as

$$\begin{aligned}&\tau^1\circ d(x_0^*,\ldots,x_3^*)\\
&=\widetilde{\text{Alt}}_{(0123)}\left\{\sum_{i=0}^{2}(-1)^i\left(\left(2\frac{(x_i^*,x_{i+1}^*)_{\epsilon^2}}{(x_i,x_{i+1})}-\frac{(x_i^*,x_{i+1}^*)^2{}_\epsilon}{(x_i,x_{i+1})^2}\right)\otimes\frac{(x_i,x_{i+2})}{(x_{i+1},x_{i+2})}\right)\right\};i\quad mod\quad 3\end{aligned} \tag{4. 22}$$

where $\widetilde{\text{Alt}}_{(0123)}$ denotes the alternation sum.

The expansion of inner expression give us a total of six terms. From which three will be of the form $2\frac{(x_i^*,x_j^*)_{\varepsilon^2}}{(x_i,x_j)}\otimes\frac{x}{y}$ and other three will of the form $\frac{(x_i^*,x_j^*)^2_\varepsilon}{(x_i,x_j)}\otimes\frac{x}{y}$. Using the property $a\otimes\frac{b}{c}=a\otimes b-a\otimes c$, terms will become double and form of the terms will be $2\frac{(x_i^*,x_j^*)_{\varepsilon^2}}{(x_i,x_j)}\otimes x$ and $\frac{(x_i^*,x_j^*)^2_\varepsilon}{(x_i,x_j)}\otimes y$. Then we will expand the alternation sum which gives us a total of 48 terms. After simplification we will obtain an expression identical with (4. 20).

The second part of $\tau^2_{0,\epsilon^2}\circ d(x_0^*,\ldots,x_3^*)$ can be written as

$$\tau^2 \circ d(x_0^*, \ldots, x_3^*) = \widetilde{\text{Alt}}_{(0123)} \sum_{i=0}^{2} (-1)^i \Bigg\{ \left(2\frac{(x_i^*, x_{i+1}^*)_{\epsilon^2}}{(x_i, x_{i+1})} - \frac{{(x_i^*, x_{i+1}^*)^2}_{\epsilon}}{(x_i, x_{i+1})^2} \right) \wedge \left(2\frac{(x_i^*, x_{i+2}^*)_{\epsilon^2}}{(x_i, x_{i+2})} - \frac{{(x_i^*, x_{i+2}^*)^2}_{\epsilon}}{(x_i, x_{i+2})^2} \right) \Bigg\}; i \quad mod \quad 3 \qquad (4.\ 23)$$

The expansion of inner sum will give us a total of 12 terms . From which six will be of the form $2\frac{(x_i^*, x_j^*)_{\varepsilon^2}}{(x_i, x_j)} \wedge x$ and remaining six will be of the form $\frac{(x_i^*, x_j^*)^2_{\varepsilon}}{(x_i, x_j)} \wedge y$. Applying the alternation sum number of trms will increase up to 48. After cancellation of like terms with opposite signs we obtain an expression identical to (4. 21).

Since the sum of (4. 20) and (4. 21) represents the value of $\partial^2_{\epsilon^2} \circ \tau^2_{1,\epsilon^2}$ which is equal to the sum of the values of $\tau^1 \circ d(x_0^*, \ldots, x_3^*)$ and $\tau^2 \circ d(x_0^*, \ldots, x_3^*)$.

□

Corollary 4.3. *The following are complexes.*

(1) $C_4(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{d'} C_3(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{\tau^2_{0,\varepsilon^2}} F \otimes F^\times \oplus \bigwedge^2 F$

(2) $C_5(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{d'} C_4(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{\tau^2_{1,\varepsilon^2}} T\mathcal{B}_2(F)$

Proof. To prove above result we have to show $\tau^2_{0,\varepsilon^2} \circ d' = 0$ and $\tau^2_{1,\varepsilon^2} \circ d' = 0$ which requires direct calculations. □

5. Trilogarathmic Complexes of Tangent Groups

This section is devoted to discuss about the suitable maps which connects the Grassmannian sub-complex and Cathelineau's tangential complex of second order for the case weight 3. We also show the commutativity of the resulting diagram.

5.1. **Definition of $T\mathcal{B}_3^2(F)$.** The second order tangent group for weight 3 is denoted by $T\mathcal{B}_3^2(F)$ is a $\mathbb{Z}$-module over the truncated polynomial ring $F[\varepsilon]_3$ whose generators are the elements of the form $\langle a; a', a''] \in \mathbb{Z}[F[\varepsilon]_3]$, where $\langle a; b_1, b_2] = [a + b_1\varepsilon + b_2\varepsilon^2] - [a], (a, b_1, b_2 \in F)$ and quotient by the $ker\partial$ where

$$\partial : \mathbb{Z}[F[\varepsilon]_3] \longrightarrow \left(T\mathcal{B}_2^2(F) \otimes F^\times\right) \oplus (F \otimes \mathcal{B}_2(F))$$

where the map ∂ is defined as

$$\begin{aligned} &\partial\left(\langle a; b_1, b_2]_2 \otimes c + u \otimes [v]_2\right) \\ &= \left(\frac{2b_2}{a} - \frac{b_1^2}{a^2}\right) \otimes (1-a) \wedge c - \left(\frac{2b_2}{(1-a)} + \frac{b_1^2}{(1-a)^2}\right) \otimes a \wedge c \\ &+ u \otimes (1-v) \wedge v + \left(\frac{2b_2}{a} - \frac{b_1^2}{a^2}\right) \wedge \left(\frac{2b_2}{(1-a)} + \frac{b_1^2}{(1-a)^2}\right) \end{aligned} \qquad (5.\ 24)$$

5.2. **Projected Five Term relations in $T\mathcal{B}_2^2(F)$.** We already have projected five term relations in $\mathcal{B}_2(F)$ (see [5]) and $T\mathcal{B}_2(F)$ (see [9]). Here we will prove the existence of projected five term relation in $T\mathcal{B}_2^2(F)$.

Lemma 5.3. *Let $l_0^*, \dots, l_4^* \in \mathbb{P}^2_{F[\varepsilon]_3}$ be 5 points in generic position, then*

$$\sum_{i=0}^{4}(-1)^i \left\langle r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4); r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*), r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)\right]_2^2 = 0 \tag{5. 25}$$

in $T\mathcal{B}_2^2(F)$, where $l_i^* = l_i + l_i'\varepsilon + l_i''\varepsilon^2 \in \mathbb{P}^2_{F[\varepsilon]_3}$, $\quad l_i, l_i', l_i'' \in \mathbb{P}^2_F$ and

$$\begin{aligned}\boldsymbol{r}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*) =& r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4) + r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)\varepsilon\\ &+ r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)\varepsilon^2\end{aligned}$$

where the LHS denotes the projected cross-ratio of any four points from $l_0^*, \dots, l_4^* \in \mathbb{P}^2_{F[\varepsilon]_3}$ projected from the fifth one.

Proof. See [[7]] for proof. ◻

5.4. **Triple-ratio in $F[\varepsilon]_3$:** Let $C_6(\mathbb{A}^2_{F[\varepsilon]_3})$ be an abelian group ,preferably free abelian, whose generators are configurations of six points. Then for $(l_0^*, \dots, l_5^*) \in C_6(\mathbb{A}^3_{F[\varepsilon]_3})$ the triple-ratio for $\nu = 3$ can be define as(see article (4.1.2) [8])

$$\begin{aligned} r_{3,\varepsilon^2} = \text{Alt}_6\Bigg\{&\frac{\{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)\}_{\varepsilon^2}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} - r_{3,\varepsilon}(l_0^*, l_1^*, l_2^*, l_3^*, l_4^*, l_5^*)\frac{\{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_{\varepsilon}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\\ &-r_3(l_0,\dots,l_5)\frac{\{(l_1^*l_2^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_{\varepsilon^2}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\Bigg\} \end{aligned} \tag{5. 26}$$

where

$$(abc)_{\varepsilon^2} = a_{\varepsilon^0}b_{\varepsilon^0}c_{\varepsilon^2} + a_{\varepsilon^0}b_\varepsilon c_\varepsilon + a_{\varepsilon^0}b_{\varepsilon^2}c_{\varepsilon^0} + a_\varepsilon b_{\varepsilon^0}c_\varepsilon + a_\varepsilon b_\varepsilon c_{\varepsilon^2} + a_{\varepsilon^2}b_{\varepsilon^0}c_{\varepsilon^0} \tag{5. 27}$$

Using above constructions we are now able to write maps to relate Grassmannian complex with the famous Cathelineau's tangent complex for weight 3 and $\nu = 3$.
After connecting we obtain the following diagram

$$\begin{array}{ccccc} C_6(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & C_5(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & C_4(\mathbb{A}^3_{F[\varepsilon]_3}) \\ \downarrow{\scriptstyle \tau^3_{2,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^3_{1,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^3_{0,\varepsilon^2}} \\ T\mathcal{B}_3^2(F) & \xrightarrow{\partial_{\varepsilon^2}} & \left(T\mathcal{B}_2^2(F)\otimes F^\times\right)\oplus(F\otimes\mathcal{B}_2(F)) & \xrightarrow{\partial_{\varepsilon^2}} & \left(F\otimes \bigwedge^2 F^\times\right)\oplus\left(\bigwedge^3 F\right) \end{array} \tag{A}$$

where

$$\begin{aligned}\tau^3_{0,\varepsilon^2}(l_0^*,\dots,l_3^*) = \sum_{i=0}^{3}(-1)^i&\left(2\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,l_3^*)_{\varepsilon^2}}{\Delta(l_0,\dots,\hat{l}_i,\dots,l_3)} - \frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,l_3^*)^2_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,l_3)}\right)\\ &\otimes\frac{\Delta(l_0,\dots,\hat{l}_{i+1},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\wedge\frac{\Delta(l_0,\dots,\hat{l}_{i+3},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\\ &+\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\left(2\frac{\Delta(l_0^*,\dots,\hat{l}_j^*,\dots,l_3^*)_{\varepsilon^2}}{\Delta(l_0,\dots,\hat{l}_j,\dots,l_3)} - \frac{\Delta(l_0^*,\dots,\hat{l}_j^*,\dots,l_3^*)^2_\varepsilon}{\Delta(l_0,\dots,\hat{l}_j,\dots,l_3)}\right),\quad i \mod 4,\end{aligned} \tag{5. 28}$$

$$\begin{aligned}
&\tau^3_{1,\varepsilon^2}(l_0^*,\dots,l_4^*)\\
&= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(\Big\langle r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4); r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l^*}_i,\dots,l_4^*), r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l^*}_i,\dots,l_4^*)\Big]_2^2\\
&\otimes\prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)+\sum_{\substack{k=0\\k\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_k^*,\dots,l_4^*)_{\varepsilon^2}}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_k,\dots,l_4)}\right)\otimes\Big[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big]_2\\
&-\sum_{\substack{m=0\\m\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_m^*,\dots,l_4^*)_\varepsilon^2}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_m,\dots,l_4)}\right)\otimes\Big[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big]_2\Bigg)
\end{aligned}\tag{5. 29}$$

and

$$\tau^3_{2,\varepsilon^2}(l_0^*,\dots,l_5^*)=\frac{2}{45}\mathrm{Alt}_6\Big\langle r_3(l_0,\dots,l_5); r_{3,\varepsilon}(l_0^*,l_1^*,l_2^*,l_3^*,l_4^*,l_5^*), r_{3,\varepsilon^2}(l_0^*,l_1^*,l_2^*,l_3^*,l_4^*,l_5^*)\Big]_3 \tag{5. 30}$$

where

$$r_3(l_0,\dots,l_5)=\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} \tag{5. 31}$$

and

$$\begin{aligned}
&r_{3,\varepsilon}(l_0^*,l_1^*,l_2^*,l_3^*,l_4^*,l_5^*)\\
&=\frac{\{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}-\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\frac{\{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}
\end{aligned}\tag{5. 32}$$

$$\begin{aligned}
r_{3,\varepsilon^2}(l_0^*,\dots,l_5^*)&=\frac{\{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)\}_{\varepsilon^2}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}-r_{3,\varepsilon}(l_0^*,\dots,l_5^*)\frac{\{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\\
&-\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\frac{\{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_{\varepsilon^2}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}
\end{aligned}\tag{5. 33}$$

we define the map ∂_{ε^2} as

$$\begin{aligned}
&\partial^1_{\varepsilon^2}\left(\langle a;b_1,b_2]_2\otimes c+x\otimes[y]_2\right)\\
&=\left(\frac{2b_2}{a}-\frac{b_1^2}{a^2}\right)\otimes(1-a)\wedge c-\left(\frac{2b_2}{(1-a)}-\frac{b_1^2}{(1-a)^2}\right)\otimes a\wedge c\\
&+x\otimes(1-y)\wedge y+\left(\frac{2b_2}{a}-\frac{b_1^2}{a^2}\right)\wedge\left(\frac{2b_2}{(1-a)}+\frac{b_1^2}{(1-a)^2}\right)\wedge x
\end{aligned}\tag{5. 34}$$

and for $a,b_1,b_2,c,c_1,c_2\in F^\times$

$$\partial_{\varepsilon^2}(\langle c;c_1,c_2]_3^2)=\langle a;b_1,b_2]_2^2\otimes a+\left(\frac{2c_2}{c}-\frac{c_1^2}{c^2}\right)\otimes[a]_2 \tag{5. 35}$$

Theorem 5.5. *The square right to the diagram (A) is commutes, i.e.*

$$\tau^3_{0,\varepsilon}\circ d=\partial_\varepsilon\circ\tau^3_{1,\varepsilon}$$

Proof. The map $\tau^3_{0,\varepsilon}$ is defined in (5.28) which is too lengthy for calculations therefore we write the map $\tau^3_{0,\varepsilon}$ as sum of two maps $\tau^{(1)}$ and $\tau^{(2)}$ then we evaluate $\tau^{(1)} \circ d$ and $\tau^{(2)} \circ d$ separately. First we find

$$\tau^{(1)} \circ d(l_0^*, \ldots, l_4^*) = \tau^3_{0,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)\right)$$

$$= \widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i \left\{2\frac{\Delta(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_3^*)_{\varepsilon^2}}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_3)} - \frac{\Delta(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_3^*)^2_{\varepsilon}}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_3)}\right\} \otimes \frac{\Delta(l_0, \ldots, \hat{l}_{i+1}, \ldots, l_3)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3)} \wedge \frac{\Delta(l_0, \ldots, \hat{l}_{i+3}, \ldots, l_3)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3)}\Big), \quad i \mod 4\right) \tag{5. 36}$$

where $\widetilde{\mathrm{Alt}}_{(01234)}$ represents the alternation sum. We start our calculation from the expansion of the inner sum. The inner sum gives us four summands and each summand can further be simplified by using the relations $p\otimes(q+r) = p\otimes q+p\otimes r$; $p\otimes(qr) = p\otimes q+p\otimes r$ and $p\otimes \frac{q}{r} = p\otimes q-p\otimes r$ in order to obtain 24 terms. And after applying the alternation sum we have a total of 120 terms containing 60 terms of the form $p^2_\varepsilon\otimes q\wedge r$ and remaining 60 of the form $2p_{\varepsilon^2} \otimes q \wedge r$. After that we combine the terms having common factor $\frac{\Delta(l_i^*,l_j^*,l_k^*)}{\Delta(l_i,l_j,l_k)} \otimes \cdots$. For example the terms having common factor $\frac{\Delta(l_4^*,l_2^*,l_3^*)^2_\varepsilon}{\Delta(l_4,l_2,l_3)} \otimes \cdots$ will be

$$\frac{\Delta(l_4^*, l_2^*, l_3^*)^2_\varepsilon}{\Delta(l_4, l_2, l_3)} \otimes \Big(\Delta(l_0, l_3, l_4) \wedge \Delta(l_0, l_2, l_3) - \Delta(l_0, l_3, l_4) \wedge \Delta(l_0, l_2, l_4) - \Delta(l_0, l_2, l_4) \wedge \Delta(l_0, l_2, l_3)$$
$$+\Delta(l_1, l_3, l_4) \wedge \Delta(l_1, l_2, l_4) - \Delta(l_1, l_3, l_4) \wedge \Delta(l_1, l_2, l_3) + \Delta(l_1, l_2, l_4) \wedge \Delta(l_1, l_2, l_3)\Big)$$

an the terms with common factor $\frac{\Delta(l_1^*,l_2^*,l_3^*)_{\varepsilon^2}}{\Delta(l_1,l_2,l_3)}\otimes$ will be

$$\frac{\Delta(l_1^*, l_2^*, l_3^*)_{\varepsilon^2}}{\Delta(l_1, l_2, l_3)} \otimes \Big(\Delta(l_0, l_2, l_3) \wedge \Delta(l_0, l_1, l_2) - \Delta(l_0, l_2, l_3) \wedge \Delta(l_0, l_1, l_3) - \Delta(l_0, l_1, l_3) \wedge \Delta(l_0, l_1, l_2)$$
$$+\Delta(l_2, l_3, l_4) \wedge \Delta(l_1, l_3, l_4) - \Delta(l_2, l_3, l_4) \wedge \Delta(l_1, l_2, l_4) + \Delta(l_1, l_3, l_4) \wedge \Delta(l_1, l_2, l_4)\Big)$$

and so on. This completes the calculation of first part $\tau^{(1)} \circ d$.
The second part $\tau^{(2)} \circ d$ can be written as

$$= \widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i \bigwedge_{\substack{j=0\\ j\neq i}}^{4}\left(2\frac{\Delta(l_0^*, \ldots, \hat{l}_j^*, \ldots, l_3^*)_{\varepsilon^2}}{\Delta(l_0, \ldots, \hat{l}_j, \ldots, l_3)} - \frac{\Delta(l_0^*, \ldots, \hat{l}_j^*, \ldots, l_3^*)^2_\varepsilon}{\Delta(l_0, \ldots, \hat{l}_j, \ldots, l_3)}\right), \quad i \mod 4\right) \tag{5. 37}$$

By expanding the inner sum and wedge product for $i = 0, \ldots, 3$; $j = 0, \ldots, 3$, we get eight terms of the form $a \wedge b \wedge c$ then pass each sum through $\widetilde{\mathrm{Alt}}_{(01234)}$ the terms will increase up to 40. This is the final value of $\tau^{(2)} \circ d(l_0^*, \ldots, l_4^*)$.

Now we come to evaluate $\partial_{\varepsilon^2} \circ \tau^3_{1,\varepsilon^2}$.

First we split this map into two parts $\partial^1_{\varepsilon^2}$ and $\partial^2_{\varepsilon^2}$ then $\partial_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2} = \partial^1_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2} + \partial^2_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2}$ and then using (5.29) we can write

$$\begin{aligned}
&\partial^1_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2}\\
&=\partial_{\varepsilon^2}\Bigg(-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(\Big\langle r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4); r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}^*{}_i,\dots,l_4^*), r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}^*{}_i,\dots,l_4^*)\Big]_2^2\\
&\otimes\prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)+\sum_{\substack{k=0\\k\neq i}}^{4}\left(2\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_k^*,\dots,l_4^*)_{\varepsilon^2}}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_k,\dots,l_4)}\right)\otimes\Big[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big]_2\\
&-\sum_{\substack{m=0\\m\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_m^*,\dots,l_4^*)^2_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_m,\dots,l_4)}\right)\otimes\Big[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big]_2\Bigg)
\end{aligned}$$

$$\begin{aligned}
&=-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg\{\left(2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2\right)\otimes\\
&\Big(1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big)\wedge\prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)\\
&-\left(2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2\right)\\
&\otimes\Big(r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big)\wedge\prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)\\
&+\left(\sum_{\substack{k=0\\k\neq i}}^{4}\left(2\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_k^*,\dots,l_4^*)_{\varepsilon^2}}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_k,\dots,l_4)}\right)-\sum_{\substack{m=0\\m\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_m^*,\dots,l_4^*)^2_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_m,\dots,l_4)}\right)\right)\\
&\otimes\Big(1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big)\wedge r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Bigg\}
\end{aligned}\tag{5. 38}$$

The second part of the map $\partial_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2}$ will be

$$\begin{aligned}
&\partial^2_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2}(l_0^*,\dots,l_4^*)\\
&=-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg\{\left(2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2\right)\\
&\wedge\left(2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2\right)\\
&\wedge\left(\sum_{\substack{k=0\\k\neq i}}^{4}\left(2\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_k^*,\dots,l_4^*)_{\varepsilon^2}}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_k,\dots,l_4)}\right)-\sum_{\substack{m=0\\m\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_m^*,\dots,l_4^*)^2_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_m,\dots,l_4)}\right)\right)\Bigg\}
\end{aligned}\tag{5. 39}$$

First we calculate $\partial^1_{\varepsilon^2}\circ\tau^3_{1,\varepsilon^2}(l_0^*,\dots,l_4^*)$. For this purpose we have to calculate the values of $2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2$ and $2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2$. Take the case $i=4$ and using (2.5) , (2.6) and (2.7) we have

$$\left(\frac{r_\varepsilon(l_4^*|l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_4|l_0,l_1,l_2,l_3)}\right)^2$$
$$=\frac{r_\varepsilon(l_4^*|l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_4|l_0,l_1,l_2,l_3)}\cdot\frac{r_\varepsilon(l_4^*|l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_4|l_0,l_1,l_2,l_3)}=\left(\frac{(l_4^*l_0^*l_3^*)_\varepsilon}{(l_4l_0l_3)}+\frac{(l_4^*l_1^*l_2^*)_\varepsilon}{(l_4l_1l_2)}-\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}-\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}\right)^2$$
$$=\frac{(l_4^*l_0^*l_3^*)^2_\varepsilon}{(l_4l_0l_3)^2}+\frac{(l_4^*l_1^*l_2^*)^2_\varepsilon}{(l_4l_1l_2)^2}+\frac{(l_4^*l_0^*l_2^*)^2_\varepsilon}{(l_4l_0l_2)^2}+\frac{(l_4^*l_1^*l_3^*)^2_\varepsilon}{(l_4l_1l_3)^2}+2\frac{(l_4^*l_0^*l_3^*)_\varepsilon}{(l_4l_0l_3)}\frac{(l_4^*l_1^*l_2^*)_\varepsilon}{(l_4l_1l_2)}+2\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}$$
$$-2\frac{(l_4^*l_0^*l_3^*)_\varepsilon}{(l_4l_0l_3)}\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}-2\frac{(l_4^*l_0^*l_3^*)_\varepsilon}{(l_4l_0l_3)}\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}-2\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}\frac{(l_4^*l_1^*l_2^*)_\varepsilon}{(l_4l_1l_2)}-2\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}\frac{(l_4^*l_1^*l_2^*)_\varepsilon}{(l_4l_1l_2)} \quad (5.\ 40)$$

and

$$=\frac{r_{\varepsilon^2}(l_4^*|l_0^*,l_1^*,l_2^*,l_3^*)}{r(l_4|l_0,l_1,l_2,l_3)}$$
$$=\frac{(l_4^*l_0^*l_3^*)_{\varepsilon^2}}{(l_4l_0l_3)}+\frac{(l_4^*l_1^*l_2^*)_{\varepsilon^2}}{(l_4l_1l_2)}-\frac{(l_4^*l_0^*l_2^*)_{\varepsilon^2}}{(l_4l_0l_2)}-\frac{(l_4^*l_1^*l_3^*)_{\varepsilon^2}}{(l_4l_1l_3)}+\frac{(l_4^*l_0^*l_3^*)_\varepsilon}{(l_4l_0l_3)}\frac{(l_4^*l_1^*l_2^*)_\varepsilon}{(l_4l_1l_2)}-\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}$$
$$-\left(\frac{(l_4^*l_0^*l_3^*)_\varepsilon}{(l_4l_0l_3)}+\frac{(l_4^*l_1^*l_2^*)_\varepsilon}{(l_4l_1l_2)}-\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}-\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}\right)\left(\frac{(l_4^*l_0^*l_2^*)_\varepsilon}{(l_4l_0l_2)}+\frac{(l_4^*l_1^*l_3^*)_\varepsilon}{(l_4l_1l_3)}\right) \quad (5.\ 41)$$

After a simple calculation we obtain

$$2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2$$
$$=2\left(\frac{(l_4^*l_0^*l_3^*)_{\varepsilon^2}}{(l_4l_0l_3)}+\frac{(l_4^*l_1^*l_2^*)_{\varepsilon^2}}{(l_4l_1l_2)}-\frac{(l_4^*l_0^*l_2^*)_{\varepsilon^2}}{(l_4l_0l_2)}-\frac{(l_4^*l_1^*l_3^*)_{\varepsilon^2}}{(l_4l_1l_3)}\right)$$
$$-\frac{(l_4^*l_0^*l_3^*)^2_\varepsilon}{(l_4l_0l_3)^2}-\frac{(l_4^*l_1^*l_2^*)^2_\varepsilon}{(l_4l_1l_2)^2}+\frac{(l_4^*l_0^*l_2^*)^2_\varepsilon}{(l_4l_0l_2)^2}+\frac{(l_4^*l_1^*l_3^*)^2_\varepsilon}{(l_4l_1l_3)^2} \quad (5.\ 42)$$

similarly

$$2\frac{r_{\varepsilon^2}(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}-\left(\frac{r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)}{1-r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)}\right)^2$$
$$=2\left(\frac{(l_4^*l_0^*l_2^*)_{\varepsilon^2}}{(l_4l_0l_2)}+\frac{(l_4^*l_1^*l_3^*)_{\varepsilon^2}}{(l_4l_1l_3)}-\frac{(l_4^*l_0^*l_1^*)_{\varepsilon^2}}{(l_4l_0l_1)}-\frac{(l_4^*l_2^*l_3^*)_{\varepsilon^2}}{(l_4l_2l_3)}\right)$$
$$-\frac{(l_4^*l_0^*l_2^*)^2_\varepsilon}{(l_4l_0l_2)^2}-\frac{(l_4^*l_1^*l_3^*)^2_\varepsilon}{(l_4l_1l_3)^2}+\frac{(l_4^*l_0^*l_1^*)^2_\varepsilon}{(l_4l_0l_1)^2}+\frac{(l_4^*l_2^*l_3^*)^2_\varepsilon}{(l_4l_2l_3)^2} \quad (5.\ 43)$$

By the substitution of (5. 42)and (5. 43) we can easily find the values of (5. 38)and (5. 39) for $i = 4$. Where (5. 38) contains 384 terms of type $\frac{\Delta(l_i^*,l_j^*,l_k^*)^2_\varepsilon}{\Delta(l_i,l_j,l_k)} \otimes l \wedge m$ and $2\frac{\Delta(l_i^*,l_j^*,l_k^*)_{\varepsilon^2}}{\Delta(l_i,l_j,l_k)} \otimes l \wedge m$. In the same way we can calculate the value of (5. 38) for $i = 0, 1, 2, 3$ and get a total of $5 \times 384 = 1920$ terms. Then we recombine the terms having common factor $\frac{\Delta(l_i^*,l_j^*,l_k^*)}{\Delta(l_i,l_j,l_k)} \otimes \cdots$. e.g.

The terms having common factor $\frac{\Delta(l_4^*,l_2^*,l_3^*)^2_\varepsilon}{\Delta(l_4,l_2,l_3)}\otimes\cdots$ will be of the form

$$-3\frac{\Delta(l_4^*,l_2^*,l_3^*)^2_\varepsilon}{\Delta(l_4,l_2,l_3)}\otimes\Big(\Delta(l_0,l_3,l_4)\wedge\Delta(l_0,l_2,l_3)-\Delta(l_0,l_3,l_4)\wedge\Delta(l_0,l_2,l_4)$$
$$-\Delta(l_0,l_2,l_4)\wedge\Delta(l_0,l_2,l_3)+\Delta(l_1,l_3,l_4)\wedge\Delta(l_1,l_2,l_4)-\Delta(l_1,l_3,l_4)\wedge\Delta(l_1,l_2,l_3)$$
$$+\Delta(l_1,l_2,l_4)\wedge\Delta(l_1,l_2,l_3)\Big)$$

the coefficient -3 can be canceled with the first factor $-\frac{1}{3}$ of (5. 38) and it becomes identical with that of (5. 36).

Similarly the value of (5.39) can be calculated by expanding the sum for $i=0,\ldots,4$ which gives us terms of the form $a\wedge b\wedge c$. After direct calculation we get an expression equal to (5.37). □

Theorem 5.6. *The left part of the diagram (A) commutes, i.e.*

$$\tau^3_{2,\varepsilon^2}\circ\partial_{\varepsilon^2}=d\circ\tau^3_{1,\varepsilon^2}$$

Proof. The map $\tau^3_{2,\varepsilon}$ gives us a large number of terms but most of them are identical due to symmetry an remains only 120 different terms . Due to the long calculations we will use the technique of combinatorics. From the definition (5. 30), we have

$$\tau^3_{2,\varepsilon^2}(l_0^*,\ldots,l_5^*)=\frac{2}{45}\mathrm{Alt}_6\left\langle r_3(l_0,\ldots,l_5);r_{3,\varepsilon}(l_0^*,\ldots,l_5^*),r_{3,\varepsilon^2}(l_0^*,\ldots,l_5^*)\right]^2_3 \tag{5. 44}$$

From now we use a short hand,that is ,we write $r_3(l_il_jl_kl_ll_ml_n)$instead of $r_3(l_i,l_j,l_k,l_l,l_m,l_n)$ and $(l_i^*l_j^*l_k^*)_{\varepsilon^2}$ in place of $\Delta(l_i^*l_j^*l_k^*)_{\varepsilon^2}$. In other words we ignore Δ and commas for simplicity. Now

$$\begin{aligned}&\partial_{\varepsilon^2}\circ\tau^3_{2,\varepsilon^2}(l_0^*\ldots l_5^*)\\&=\frac{2}{45}\mathrm{Alt}_6\Bigg\{\left\langle r_3(l_0\ldots l_5);r_{3,\varepsilon}(l_0^*\ldots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\right]^2_2\otimes r_3(l_0\ldots l_5)\\&+\left(2\frac{r_{3,\varepsilon^2}(l_0^*\ldots l_5^*)}{r_3(l_0\ldots l_5)}-\frac{r^2_{3,\varepsilon}(l_0^*\ldots l_5^*)}{r_3^2(l_0\ldots l_5)}\right)\otimes[r_3(l_0\ldots l_5)]_2\Bigg\}\end{aligned}\tag{5. 45}$$

We are going to evaluate the value of $2\frac{r_{3,\varepsilon^2}(l_0^*\ldots l_5^*)}{r_3(l_0\ldots l_5)}-\frac{r^2_{3,\varepsilon}(l_0^*\ldots l_5^*)}{r_3^2(l_0\ldots l_5)}$.
Using (5. 31) , (5. 32) and (5. 33) we get

$$\begin{aligned}&2\frac{r_{3,\varepsilon^2}(l_0^*\ldots l_5^*)}{r_3(l_0\ldots l_5)}-\frac{r^2_{3,\varepsilon}(l_0^*\ldots l_5^*)}{r_3^2(l_0\ldots l_5)}\\&=\left(\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)}+\frac{(l_1^*l_2^*l_4^*)_{\varepsilon^2}}{(l_1l_2l_4)}+\frac{(l_2^*l_0^*l_5^*)_{\varepsilon^2}}{(l_2l_0l_5)}-\frac{(l_0^*l_1^*l_4^*)_{\varepsilon^2}}{(l_0l_1l_4)}-\frac{(l_1^*l_2^*l_5^*)_{\varepsilon^2}}{(l_1l_2l_5)}-\frac{(l_2^*l_0^*l_3^*)_{\varepsilon^2}}{(l_2l_0l_3)}\right)\\&+\frac{(l_0^*l_1^*l_4^*)^2_\varepsilon}{(l_0l_1l_4)^2}+\frac{(l_1^*l_2^*l_5^*)^2_\varepsilon}{(l_1l_2l_5)^2}+\frac{(l_2^*l_0^*l_3^*)^2_\varepsilon}{(l_2l_0l_3)^2}-\frac{(l_0^*l_1^*l_3^*)^2_\varepsilon}{(l_0l_1l_3)^2}-\frac{(l_1^*l_2^*l_4^*)^2_\varepsilon}{(l_1l_2l_4)^2}-\frac{(l_2^*l_0^*l_5^*)_{\varepsilon^2}}{(l_2l_0l_5)}\end{aligned}\tag{5. 46}$$

Then (5. 45) becomes

$$=\frac{2}{45}\text{Alt}_6\Bigg\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2^2\otimes\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}$$
$$+\Bigg(2\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)}+2\frac{(l_1^*l_2^*l_4^*)_{\varepsilon^2}}{(l_1l_2l_4)}+2\frac{(l_2^*l_0^*l_5^*)_{\varepsilon^2}}{(l_2l_0l_5)}-2\frac{(l_0^*l_1^*l_4^*)_{\varepsilon^2}}{(l_0l_1l_4)}-2\frac{(l_1^*l_2^*l_5^*)_{\varepsilon^2}}{(l_1l_2l_5)}-2\frac{(l_2^*l_0^*l_3^*)_{\varepsilon^2}}{(l_2l_0l_3)}$$
$$+\frac{(l_0^*l_1^*l_4^*)_\varepsilon^2}{(l_0l_1l_4)^2}+\frac{(l_1^*l_2^*l_5^*)_\varepsilon^2}{(l_1l_2l_5)^2}+\frac{(l_2^*l_0^*l_3^*)_\varepsilon^2}{(l_2l_0l_3)^2}-\frac{(l_0^*l_1^*l_3^*)_\varepsilon^2}{(l_0l_1l_3)^2}-\frac{(l_1^*l_2^*l_4^*)_\varepsilon^2}{(l_1l_2l_4)^2}-\frac{(l_2^*l_0^*l_5^*)_{\varepsilon^2}}{(l_2l_0l_5)}\Bigg)\otimes[r_3(l_0\dots l_5)]_2\Bigg\} \quad (5.\ 47)$$

Above expression consist of two summands of the form $a\otimes b$ and $c\otimes[b]_2$.To discuss both separately, consider the first summand

$$=\frac{2}{45}\text{Alt}_6\Bigg\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2^2\otimes\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} \quad (5.\ 48)$$

Which can further be written as

$$\begin{aligned}
=&\text{Alt}_6\Big\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_0l_1l_3)\Big\}\\
+&\text{Alt}_6\Big\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_1l_2l_4)\Big\}\\
+&\text{Alt}_6\Big\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_2l_0l_5)\Big\}\\
-&\text{Alt}_6\Big\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_0l_1l_4)\Big\}\\
-&\text{Alt}_6\Big\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_1l_2l_5)\Big\}\\
-&\text{Alt}_6\Big\{\Big\langle r_3(l_0\dots l_5);r_{3,\varepsilon}(l_0^*\dots l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_2l_0l_3)\Big\}
\end{aligned} \quad (5.\ 49)$$

According to the technique used in ([8]) and ([9]) we have

$$\begin{aligned}
&\text{Alt}_6\Big\{\Big\langle r_3(l_0l_1l_2l_3l_4l_5);r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*),,r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_0l_1l_3)\Big\}\\
=&\text{Alt}_6\Big\{\Big\langle r_3(l_1l_2l_0l_4l_5l_3);r_{3,\varepsilon}(l_1^*l_2^*l_0^*l_4^*l_5^*l_3^*),,r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_1l_2l_4)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\langle r_3(l_0l_1l_2l_3l_4l_5);r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\\
=&\Big\langle r_3(l_1l_2l_0l_4l_5l_3);r_{3,\varepsilon}(l_1^*l_2^*l_0^*l_4^*l_5^*l_3^*),r_{3,\varepsilon^2}(l_1^*l_2^*l_0^*l_4^*l_5^*l_3^*)\Big]_2
\end{aligned} \quad (5.\ 50)$$

And this symmetry is true for all other ratios obtained by going through the alternation. Therefore (5. 49) becomes

$$\begin{aligned}
=\frac{2}{15}\text{Alt}_6\Big\{&\Big\langle r_3(l_0l_1l_2l_3l_4l_5);r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_0l_1l_3)\\
&-\Big\langle r_3(l_0l_1l_2l_3l_4l_5);r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_0l_1l_4)\Big\}
\end{aligned}$$

Applying the odd permutation (l_3l_4) (or $(l_3^*l_4^*)$)

$$=\frac{4}{15}\text{Alt}_6\Big\{\Big\langle r_3(l_0l_1l_2l_3l_4l_5);r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*),r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2\otimes(l_0l_1l_3)\Big\}$$

Once again using the permutation (l_0l_3) (or $(l_0^*l_3^*)$) we get

$$= \frac{2}{15}\mathrm{Alt}_6\Big\{\Big\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*), r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2 \otimes (l_0l_1l_3)$$
$$-\Big\langle r_3(l_3l_1l_2l_0l_4l_5); r_{3,\varepsilon}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*), r_{3,\varepsilon^2}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*)\Big]_2 \otimes (l_3l_1l_0)\Big\}$$

we have another symmetry $(l_0l_1l_3) = (l_3l_1l_0) = (l_0l_3l_1)$ up to 2-torsion, then

$$= \frac{2}{15}\mathrm{Alt}_6\Big\{\Big(\Big\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*), r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big]_2$$
$$-\Big\langle r_3(l_3l_1l_2l_0l_4l_5); r_{3,\varepsilon}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*), r_{3,\varepsilon^2}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*)\Big]_2\Big) \otimes (l_0l_1l_3)\Big\} \tag{5. 51}$$

Here again we follow the method used in ([8],[9]) that is we can express the triple ratio as the ratio of two cross ratios. In ([8],[9]) it has been done for $r_3(l_3l_1l_2l_0l_4l_5)$ and $r_{3,\varepsilon}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*)$, so we will do it only for $r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)$. Consider

$$r_3(l_0l_1l_2l_3l_4l_5) = \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}$$

Since

$$\mathbf{r}_3(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = r_3(l_0l_1l_2l_3l_4l_5) + \Big(r_3(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big)_\varepsilon \varepsilon + \Big(r_3(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\Big)_{\varepsilon^2} \varepsilon^2$$

so we can write

$$r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = \left(\frac{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)}{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)}\right)_{\varepsilon^2}$$

Using the projection here by l_1^* and l_2^*

$$r_{3,\varepsilon^2}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = \left(\frac{(l_1^*l_2^*l_3^*)(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)}{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)(l_1^*l_2^*l_3^*)}\right)_{\varepsilon^2} = \left(\frac{r(l_2^*|l_1^*l_0^*l_5^*l_3^*)}{r(l_1^*|l_0^*l_2^*l_3^*l_4^*)}\right)_{\varepsilon^2}$$

Therefore (5. 51) can be written as

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left\langle\frac{r(l_2|l_1l_0l_5l_3)}{r(l_1|l_0l_2l_3l_4)}; \left(\frac{r(l_2^*|l_1^*l_0^*l_5^*l_3^*)}{r(l_1^*|l_0^*l_2^*l_3^*l_4^*)}\right)_\varepsilon, \left(\frac{r(l_2^*|l_1^*l_0^*l_5^*l_3^*)}{r(l_1^*|l_0^*l_2^*l_3^*l_4^*)}\right)_{\varepsilon^2}\right]_2^2 \otimes (l_0l_1l_3)\right.$$
$$\left.-\left\langle\frac{r(l_2|l_1l_3l_5l_0)}{r(l_1|l_3l_2l_0l_4)}; \left(\frac{r(l_2^*|l_1^*l_3^*l_5^*l_0^*)}{r(l_1^*|l_3^*l_2^*l_0^*l_4^*)}\right)_\varepsilon, \left(\frac{r(l_2^*|l_1^*l_3^*l_5^*l_0^*)}{r(l_1^*|l_3^*l_2^*l_0^*l_4^*)}\right)_{\varepsilon^2}\right]_2^2 \otimes (l_0l_1l_3)\right\}$$

Applying the five term relation in $T\mathcal{B}_2^2(F)$

$$= \frac{2}{15}\mathrm{Alt}_6\{\Big(\Big\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*), r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_5^*l_3^*)\Big]_2^2$$
$$-\Big\langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_1^*|l_0^*l_2^*l_3^*l_4^*)\Big]_2^2$$
$$-\left\langle\frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)}; \left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_\varepsilon, \left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon^2}\right]_2^2\Big) \otimes (l_0l_1l_3)\} \tag{5. 52}$$

From above three terms of individual determinant $(l_0l_1l_3)$,Consider the last term. Which is

$$\frac{2}{15}\mathrm{Alt}_6\left\{\left\langle\frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)}; \left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_\varepsilon, \left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon^2}\right]_2^2 \otimes (l_0l_1l_3)\right\}$$

Which can further be written as

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left(\left\langle\frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)};\left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_\varepsilon,\left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon^2}\right]_2^2\otimes(l_0l_1l_3)\right)\right\}$$

Consider a permutation subgroup $S_3{\times}S_3$ in S_6, Where S_3 permuting $\{l_0,l_1,l_3\}$ and $\{l_2,l_4,l_5\}$, which fixes the determinant $(l_0l_1l_3)$ as $(l_0l_1l_3)\sim(l_3l_1l_0)\sim(l_3l_0l_1)\cdots$. Take

$$\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left\{\left\langle\frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)};\left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_\varepsilon,\left(\frac{r(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{r(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon^2}\right]_2^2\otimes(l_0l_1l_3)\right\}$$
$$=\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left\{\left\langle\frac{(l_2l_5l_3)(l_1l_0l_4)}{(l_2l_5l_0)(l_1l_3l_4)};\left(\frac{(l_2^*l_5^*l_3^*)(l_1^*l_0^*l_4^*)}{(l_2^*l_5^*l_0^*)(l_1^*l_3^*l_4^*)}\right)_\varepsilon,\left(\frac{(l_2^*l_5^*l_3^*)(l_1^*l_0^*l_4^*)}{(l_2^*l_5^*l_0^*)(l_1^*l_3^*l_4^*)}\right)_{\varepsilon^2}\right]_2^2\otimes(l_0l_1l_3)\right\}$$

This becomes zero if we use odd permutation (l_2l_5) and (5. 52) will be of the form

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\left(\left\langle r(l_2|l_1l_0l_5l_3);r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*),r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_5^*l_3^*)\right]_2^2\right.\right.$$
$$\left.\left.-\left\langle r(l_1|l_0l_2l_3l_4);r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*),r_{\varepsilon^2}(l_1^*|l_0^*l_2^*l_3^*l_4^*)\right]_2^2\right)\otimes(l_0l_1l_3)\right\} \qquad (5.\ 53)$$

Take first term of above

$$\frac{2}{15}\mathrm{Alt}_6\left\{\left\langle r(l_2|l_1l_0l_5l_3);r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*),r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_5^*l_3^*)\right]_2^2\otimes(l_0l_1l_3)\right\}$$

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left\{\left\langle r(l_2|l_1l_0l_5l_3);r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*),r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_5^*l_3^*)\right]_2^2\otimes(l_0l_1l_3)\right\}\right\}$$

Since the ratio is projected by 2 so, the the permutation $(l_0l_1l_3)$ will have no influence in above ratio. Therefore we can write above as

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{6}\mathrm{Alt}_{(l_2l_4l_5)}\left\{\left\langle r(l_2|l_1l_0l_5l_3);r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*),r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_5^*l_3^*)\right]_2^2\otimes(l_0l_1l_3)\right\}\right\}$$

By the expansion of inner alternation sum we get

$$=\frac{1}{45}\mathrm{Alt}_6\left\{\left(\left\langle r(l_4|l_1l_0l_2l_3);r_\varepsilon(l_4^*|l_1^*l_0^*l_2^*l_3^*),r_{\varepsilon^2}(l_4^*|l_1^*l_0^*l_2^*l_3^*)\right]_2^2\right.\right.$$
$$-\left\langle r(l_2|l_1l_0l_4l_3);r_\varepsilon(l_2^*|l_1^*l_0^*l_4^*l_3^*),r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_4^*l_3^*)\right]_2^2+\left\langle r(l_5|l_1l_0l_4l_3);r_\varepsilon(l_5^*|l_1^*l_0^*l_4^*l_3^*),r_{\varepsilon^2}(l_5^*|l_1^*l_0^*l_4^*l_3^*)\right]_2^2$$
$$-\left\langle r(l_4|l_1l_0l_5l_3);r_\varepsilon(l_4^*|l_1^*l_0^*l_5^*l_3^*),r_{\varepsilon^2}(l_4^*|l_1^*l_0^*l_5^*l_3^*)\right]_2^2+\left\langle r(l_2|l_1l_0l_5l_3);r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*),r_{\varepsilon^2}(l_2^*|l_1^*l_0^*l_5^*l_3^*)\right]_2^2$$
$$\left.\left.-\left\langle r(l_5|l_1l_0l_2l_3);r_\varepsilon(l_5^*|l_1^*l_0^*l_2^*l_3^*),r_{\varepsilon^2}(l_5^*|l_1^*l_0^*l_2^*l_3^*)\right]_2^2\right)\otimes(l_0l_1l_3)\right\}$$

Using projected five-term relation for $T\mathcal{B}_2^2(F)$

$$= \frac{1}{45}\mathrm{Alt}_6\Bigg\{\Bigg(\Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2$$
$$- \Big\langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_1^*|l_0^*l_2^*l_3^*l_4^*)\Big]_2^2$$
$$- \Big\langle r(l_3|l_0l_1l_2l_4); r_\varepsilon(l_3^*|l_0^*l_1^*l_2^*l_4^*), r_{\varepsilon^2}(l_3^*|l_0^*l_1^*l_2^*l_4^*)\Big]_2^2 + \Big\langle r(l_0|l_1l_4l_3l_5); r_\varepsilon(l_0^*|l_1^*l_4^*l_3^*l_5^*), r_{\varepsilon^2}(l_0^*|l_1^*l_4^*l_3^*l_5^*)\Big]_2^2$$
$$- \Big\langle r(l_1|l_0l_4l_3l_5); r_\varepsilon(l_1^*|l_0^*l_4^*l_3^*l_5^*), r_{\varepsilon^2}(l_1^*|l_0^*l_4^*l_3^*l_5^*)\Big]_2^2 - \Big\langle r(l_3|l_0l_1l_4l_5); r_\varepsilon(l_3^*|l_0^*l_1^*l_4^*l_5^*), r_{\varepsilon^2}(l_3^*|l_0^*l_1^*l_4^*l_5^*)\Big]_2^2$$
$$+ \Big\langle r(l_0|l_1l_5l_3l_2); r_\varepsilon(l_0^*|l_1^*l_5^*l_3^*l_2^*), r_{\varepsilon^2}(l_0^*|l_1^*l_5^*l_3^*l_2^*)\Big]_2^2 - \Big\langle r(l_1|l_0l_5l_3l_2); r_\varepsilon(l_1^*|l_0^*l_5^*l_3^*l_2^*), r_{\varepsilon^2}(l_1^*|l_0^*l_5^*l_3^*l_2^*)\Big]_2^2$$
$$- \Big\langle r(l_3|l_0l_1l_5l_2); r_\varepsilon(l_3^*|l_0^*l_1^*l_5^*l_2^*), r_{\varepsilon^2}(l_3^*|l_0^*l_1^*l_5^*l_2^*)\Big]_2^2\Bigg) \otimes (l_0l_1l_3)\Bigg\} \tag{5. 54}$$

Use the cyclic permutation $(l_0l_1l_3)(l_2l_4l_5)$ we get

$$= \frac{1}{45}\cdot 9\mathrm{Alt}_6\left\{\Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_3)\right\} \tag{5. 55}$$

Similarly we will write the second term of (5. 53) as

$$\frac{1}{45}\cdot -6\mathrm{Alt}_6\left\{\Big\langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_1^*|l_0^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_3)\right\} \tag{5. 56}$$

Using (5. 55) and (5. 56) we can write (5. 53) as

$$=\frac{1}{45}\mathrm{Alt}_6\Bigg\{\Bigg(9\Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2$$
$$-6\Big\langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_1^*|l_0^*l_2^*l_3^*l_4^*)\Big]_2^2\Bigg) \otimes (l_0l_1l_3)\Bigg\} \tag{5. 57}$$

The permutation $(l_0l_1l_3)(l_2l_4l_5)$ will give us

$$= \frac{1}{3}\mathrm{Alt}_6\left\{\Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_3)\right\} \tag{5. 58}$$

This is the final value of the first summand of (5. 45). Now we proceed for the second summand of (5. 45). Since $\mathcal{B}_2(F)$ satisfies five-term relation therefore second summand can be written as

$$= \frac{1}{3}\mathrm{Alt}_6\left\{\left(\frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} + 2\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)} - \frac{(l_0^*l_1^*l_3^*)_\varepsilon^2}{(l_0l_1l_3)^2}\right)\otimes [r(l_0|l_1l_2l_3l_4)]_2\right\} \tag{5. 59}$$

Use (5. 58) and (5. 59) to write (5. 45) as

$$=\frac{1}{3}\mathrm{Alt}_6\Bigg\{\Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_3)$$
$$+\left(2\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)} - \frac{(l_0^*l_1^*l_3^*)_\varepsilon^2}{(l_0l_1l_3)^2}\right)\otimes [r(l_0|l_1l_2l_3l_4)]_2\Bigg\} \tag{5. 60}$$

This completes the calculations of one side of the proof.
Now to compute other side $\tau^3_{1,\varepsilon^2} \circ d(l_0^* \dots l_5^*)$ we will write the map $\tau^3_{1,\varepsilon}$ in the form of

alternation sum. That is

$$\tau^3_{1,\varepsilon^2}(l_0^* \dots l_4^*) = \frac{1}{3}\mathrm{Alt}_6\Big\{ \Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_2)$$
$$+\left(2\frac{(l_0^*l_1^*l_2^*)_{\varepsilon^2}}{(l_0l_1l_2)} - \frac{(l_0^*l_1^*l_2^*)^2_\varepsilon}{(l_0l_1l_2)^2}\right)\otimes [r(l_0|l_1l_2l_3l_4)]_2\Big\} \tag{5. 61}$$

Applying the cycle $(l_0l_1l_2l_3l_4l_5)$ for the map d and the expansion through the alternation Alt_5 we get

$$\tau^3_{1\varepsilon}\circ d(l_0^* \dots l_5^*) = \frac{1}{3}\mathrm{Alt}_6\Big\{ \Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_2)$$
$$+\left(2\frac{(l_0^*l_1^*l_2^*)_{\varepsilon^2}}{(l_0l_1l_2)} - \frac{(l_0^*l_1^*l_2^*)^2_\varepsilon}{(l_0l_1l_2)^2}\right)\otimes [r(l_0|l_1l_2l_3l_4)]_2\Big\} \tag{5. 62}$$

The odd cycle (l_2l_3) will make (5. 62) like

$$\tau^3_{1\varepsilon}\circ d(l_0^* \dots l_5^*) = \frac{1}{3}\mathrm{Alt}_6\Big\{ \Big\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*), r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)\Big]_2^2 \otimes (l_0l_1l_3)$$
$$+\left(2\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)} - \frac{(l_0^*l_1^*l_3^*)^2_\varepsilon}{(l_0l_1l_3)^2}\right)\otimes [r(l_0|l_1l_2l_3l_4)]_2\Big\} \tag{5. 63}$$

At last two-term relation in $T\mathcal{B}_2^2(F)$ and $\mathcal{B}_2(F)$ will give us the correct sign. The final result obtained is same as (5. 60).

□

Corollary 5.7. *The maps defined below are zero*

(1) $C_5(\mathbb{A}^4_{F[\varepsilon]_3}) \xrightarrow{d'} C_4(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{\tau^3_{0,\varepsilon^2}} F\otimes F^\times \oplus \bigwedge^3 F$

(2) $C_6(\mathbb{A}^4_{F[\varepsilon]_3}) \xrightarrow{d'} C_5(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{\tau^3_{1,\varepsilon^2}} \left(T\mathcal{B}_2^2(F)\otimes F^\times\right)\oplus (F\otimes \mathcal{B}_2(F))$

Proof. Simply we have to show $\tau^3_{0,\varepsilon^2}\circ d' = 0$ and $\tau^3_{1,\varepsilon^2}\circ d' = 0$ □

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References

[1] J. L. Cathelineau, *Infinitesimal Polylogarithms, multiplicative Presentations of Kähler Differentials and Goncharov complexes*, talk at the workshop on polylogarthms, Essen, May 1-4 (1997).
[2] J. L. Cathelineau, *The tangent complex to the Bloch-Suslin complex*, Bull. Soc. Math. France **135,** (2007) 565-597.
[3] J. L. Cathelineau, *Projective Configurations, Homology of Orthogonal Groups, and Milnor K-theory*, Duke Mathematical Journal **2**, No. 121 (2004).
[4] Ph. Elbaz-Vincent and H. Gangl, *On Poly(ana)logs I*, Compositio Mathematica, **130**, (2002) 161-210.
[5] A. B. Goncharov, *Geometry of Configurations, Polylogarithms and Motivic Cohomology*, Adv. Math., **114**, (1995) 197-318.
[6] A. B. Goncharov and J. Zhao, *Grassmannian Trilogarithms*, Compositio Mathematica, **127**, (2001) 83-108.
[7] S. Hussain and R. Siddiqui, *Projected Five Term Relation in* $T\mathcal{B}_2^2(F)$,International Journal of Algebra, **6**, No. 28 (2012) 1353-1363.

[8] R. Siddiqui, *Configuration Complexes and Tangential and Infinitesimal versions of Polylogarithmic Complexes*, Doctoral thesis, Durham University. Available at Durham E-Theses Online: http://etheses.dur.ac.uk/586/ (2010).

[9] R. Siddiqui, *Tangent to Bloch-Suslin and Grassmannian Complexes over the dual numbers*, (2012) arXiv:1205.4101v2 [math.NT].