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The Edgeworth Type Expansion of Spacings Based Statistics

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Abstract. In this article, we study the asymptotic approximation of spacings based statistic. Using an appropriate version of Cramer's-type condition, we derive the Edgeworth Expansion of entropy statistics based on uniform spacings. The simulated values are also presented in the form of a table.

AMS (MOS) Subject Classification Codes: Primary 62G20;Secondary 62E20.
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i.i.d Random Variable; Cramer condition.

1. INTRODUCTION

Consider a population with continuous cumulative distribution function (cdf) $G(z)$ and probability density function (pdf) $g(z)$. We select an increasing order sample $Z_1^{'}, Z_2^{'}, ..., Z_n^{'}$ from this population. The sample spacings are defined as $W_m = Z_m^{'} - Z_{(m-1)}^{'}$, $m = 1, 2, ..., n$ with notations $Z_0^{'} = 0$ and $Z_{n+1}^{'} = 1$. The statistics based on spacings that we will study is given by

$$E_n = \sum_{m=1}^{n} (nW_m) \log(nW_m) \tag{1.1}$$

The statistics (1.1) is based on simple spacings, and such type of statistics have been studied extensively in literature see [25] and references contained there in. The random variable mentioned in (1.1) is called entropy-type spacings statistics and is studied by several authors [2, 13, 14]. The statistics based on spacings are, particularly, used for testing the goodness-of-fit problems in which the null hypothesis H_0 that the distribution is equal to a specified one is tested against the alternative $H_{1,n}$ that it is not so. It is a well-known fact that for such type of problem one transform the data via the probability integral transformation $U = G(Z^{'})$ that reduces the support of *G* to [0,1] and the specified cdf is reduced to that of a uniform random variable on [0,1]. The problem of goodness-of-fit tests based on uniform spacings received great attention from researchers after the introduction of Greenwood statistics [11]. The distribution theory of spacings based statistics is proved by many authors, see, for instance, [12, 20, 26]. The statistics based on simple spacings is also used for the analysis of circular data on the circumference [22, 23, 24]. For the random variable based on spacings, Mirakhmedov [18] obtained lower estimation in the remainder term of CLT by proving the Lindeberg type condition, also it is worth noticing that the probability of large deviations of a random variable based on simple spacings, a problem less investigated earlier, is proved by Mirakhmedov [19]. The research articles [5, 6, 15] provide a unified treatment of the distributions of spacings based statistics. Sometimes the exact distribution of the random variable in tractable form is not available. Even if it exists often its rate of convergence to the normal form is very slow [8]. That is why the researchers have shown considerable interest into the asymptotic distribution theory for the statistics based on spacings. One of the most famous among others is the Edgeworth Expansion used for approximation. As compared to normal approximation in which only the mean and variance play a role, the approximation by Edgeworth type Expansion is more appealling as involves the first four moments of the statistics. For this reason, sometimes better approximations for the distribution function of spacings based statistics may be obtained easily by using Edgeworth expansions. The advantage of the Edgeworth series is that the error is controlled, so it is a true asymptotic expansion. Some authors calculated Edgeworth expansion of spacing statistics for small to moderate sample sizes [9]. The Edgeworth series approximation for large sample sizes is also available in literature [16]. Bhattacharya and Ghosh [3] has shown the validity of formal Edgeworth expansions under suitable assumptions. In their paper Does et. al. [7] used a special condition and derived the Edgeworth expansions of spacings based statistics. By using the characterization of [7], we aim to find the Edgeworth type expansion with uniform remainder $o(1/n)$ for the distribution function of $\xi_n(z) = P\left\{n\left(\sigma^2-\tau^2\right)^{-1/2}(E_n-(n+1)\mu \leq z)\right\},\ z \in R$ where μ is the mean value, σ^2 is the variance and τ is the correlation coefficient of the random variable E_n. The paper is organized as, in section (Asymptotic Normality) we discuss the limit theorem for our statistics, in section (Edgeworth type Expansion) we formulate our theorem and recall some preliminary results and state two lemmas (without proof) necessary for the proof of our Theorem, in section (Proof of Theorem 3.1) we will prove our result.

2. Asymptotic Normality

Let $U_1, U_2, ..., U_n$, be a sample from uniform [0,1] distribution with $U_{1,n} \leq U_{2,n} \leq ... \leq U_{n-1,n}$ its order statistics and $M_m = U_m - U_{m-1}$ are thier uniform spacings. Let $g_m(u), m = 1, 2, ..., n$ be a sequence of real-valued measurable functions. Consider the statistics

$$R_n = \sum_{m=1}^{n} g_m(nM_m),\ n = 1, 2, ..., \tag{2.1}$$

The statistics given in (2.1) is called non-symetric. When all of $g_1, g_2, ..., g_n$ have the same value, statistics (2.1) is called symmetric. In this section we consider non symmetric form of (2.1) but will be specialized to symmetric case when focus on our statistics. The asymptotic normality and Cramer's type large deviation theorem for statistics (2.1) have been obtained in [18] and [21] repectively for the sum of functions of uniform spacings (i.e. under null hypothesis). Note that under alternatives converging to uniform null hypothesis the spacings W_m can be reduced to uniform spacings for details see [20]. Let $Z_m,\ m =$

$1, 2, ..., n$ be exponential random variables distributed identically and independently (i.i.d) with mean 1 and $\Im(Z)$ denote distribution of a random vector Z, then it is well known that $\Im(M_1, M_2, ..., M_n) = \Im(Z_1, Z_2, ..., Z_n | \sum_{m=1}^{n} Z_m = n)$ [17]. We suppose that the moments used below exist

$$R_n(Z) = \sum_{m=1}^{n} g_m(Z_m)\ ,\ S_n = Z_1 + Z_2 + ... + Z_n\ ,\ \rho = corr(R_n(Z), S_n),$$

$$f_m(Z) = g_m(Z_m) - Eg_m(Z_m) - (Z-1)\rho\sqrt{\frac{VarR_n(Z)}{n}}\ ,\ H_n(M) = \sum_{m=1}^{n} f_m(nM_m),$$

$$H_n(Z) = \sum_{m=1}^{n} f_m(Z_m)\ ,\ \mu_n = \sum_{m=1}^{n} Eg_m(Z_m)\ ,\ \sigma_n^2 = \sum_{m=1}^{n} var\left(f_m(Z_m)\right) \quad (2.2)$$

From the above moments it can be seen that $\sum_m Ef_m(Z_m) = 0$ and $\sigma_n^2 \equiv Var\left(H_n(Z)\right) = (1-\rho^2)Var\left(R_n(Z)\right)$, and $Cov\left(H_n(Z), S_n\right) = 0$. It is clear that $H_n(M) = R_n(M) - E(R_n(M))$. So without hesitation one may consider the statistics $H_n(M)$ instead of $R_n(M)$. From the above one notice that $\sigma_n^2 = 0$ if and only if $g_m(Z) = \lambda Z + C_m$, where C_m are arbitrary constants and λ does not depend on *m* for all $m = 1, ..., n$. We suppose that $\sigma_n^2 > 0\ \forall\ n = 1, 2, ...$. Since $g_m(Z)$ are random functions so we can suppose that $g_1(Z_1), ..., g_n(Z_n)$ is sequence of independent random variables not depending on M or Z. Let $\Phi(z)$ represent standard normal distribution and by putting

$$\tilde{f}_m = \frac{f_m(Z_m)}{\sigma_n}\ ,\ \beta_{3,n} = \sum_{m=1}^{n} E\left|\tilde{f}_m\right|^3\ and\ P_n(Z) = P\left\{T_n(Z) \le Z\sigma_n\right\}$$

also by well known inequality $\beta_{3,n} \le \beta_{4,n}^{1/2}$ where $\beta_{4,n}$ can be easily calculated from corresponding statistic, (for further details of the above stated moments, see, [20]), we have the following lemma

Lemma 2.1([18]).There exists a positive constant C such that

$$sup\left|P_n(z) - \Phi(z)\right| \le C\beta_{3,n}\ z \in R\ as\ n \to \infty$$

then the random variable R_n has asymptotically normal distribution with expectation μ_n and variance σ_n^2.
The random variable E_n is a special case of (2.1) with

$$g_m(u) = g(u) = u\log u \quad (2.3)$$

Therefore, by lemma 2.1, the following theorem establishes the asymptotic normality of our statistics

Theorem 2.1 The Statistics E_n has asymptotically normal distribution with expectation $n\mu$ and variance $n\sigma^2$ as $n \to \infty$ where $\mu = 1 - \gamma$ and $\sigma^2 = 2\zeta(2) + (1-\gamma)^2 - 2\gamma$ while $\gamma = 0.5772...$ is the celebrated Euler's gamma.

Proof. It is obvious that $E\left(g^2(z)\right) < \infty$, so by by lemma 2.1 the statistic E_n is asymptotically normal with parameters $nE\left(g(z)\right)$ and $n\left(1-\rho^2\right)Var\left(g(z)\right)$ where ρ is the correlation between $g(z)$ and z. By direct calculations it is easy to find $\mu = E\left(g(z)\right)$, $\left(1-\rho^2\right)$ and $\sigma^2 = Var\left(g(z)\right)$.

3. EDGEWORTH TYPE EXPANSION

The asymptotic expansions of various spacing statistics of type (2.1) have been derived by several authors under different conditions see, for example, [9, 16]. In particular, we refer to Does et. al. [7] in which the authors established a general formula for the Edgeworth expansions of spacings based statistics under a natural moment assumption and an appropriate version of Cramer's- type condition. We state two results form [7] that we use to derive the Edgeworth type expansion of E_n. We have

Lemma 3.1 ([7]): Let $g : [0, \infty) \to R$ be a non linear measurable function whose derivative exists and is not necessarily constant on $(c, d) < (0, \infty)$ such that $E\left(g^4(Z)\right) < \infty$ where Z is distributed Exp $\{1\}$. The random variable R_n is the sum of all g functions of normed uniform spacings. If F_n is the distribution function of $(R_n - ER_n/\sqrt{varR_n}$ and F_n^* is the Edgeworth type expansion of R_n then

$$\lim_{n\to\infty} n \sup_{z\epsilon R} |F_n(z) - F_n^*(z)| = 0.$$

This Lemma forms the basic result for deriving Edgeworth type expansion of spacings statistics of the form (2.1).

Lemma 3.2 ([7]): Let Z be a random variable taking values in R^m , the distribution of which is absolutely continuous on some Borel set B with $P\left(Z\varepsilon B\right) > 0$. Let $h : R^m \to R^k$ be a measurable function which is Lebesgue almost everywhere differentiable on B with $k\times m$ matrix $h^{'}$ as differential. If all $\chi\varepsilon\left(R^k - \{0\}\right)$ satisfy $P\left\{\left(h^{'}(Z)\right)^T \chi = 0|Z\varepsilon B\right\} < 1$. Then $\lim_{|v|\to\infty} \sup\left|E\left(e^{iv^T h(Z)}\right)\right| < 1$ holds.

This Lemma provides the neccessary condition for the application of Lemma 3.1. As a consequence of Lemma 3.1 and Lemma 3.2, keeping in view (2.3), we formulate our Theorem as under
Let $\Phi(Z)$ be the standard normal distribution, $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ and

$$\begin{aligned}\xi_n(z) = \Phi(z) - \phi(z)&\left[n^{-\frac{1}{2}}\left\{\frac{1}{6}\left(-\frac{16343}{5000}\right)(z^2-1) + \left(\frac{4643}{5000}\right)\right\}\right.\\ &+n^{-1}\left\{\frac{1}{24}\left(\frac{45563}{2000}\right)(z^3-3z) + \frac{1}{72}\left(\frac{16343}{5000}\right)^2(z^5-10z^3+15z)\right\}\\ &\left.+\frac{1}{8}\left\{-4\left(\frac{-4643}{5000}\right)\left(\frac{16343}{5000}\right) + \left(\frac{32753}{25000}\right)\right\}z + \frac{1}{6}\left(\frac{-4643}{5000}\right)\left(\frac{16343}{5000}\right)z^3\right]\end{aligned}$$

or we can write

$$\begin{aligned}\xi_n(z) = \Phi(z) - \phi(z)&\left[n^{-\frac{1}{2}}\left\{\left(\frac{681}{1250}\right)(z^2-1) - \frac{4643}{5000}\right\}\right.\\ &+n^{-1}\left\{\left(\frac{2373}{2500}\right)(z^3-3z)\right\} + \frac{371}{2500}(z^5-10z^3+15z)\\ &\left.+\left(\frac{8407}{5000}\right)x - \left(\frac{2529}{5000}\right)z^3\right] \qquad (3.1)\end{aligned}$$

Theorem 3.1 Let $\nabla_n = n - E_n$ with $\xi_n(z)$ as given in (3.1) while μ_n and σ_n^2 are as given in (2.2). Then
$P\left\{(\nabla_n - \mu_n)\,\sigma_n \le z\right\} = \xi_n(x) + o\left(\frac{1}{n}\right)$, $z\epsilon R$ as $n \to \infty$.

4. Proof of Theorem 3.1

By using basic definition of well known gamma fuction we have $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Since

$$\frac{1}{\Gamma(u+p)}n^{u+p-1}e^{-n} = \frac{1}{\Gamma(u+p)}\exp\left\{u\log n(p-1)\log n - n\right\}$$

i.e. the integrand is of exponential family so it is continous and derivatives of all order for this function exists. Thus for $n \ge 0$ one can write

$$\frac{d^n}{dx^n}\Gamma(x) = \int_0^\infty t^{x-1}(\log t)^n e^{-t}dt.$$

This result is the same as stated by Cramer that Gamma function is continuous and possesses continuous derivatives of all orders (see , for example , p-125 [4]). Also, for even n one has the zeta function as (see, for example, p-1080 [10])

$$\zeta(n) = \frac{2^{n-1}|B_n|\pi^n}{n!}$$

Particularly, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}, \ldots$ B_n are the well known Bernoulli numbers with $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$. Although no analytic form for $\zeta(n)$ is known for odd n but the one we need here is

$$\zeta(3) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^2} \approx 1.2020569032...$$

where $H_n = \gamma + \psi_0(n+1)$ are harmonic numbers with $\psi(n)$ is digamma function and $\gamma = 0.577215664...$ is the Euler constant. If we use the recurrence relation of well known polygamma function $\psi^{(k)}(z)$ and its relation with the logarithmic derivatives of Gamma function as given below

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + (-1)^k\frac{k!}{z^{k+1}}\ and$$

$$\psi^{(k)}(z+n) = \frac{d^{n+1}}{dz^{n+1}}\log\Gamma(z+n),\ n = 1, 2, 3, ...$$

an obvious relation between Gamma function and Riemann Zeta function (see, for example, P-260 [1]) is obtained as below

$$\frac{d^{n+1}}{dz^{n+1}}\log\Gamma(z+1) = (-1)^n n!\left[-\zeta(n+1) + 1 + \frac{1}{2^{n+1}} + ... + \frac{1}{z^{n+1}}\right].$$

After complicated and long but manageable derivatives and calculations we have

$$\Gamma^{(4)}(1) = \frac{27}{2}\zeta(4) + 8\gamma\zeta(3) + 6\gamma^2\zeta(2) + \gamma^4$$

$$\Gamma^{(4)}(2) = 6\zeta(4) - 8(1-\gamma)\zeta(3) + 6\gamma(\gamma-2)\zeta(2) + \gamma^3(\gamma-4)$$

$$\Gamma^{(4)}(3) = 27\zeta(4) + (16\gamma - 24)\zeta(3) + 12(\gamma^2 - 3\gamma + 1)\zeta(2) + 2\gamma^2(\gamma^2 - 6\gamma + 6)$$

$$\Gamma^{(4)}(4) = 81\zeta(4) + 8(6\gamma - 11)\zeta(3) + 12(3\gamma^2 - 11\gamma + 6)\zeta(2) + 2\gamma(3\gamma^3 - 22\gamma^2 + 36\gamma - 12)$$

$$\Gamma^{(4)}(5) = 324\zeta(4) + 16(12\gamma - 25)\zeta(3) + 4(36\gamma^2 - 150\gamma + 105)\zeta(2)$$

$$+4(6\gamma^4 - 50\gamma^3 + 105\gamma^2 - 60\gamma + 6)$$

It is to be noted that all the conditions settled in the two lemmas 3.1 and 3.2 are satisfied by the statistic given in (2.1) with $g(u) = u \log u$. Therefore, if $\tilde{F}_n$ is the distribution of $\tilde{E}_n = (E_n - E(E_n))/\sqrt{varE_n}$ and $\xi_n(z)$ is as in (3.1) then

$$\lim_{n\to\infty} n \sup_{z\varepsilon R} |\tilde{F}_n(z) - \xi_n(z)| = O(1).$$

For the symmetric case keeping in view (2.2) and (2.3) we replace the function $g(Z)$ by $\tilde{g}(Z) = (g(Z) - \mu - \tau(Z-1))\left(\sigma^2 - \tau^2\right)^{-1/2}$ which is merely a sort of centralization and does not affect the distribution of $\tilde{F}_n(z)$. We get different parameters as

$\mu = 1 - \gamma,\ \sigma^2 = 2\zeta(2) + (1-\gamma)^2 - 2\gamma,\ \tau^2 = 2\zeta(2) - 3$

$\kappa_3 = E(\tilde{g}(Z))^3 = \{2\zeta(2) - 3\}^{-3/2}\{3\zeta(2) - 12\zeta(3) + 10\} \approx 3.268670146...$

$a = -\frac{1}{2}E\tilde{g}(Z)(Z-1)^2 = -\frac{1}{2}\{2\zeta(2) - 3\}^{-1/2} \approx 0.928687857...$

$\kappa_4 = E(\tilde{g}(Z))^4 - 3 - 3\left(E\tilde{g}^2(Z)(Z-1)\right)^2$

$= \{2\zeta(2) - 3\}^{-2}\{324\zeta(4) - 64\zeta(3) + 204\zeta(2) + 65 - 15\} \approx 22.78151277...$

$b = 3\left(E\tilde{g}(Z)(Z-1)^2\right)^2 - 2E\tilde{g}^2(Z)(Z-1)^2 + 4E\tilde{g}(Z)(Z-1)^3 + 6$

$= 3\{\zeta(2) - 3\}^{-1} \approx 1.310119109...$

so that the Edgeworth type expansion $\xi_n(z)$ of function $\tilde{g}(z)$ is as given in (3.1). Note that $E\left(g^4(Z)\right) = 9639/50 < \infty$ so that the first condition of Lemma 3.1 is satisfied. By taking m = 1, k = 1, $h(Z) = \left(Z, \frac{g(Z)-1.4228Z+1}{\sqrt{2\zeta(2)-3}}\right)$ and $B = (0,\infty)$ in Lemma 3.2 and let $A = (c,d)$ then $(h(Z))^T A = \begin{bmatrix} Z & \frac{g(Z)-1.4228Z+1}{\sqrt{2\zeta(2)-3}}\end{bmatrix}\begin{bmatrix} c & d\end{bmatrix}^T = cZ + \frac{g(Z)-1.4228Z+1}{\sqrt{2\zeta(2)-3}}d.$ For $(h(Z))^T A = 0$,three cases arises (i) c = 0,$d \neq 0$, (ii) $c \neq 0$, d=0, (iii) $c \neq 0$,$d \neq 0$. For all the three possible cases $P\left\{(h(Z))^T A = 0/Z\epsilon B\right\} < 1$. So if $Q(s,t)$ is the characteristic function of $(Z, \tilde{g}(Z))$ then by lemma 3.2 $\lim_{(s,t)\to\infty} \sup|Q(s,t)| < 1$ that is the Cramer condition is satisfied. Hence by lemma 3.1

$$\lim_{n\to\infty} n \sup_{Z\epsilon R} |\tilde{F}_n(Z) - \xi_n(Z)| = O(1).$$

where $\tilde{F}_n(Z)$ is the distribution of $(\nabla_n - \mu)\,\sigma_n^{-1}$ that is
$P\left\{(\nabla_n - \mu)\,\sigma_n^{-1} \leq Z\right\} = \xi_n(Z) + o\left(\frac{1}{n}\right)$, $Z\epsilon R$ as $n \to \infty$. This complete the proof.

The Edgeworth expansions of $\tilde{F}_n$ is calculated using Mathematica for n=10, 20, 30, 50, 70, 100, 250, 300, 500 and 11000 in the region $|Z| \leq 3$. and the values are tabulated as under.

Table-1

Z	-3	-2.5	-2	-1.5	-1	-.5	0	.5	1	1.5	2	2.5	3
ξ_{10}	.001	-.003	.003	.067	.234	.472	.686	.827	.908	.954	.973	.981	.990
ξ_{20}	.000	-.001	.010	.069	.211	.421	.631	.790	.889	.945	.972	.985	.994
ξ_{30}	.000	.000	.013	.069	.201	.400	.607	.773	.881	.942	.973	.987	.995
ξ_{50}	.000	.001	.016	.069	.192	.378	.583	.756	.872	.940	.972	.989	.996
ξ_{70}	.000	.002	.017	.069	.186	.367	.570	.746	.867	.938	.974	.990	.997
ξ_{100}	.000	.003	.018	.069	.182	.357	.559	.738	.863	.937	.974	.990	.997
ξ_{150}	.000	.003	.019	.069	.178	.348	.548	.729	.859	.936	.974	.991	.998
ξ_{250}	.000	.004	.020	.069	.173	.339	.537	.721	.855	.935	.975	.992	.998
ξ_{300}	.000	.004	.021	.069	.172	.337	.534	.719	.854	.935	.975	992	.998
ξ_{500}	.000	.004	.021	.068	.169	.330	.526	.713	.851	.935	.975	992	.998
ξ_{11000}	.000	.005	.022	.068	.166	.323	.518	.706	.848	.934	.976	993	.999
Φ	.001	.006	.023	.067	.159	.309	.500	.692	.841	.933	.977	.994	.999

5. Conclusion

The Edgeworth type Expansion for entropy statistics based on uniform spacings is derived. This can be used for testing problems especially in such a case that the exact distribution is not available. From the table we observe that although the conditions used by Does et.al. [7] are not strong enough even then Edgeworth type Expansion obtained by his method perform very well. His method is not hard and can be easily applied. The convergence to normal form is considerably fast.

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