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Copson, Leindler Type Inequalities For Function Of Several Variables On Time Scales

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Abstract. In the paper, Copson and Leindler type inequalities are proved for functions of n variables. Core of proves is use of mathematical induction principle. Special cases of obtained inequalities include some existing results in the literature.

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1. Introduction

Copson's Inequalities:
In 1976, Copson [7] proved that if $0 < r \leq 1$, $c < 1$, and $\lambda(\cdot), g(\cdot)$ both are positive real valued functions, then

$$\int_0^\infty \frac{\lambda(\varsigma)}{(\Lambda(\varsigma))^c}\left(\int_\varsigma^\infty \lambda(s)g(s)\,ds\right)^r \geq \left(\frac{r}{1-c}\right)^r \int_0^\infty \lambda(s)(\Lambda(\varsigma))^{r-c} g^r(\varsigma)\,d\varsigma, \tag{1. 1}$$

where $\Lambda(\varsigma) = \int_0^\varsigma \lambda(s)ds$. He also proved that if $0 < r \leq 1$ and $c > 1$, and $\Lambda(\varsigma) \to \infty$ as $\varsigma \to \infty$, then

$$\int_a^\infty \frac{\lambda(\varsigma)}{(\Lambda(\varsigma))^c}\left(\int_a^\varsigma \lambda(s)g(s)\,ds\right)^r \geq \left(\frac{r}{c-1}\right)^r \int_a^\infty \lambda(s)(\Lambda(\varsigma))^{r-c} g^r(\varsigma)\,d\varsigma. \tag{1. 2}$$

In actual, Copson's original motive was to obtain a generalization of the following inequality given by Hardy and Littlewood in [9].

$$\sum_{l=1}^\infty l^{-c}\left(\sum_{k=l}^\infty g(k)\right)^r \geq M\sum_{l=1}^\infty l^{-c}(lg(l))^r, r > 0, c < 1,$$

where M is a fixed positive number that depends on r, c and $g(l) > 0$ for $l \in \mathbb{N}$.

Leindler's Inequalities:
Leindler in [11] proved that if $c \leq 0 < r < 1$, $g(l) > 0$ and $\lambda(l) > 0$ for $l \in \mathbb{N}$, then

$$\sum_{l=1}^\infty \frac{\lambda(l)}{(\Lambda(l))^c}\left(\sum_{\iota=l}^\infty \lambda(\iota)g(\iota)\right)^r \geq \left(\frac{r}{1-c}\right)^r \sum_{l=1}^\infty \lambda(l)\left(\sum_{\iota=1}^l \lambda(\iota)\right)^{r-c} g^r(l), \tag{1. 3}$$

where $\Lambda(l) = \sum_{\iota=1}^l \lambda(\iota)$ and if $c > 1 > r > 0$, $\Lambda_n \to \infty$, then

$$\sum_{l=1}^\infty \frac{\lambda(l)}{(\Lambda(l))^c}\left(\sum_{\iota=1}^l \lambda(\iota)g(\iota)\right)^r \geq \left(\frac{rL}{c-1}\right)^r \sum_{l=1}^\infty \lambda(l)(\Lambda(l))^{r-c} g^r(l), \tag{1. 4}$$

where $L = \inf \frac{\lambda(l)}{\lambda(l+1)}$.
In 1988, S. Hilger, a German mathematician presented time scales theory which deals both discrete and continuous cases simultaneously. For introduction to time scales calculus, the readers are referred to [5, 6] and for some Hardy inequalities on time scales, we mention [2, 3, 8, 10].
Agarwal et al., extended the inequalities (1. 1)–(1. 4) in [1, Theorem 2.3.1, Theorem 2.3.2, Theorem 3.4.1, Theorem 3.4.2] (see also [12]) respectively on time scales in the following form:

Theorem 1.1. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_\mathbb{T} = (0,\infty) \cap \mathbb{T}$, $c \leq 0 < r < 1$ and $\lambda : \mathbb{T} \to \mathbb{R}^+$ is such that $\Lambda(\varsigma) = \int_a^\varsigma \lambda(s)\Delta s > 0$. Define $g : \mathbb{T} \to \mathbb{R}^+$ such that*

$$\chi(\varsigma) = \int_\varsigma^\infty \lambda(s)g(s)\Delta s \quad \textit{exists},$$

then

$$\int_a^\infty \frac{\lambda(\varsigma)(\chi(\varsigma))^r}{(\Lambda^\sigma(\varsigma))^c}\Delta\varsigma \geq \left(\frac{r}{1-c}\right)^r \int_a^\infty \lambda(\varsigma)(\Lambda^\sigma(\varsigma))^{r-c} g^r(\varsigma)\Delta\varsigma\,. \qquad (1.\ 5)$$

Theorem 1.2. *Let* $\mathbb{T}$ *be a time scale with* $a \in (0,\infty)_{\mathbb{T}}$, $0 < r \leq 1 < c$ *and* $\lambda : \mathbb{T} \to \mathbb{R}^+$ *is such that* $\Lambda(\varsigma) = \int_a^\varsigma \lambda(s)\Delta s > 0$. *Define*

$$L := inf \frac{\Lambda(\varsigma)}{\Lambda^\sigma(\varsigma)} > 0, \varsigma \in \mathbb{T} \qquad (1.\ 6)$$

and $g : \mathbb{T} \to \mathbb{R}^+$ *is such that*

$$\chi(\varsigma) = \int_a^\varsigma \lambda(s)g(s)\Delta s \quad \textit{exists},$$

then

$$\int_a^\infty \frac{\lambda(\varsigma)(\chi^\sigma(\varsigma))^r}{(\Lambda^\sigma(\varsigma))^c}\Delta\varsigma \geq \left(\frac{rL^{1-c}}{c-1}\right)^r \int_a^\infty \lambda(\varsigma)(\Lambda^\sigma(\varsigma))^{r-c} g^r(\varsigma)\Delta\varsigma\,. \qquad (1.\ 7)$$

Theorem 1.3. *Let* $\mathbb{T}$ *be a time scale with* $a \in (0,\infty)_{\mathbb{T}}$, $c \leq 0 < r < 1$ *and* $\lambda : \mathbb{T} \to \mathbb{R}^+$ *is such that* $\Lambda(\varsigma) = \int_\varsigma^\infty \lambda(s)\Delta s$. *Define* $g : \mathbb{T} \to \mathbb{R}^+$ *such that*

$$\Psi(\varsigma) = \int_a^\varsigma \lambda(s)g(s)\Delta s \quad \textit{exists},$$

then

$$\int_a^\infty \frac{\lambda(\varsigma)}{\Lambda^c(\varsigma)}(\Psi^\sigma(\varsigma))^r\Delta\varsigma \geq \left(\frac{r}{1-c}\right)^r \int_a^\infty \lambda(\varsigma)(\Lambda(\varsigma))^{r-c} g^r(\varsigma)\Delta\varsigma. \qquad (1.\ 8)$$

Theorem 1.4. *Let* $\mathbb{T}$ *be a time scale with* $a \in (0,\infty)_{\mathbb{T}}$, $0 < r < 1 < c$ *and* $\lambda : \mathbb{T} \to \mathbb{R}^+$ *is such that* $\Lambda(\varsigma) = \int_\varsigma^\infty \lambda(s)\Delta s$. *Define*

$$K := \inf_{\varsigma \in \mathbb{T}} \frac{\Lambda^\sigma(\varsigma)}{\Lambda(\varsigma)} > 0 \qquad (1.\ 9)$$

and $g : \mathbb{T} \to \mathbb{R}^+$ *is such that*

$$\bar{\Psi}(\varsigma) = \int_\varsigma^\infty \lambda(s)g(s)\Delta s \quad \textit{exists},$$

then

$$\int_a^\infty \frac{\lambda(\varsigma)}{\Lambda^c(\varsigma)}\left(\bar{\Psi}(\varsigma)\right)^r \Delta\varsigma \geq \left(\frac{rK^c}{c-1}\right)^r \int_a^\infty \lambda(\varsigma)(\Lambda(\varsigma))^{r-c} g^r(\varsigma)\Delta\varsigma. \tag{1. 10}$$

The aim of the paper is to extend Theorem 1.1– Theorem 1.4 for function of several variables by using mathematical induction principle.

2. Preliminaries

First we recall the basic concepts from [5, 6] used in the paper. An arbitrary nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$. $\mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ are the examples of time scales.
For $\varsigma \in \mathbb{T}$, the forward operator $\sigma : \mathbb{T} \to \mathbb{T}$ is:

$$\sigma(\varsigma) := inf\{r \in \mathbb{T} : r > \varsigma\}$$

and the backward operator $\rho : \mathbb{T} \to \mathbb{T}$ is:

$$\rho(\varsigma) := sup\{r \in \mathbb{T} : r < \varsigma\}.$$

If we take $\inf \phi = sup\mathbb{T}$ (i.e., $\sigma(\varsigma) = \varsigma$ for $\mathbb{T}$ having a maximum ς) and $\sup \phi = inf\mathbb{T}$ (i.e., $\rho(\varsigma) = \varsigma$ for $\mathbb{T}$ having a minimum ς), where ϕ denotes the empty set, then ς is right-scattered if $\sigma(\varsigma) > \varsigma$ and ς is stated as left-scattered if $\rho(\varsigma) < \varsigma$. Isolated points are the points that are right-scattered and left-scattered at the same time. A point $\varsigma \in \mathbb{T}$ is said to be right-dense if $\varsigma < \sup \mathbb{T}$ and $\sigma(\varsigma) = \varsigma$ and is said to be left-dense if $\varsigma > \inf \mathbb{T}$ and $\rho(\varsigma) = \varsigma$. Dense points are the points that are right dense and left dense simultaneously. The graininess function $\mu : \mathbb{T} \to \mathbb{R}$ is defined as: $\mu(\varsigma) := \sigma(\varsigma) - \varsigma$.
A real valued function η on $\mathbb{T}$ is rd-continuous (right-dense continuous), if

- η is continuous at each $k \in \mathbb{T}$ where $k < sup\mathbb{T}$ and $\sigma(k) = k$,
- left hand limits exist at each $m \in \mathbb{T}$ where $m > inf\mathbb{T}$, and $\rho(m) = m$.

The class of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$ or $\mathbb{C}_{rd}$.
Define

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \text{if} \quad \sup\mathbb{T} < \infty, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Delta Derivative
Assume $\omega : \mathbb{T} \to \mathbb{R}$ is a function and let $\varsigma \in \mathbb{T}^k$. Then we define $\omega^\Delta(\varsigma)$ to be the number (provided it exists) with the property that for given $\epsilon > 0$, there is a neighborhood P of ς (i.e., $P = (\varsigma - \delta, \varsigma + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[\omega(\sigma(\varsigma)) - \omega(r)] - \omega^\Delta(\varsigma)[\sigma(\varsigma) - r]| \leq \epsilon|\sigma(\varsigma) - r|$$

holds for all $r \in P$. We call ω is delta differentiable at ς or $\omega^\Delta(\varsigma)$ is the delta (or Hilger) derivative of ω at ς.

Antiderivative
A function $W : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $w : \mathbb{T} \to \mathbb{R}$ provided

$$W^\Delta(\varsigma) = w(\varsigma)$$

holds for all $\varsigma \in \mathbb{T}^k$.

Delta Integral

For $W^{\Delta}(\varsigma) = w(\varsigma)$, $\varsigma \in \mathbb{T}^k$, the delta integral of w is stated as:

$$\int_l^{\varsigma} w(s)\Delta s = W(\varsigma) - W(l), \quad \text{for } l \in \mathbb{T}.$$

Also for $w \in \mathbb{C}_{rd}(\mathbb{T}^k, \mathbb{R})$, the Cauchy integral

$$W(\varsigma) := \int_{\varsigma_0}^{\varsigma} w(s)\Delta s$$

exists for $\varsigma_0 \in \mathbb{T}^k$ and satisfies $W^{\Delta}(\varsigma) = w(\varsigma)$.

An indefinite integral is defined as:

$$\int_t^{\infty} w(\varsigma)\Delta\varsigma = \lim_{p\to\infty} \int_t^{p} w(\varsigma)\Delta\varsigma \quad \text{for } t \in \mathbb{T}^k.$$

2.0.1. ***Fubini's Theorem***.

Theorem 2.1. [4] *Suppose $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is an integrable function w.r.t both time scales. Define $\chi = \int\limits_{\mathbb{T}_1} f(y_1, y_2)\Delta y_1$, which exists for almost every $y_2 \in \mathbb{T}_2$ and $\omega = \int\limits_{\mathbb{T}_2} f(y_1, y_2)\Delta y_2$, which exists for almost every $y_1 \in \mathbb{T}_1$, then,*

$$\int\limits_{\mathbb{T}_2} \Delta y_1 \int\limits_{\mathbb{T}_1} f(y_1, y_2)\Delta y_2 = \int\limits_{\mathbb{T}_2} \Delta y_2 \int\limits_{\mathbb{T}_1} f(y_1, y_2)\Delta y_1. \tag{2. 11}$$

Note:

Throughout the paper, functions are considered to be non-negative and delta integrals are assumed to exist.

3. Copson-Type Inequalities for Functions of n Variables

In the sequel, following notations are used:

$\varsigma_{\mathbf{n}} = (\varsigma_1, \ldots, \varsigma_n)$, $\mathbf{s_n} = (s_1, \ldots, s_n)$ and $(0,\infty)_{\mathbb{T}_\iota} = (0,\infty) \cap \mathbb{T}_\iota$.

Theorem 3.1. *Let $\iota = 1, 2, \ldots, n$; $\mathbb{T}_\iota$ be time scales with $a_\iota \in (0,\infty)_{\mathbb{T}_\iota}$, $c_\iota \leq 0 < r < 1$ and $\lambda_\iota : \mathbb{T}_\iota \to \mathbb{R}^+$ such that $\Lambda_\iota(\varsigma_\iota) = \int\limits_{a_\iota}^{\varsigma_\iota} \lambda_\iota(s_\iota)\Delta s_\iota > 0$. Define $g : \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}^+$ such that*

$$\psi_\iota(\varsigma_{\mathbf{n}}) = \int\limits_{\varsigma_1}^{\infty} \cdots \int\limits_{\varsigma_\iota}^{\infty} \prod_{\kappa=1}^{\iota} \lambda_\kappa(s_\kappa) g(\mathbf{s_n})\Delta s_\iota \ldots \Delta s_1 \quad \textit{exists}.$$

Then

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\kappa=1}^{n}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}}\psi_n^r(\varsigma_{\mathbf{n}})\Delta\varsigma_n\ldots\Delta\varsigma_1$$
$$\geq\prod_{\kappa=1}^{n}\left(\frac{r}{1-c_\kappa}\right)^r\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\kappa=1}^{n}\lambda_\kappa(\varsigma_\kappa)(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{r-c_\kappa}g^r(\varsigma_{\mathbf{n}})\Delta\varsigma_n\ldots\Delta\varsigma_1, \quad (3.\ 12)$$

where $\Lambda^\sigma(\varsigma)=\Lambda(\sigma(\varsigma))$.

Proof. To prove the required result, we use mathematical induction principle. For $n=1$, the statement is true by Theorem 1.1. Now, suppose statement is true for $1\leq n\leq q$. To prove the result for $n=q+1$, left hand side of (3. 12) can be written as

$$\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\kappa=1}^{q}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa})^{c_\kappa}}\left\{\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{(\Lambda_{q+1}^{\sigma_{q+1}}(\varsigma_{q+1}))^{c_{q+1}}}\psi_{q+1}^r(\varsigma_{\mathbf{q+1}})\Delta\varsigma_{q+1}\right\}\Delta\varsigma_q\cdots\Delta\varsigma_1 \quad (3.\ 13)$$

where $\psi_{q+1}(\varsigma_{\mathbf{q+1}})=\int_{\varsigma_1}^{\infty}\cdots\int_{\varsigma_{q+1}}^{\infty}\prod_{\kappa=1}^{q+1}\lambda_\kappa(s_\kappa)g(\mathbf{s_{q+1}})\Delta s_{q+1}\cdots\Delta s_1$.
Denote

$$I_1=\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{(\Lambda_{q+1}^{\sigma_{q+1}}(\varsigma_{q+1}))^{c_{q+1}}}\psi_{q+1}^r(\varsigma_{\mathbf{q+1}})\Delta\varsigma_{q+1}. \quad (3.\ 14)$$

Use (1. 5) in (3. 14) with respect to $\varsigma_{q+1}\in\mathbb{T}_{q+1}$ for fix $(\varsigma_1,\ldots,\varsigma_q)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_q$ to obtain

$$(I_1)^r\geq\left(\frac{r}{1-c_{q+1}}\right)^r\int_{a_{q+1}}^{\infty}\left\{\lambda_{q+1}(\varsigma_{q+1})[\Lambda^{\sigma_{q+1}}(\varsigma_{q+1})]^{r-c_{q+1}}\psi_q^r(\mathbf{s_q},\varsigma_{q+1})\right\}\Delta\varsigma_{q+1}. \quad (3.\ 15)$$

Substitute (3. 15) in (3. 13) and use (2. 11) q-times in resultant inequality to get

$$\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\kappa=1}^{q}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa})^{c_\kappa}}\left\{\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{(\Lambda_{q+1}^{\sigma_{q+1}}(\varsigma_{q+1}))^{c_{q+1}}}\psi_{q+1}^r(\varsigma_{\mathbf{q+1}})\Delta\varsigma_{q+1}\right\}\Delta\varsigma_q\ldots\Delta\varsigma_1$$
$$\geq\left(\frac{r}{1-c_{q+1}}\right)^r$$

$$\times \int_{a_{q+1}}^{\infty} \lambda_{q+1}(\varsigma_{q+1})[\Lambda^{\sigma_{q+1}}(\varsigma_{q+1})]^{r-c_{q+1}} \left\{ \int_{a_1}^{\infty} \cdots \int_{a_q}^{\infty} \prod_{\kappa=1}^{q} \frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa})^{c_\kappa}} \psi_q^r(\mathbf{s_q}, \varsigma_{q+1}) \Delta\varsigma_q \cdots \Delta\varsigma_1 \right\} \Delta\varsigma_{q+1}. \tag{3. 16}$$

Use induction hypothesis for $\psi_q^r(\mathbf{s_q}, \varsigma_{q+1})$ with fixed $\varsigma_{q+1} \in \mathbb{T}_{q+1}$ instead of $\psi_q^r(\mathbf{s_q})$ to obtain

$$\int_{a_1}^{\infty} \cdots \int_{a_{q+1}}^{\infty} \prod_{\kappa=1}^{q+1} \frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}} \psi_{q+1}^r(\varsigma_{\mathbf{q+1}}) \Delta\varsigma_{q+1} \cdots \Delta\varsigma_1$$
$$\geq \prod_{\kappa=1}^{q+1} \left(\frac{r}{1-c_\kappa} \right)^r \int_{a_1}^{\infty} \cdots \int_{a_{q+1}}^{\infty} \prod_{\kappa=1}^{q+1} \lambda_\kappa(\varsigma_\kappa) (\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{r-c_\kappa} g^r(\varsigma_{\mathbf{q+1}}) \Delta\varsigma_{q+1} \cdots \Delta\varsigma_1.$$

Hence by induction principle, statement is true for all positive integers n. □

Example 3.2. *In Theorem 3.1, if we assume* $\mathbb{T}_\iota = \mathbb{R}_+$, $c_\iota \leq 0 < r < 1$ *and* $a_\iota = 1$ *for all* $\iota = 1, \ldots, n$. *Then, (3. 12) takes the form*

$$\int_{1}^{\infty} \cdots \int_{1}^{\infty} \prod_{\kappa=1}^{n} \frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa(\varsigma_\kappa))^{c_\kappa}} \psi_n^r(\varsigma_{\mathbf{n}}) d\varsigma_n \cdots d\varsigma_1$$
$$\geq \prod_{\kappa=1}^{n} \left(\frac{r}{1-c_\kappa} \right)^r \int_{1}^{\infty} \cdots \int_{1}^{\infty} \prod_{\kappa=1}^{n} \lambda_\kappa(\varsigma_\kappa) (\Lambda_\kappa(\varsigma_\kappa))^{r-c_\kappa} g^r(\varsigma_{\mathbf{n}}) d\varsigma_n \cdots d\varsigma_1.$$

Example 3.3. *In Theorem 3.1, choose* $\mathbb{T}_\iota = \mathbb{N}$, $a_\iota = 1$ *and* $\varsigma_\iota = m_\iota$ *for all* $\iota = 1, 2, \ldots, n$. *In this case*
$\Lambda_\iota(m_\iota) = \sum_{s_\iota=1}^{m_\iota - 1} \lambda_\iota(s_\iota)$ *and* $\Lambda_\iota^{\sigma_\iota}(m_\iota) = \sum_{s_\iota=1}^{m_\iota} \lambda_\iota(s_\iota)$, *where* $\sigma_\iota(\varsigma_\iota) = \varsigma_\iota + 1$ *and* $\mu_\iota(\varsigma_\iota) = 1$. *Also*

$$\psi_n(m_1, \ldots, m_n) = \sum_{s_1=m_1}^{\infty} \cdots \sum_{s_n=m_n}^{\infty} \lambda_1(s_1) \cdots \lambda_n(s_n) g(s_1, \ldots, s_n).$$

Therefore, (3. 12) takes the form

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \frac{\lambda_\iota(m_\iota)}{\left(\sum_{s_\iota=1}^{m_\iota} \lambda_\iota(s_\iota) \right)^{c_\iota}} \left[\sum_{p_1=m_1}^{\infty} \cdots \sum_{p_n=m_n}^{\infty} \prod_{k=1}^{n} \lambda_k(p_k) g(p_1, \ldots, p_n) \right]^r$$
$$\geq \frac{r^{nr}}{\prod_{\iota=1}^{n} (1-c_\iota)^r} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \lambda_\iota(m_\iota) \left[\sum_{s_\iota=1}^{m_\iota} \lambda_\iota(s_\iota) \right]^{r-c_\iota} g^r(m_1, \ldots, m_n).$$

Example 3.4. *In Theorem 3.1, choose* $\mathbb{T}_\iota = q_\iota^{\mathbb{N}_0}$, $a_\iota = 1$, $\varsigma_\iota = q_\iota^{m_\iota}$ *and* $s_\iota = q_\iota^{p_\iota}$ *for* $m_\iota, p_\iota \in \mathbb{N}_0$ *and* $q_\iota > 1$ *for all* $\iota = 1, 2, \ldots, n$. *In this case*
$\Lambda_\iota\left(q_\iota^{m_\iota}\right) = (q_\iota - 1)\sum\limits_{s_\iota=1}^{m_\iota-1} \lambda_\iota\left(q_\iota^{s_\iota}\right)q_\iota^{s_\iota}$ *and* $\Lambda_\iota^{\sigma_\iota}\left(q_\iota^{m_\iota}\right) = (q_\iota - 1)\sum\limits_{s_\iota=1}^{m_\iota} \lambda_\iota\left(q_\iota^{s_\iota}\right)q_\iota^{s_\iota}$,
where $\sigma_\iota(\varsigma_\iota) = q_\iota\varsigma_\iota = q_\iota^{m_\iota+1}$, $\mu_\iota(\varsigma_\iota) = (q_\iota - 1)\varsigma_\iota = (q_\iota - 1)q_\iota^{m_\iota}$.
Also

$$\psi_n\left(q_1^{m_1}, \ldots, q_n^{m_n}\right) = \prod_{k=1}^{n}(q_k - 1)\sum_{p_1=m_1}^{\infty}\cdots\sum_{p_n=m_n}^{\infty}\prod_{k=1}^{n}\lambda_k\left(q_k^{p_k}\right)q_k^{p_k}g\left(q_1^{p_1}, \ldots, q_n^{p_n}\right).$$

Therefore, (3. 12) takes the form

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\prod_{\iota=1}^{n}\frac{\lambda_\iota\left(q_\iota^{m_\iota}\right)q_\iota^{m_\iota}}{\left(\sum\limits_{s_\iota=1}^{m_\iota}\lambda_\iota\left(q_\iota^{s_\iota}\right)q_\iota^{s_\iota}\right)^{c_\iota}}\left[\sum_{p_1=m_1}^{\infty}\cdots\sum_{p_n=m_n}^{\infty}\prod_{k=1}^{n}\lambda_k\left(q_k^{p_k}\right)q_k^{p_k}g\left(q_1^{p_1}, \ldots, q_n^{p_n}\right)\right]^r$$

$$\geq \frac{r^{nr}}{\prod\limits_{\iota=1}^{n}(1 - c_\iota)^r}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\prod_{\iota=1}^{n}\lambda_\iota\left(q_\iota^{m_\iota}\right)q_\iota^{m_\iota}\left[\sum_{s_\iota=1}^{m_\iota}\lambda_\iota\left(q_\iota^{s_\iota}\right)q_\iota^{s_\iota}\right]^{r-c_\iota}g^r\left(q_1^{m_1}, \ldots, q_n^{m_n}\right).$$

Theorem 3.5. *Let* $\iota = 1, 2, \ldots, n$; $\mathbb{T}_\iota$ *be time scales with* $a_\iota \in (0, \infty)_{\mathbb{T}_\iota}$, $0 < r \leq 1 < c_\iota$ *and* $\lambda_\iota : \mathbb{T}_\iota \to \mathbb{R}^+$ *are such that* $\Lambda_\iota(\varsigma_\iota) = \int\limits_{a_\iota}^{\varsigma_\iota}\lambda_\iota(s_\iota)\Delta s_\iota > 0$. *Define*

$$L_\iota = \inf_{\varsigma_\iota \in \mathbb{T}_\iota}\frac{\Lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{\sigma_\iota}(\varsigma_\iota)}, \iota = 1, \ldots, n; \tag{3. 17}$$

and $g : \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}^+$ *is such that*

$$\psi_\iota(\varsigma_{\mathbf{n}}) = \int_{a_1}^{\varsigma_1}\cdots\int_{a_\iota}^{\varsigma_\iota}\prod_{\kappa=1}^{\iota}\lambda_\kappa(s_\kappa)g(\mathbf{s_n})\Delta s_\iota\cdots\Delta s_1 \quad \textit{exists}.$$

Then

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\kappa=1}^{n}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}}(\psi_n^{\sigma_1\cdots\sigma_n}(\varsigma_{\mathbf{n}}))^r\Delta\varsigma_n\cdots\Delta\varsigma_1$$
$$\geq \prod_{\kappa=1}^{n}\left(\frac{rL_\kappa^{1-c_\kappa}}{c_\kappa - 1}\right)^r\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\kappa=1}^{n}\lambda_\kappa(\varsigma_\kappa)(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{r-c_\kappa}g^r(\varsigma_{\mathbf{n}})\Delta\varsigma_n\cdots\Delta\varsigma_1, \tag{3. 18}$$

where $\Lambda^\sigma(\varsigma) = \Lambda(\sigma(\varsigma))$.

Proof. To prove the required result, use mathematical induction principle. For $n = 1$, the statement is true by Theorem 1.2. Now, suppose above statement is true for $1 \leq n \leq q$. To prove the result for $n = q+1$, left hand side of (3. 18) can be written as

$$\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\kappa=1}^{q}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}}\left\{\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{(\Lambda_{q+1}^{\sigma_{q+1}}(\varsigma_{q+1}))^{c_{q+1}}}\left(\psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}})\right)^r\Delta\varsigma_{q+1}\right\}\Delta\varsigma_q\cdots\Delta\varsigma_1. \tag{3. 19}$$

where $\psi_{q+1}(\varsigma_{\mathbf{q+1}}) = \int_{a_1}^{\varsigma_1}\cdots\int_{a_{q+1}}^{\varsigma_{q+1}}\prod_{\kappa=1}^{q+1}\lambda_\kappa(s_\kappa)g(\mathbf{s_{q+1}})\Delta s_{q+1}\cdots\Delta s_1$. Denote

$$I_1 = \int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(t_{q+1})}{(\Lambda_{q+1}^{\sigma_{q+1}}(\varsigma_{q+1}))^{c_{q+1}}}\left(\psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}})\right)^r\Delta\varsigma_{q+1}. \tag{3. 20}$$

Use (1. 7) in (3. 20) with respect to $\varsigma_{q+1}\in\mathbb{T}_{q+1}$ for fix $(\varsigma_1,\cdots,\varsigma_q)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_q$ to obtain

$$(I_1)^r \ge \left(\frac{rL_{q+1}^{1-c_{q+1}}}{c_{q+1}-1}\right)^r\int_{a_{q+1}}^{\infty}\left\{\lambda_{q+1}(\varsigma_{q+1})[\Lambda^{\sigma_{q+1}}(\varsigma_{q+1})]^{r-c_{q+1}}\left(\psi_q^{\sigma_1\cdots\sigma_q}(\varsigma_{\mathbf{q}},\varsigma_{q+1})\right)^r\right\}\Delta\varsigma_{q+1}. \tag{3. 21}$$

Substitute (3. 21) in (3. 19) and use (2. 11) q-times on the right hand side of resultant inequality to get

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\kappa=1}^{q}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}}\left\{\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{(\Lambda_{q+1}^{\sigma_{q+1}}(\varsigma_{q+1}))^{c_{q+1}}}\left(\psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}})\right)^r\Delta\varsigma_{q+1}\right\}\Delta\varsigma_q\cdots\Delta\varsigma_1\\
&\qquad\ge\left(\frac{rL_{q+1}^{1-c_{q+1}}}{c_{q+1}-1}\right)^r\\
&\times\left\{\int_{a_{q+1}}^{\infty}\lambda_{q+1}(\varsigma_{q+1})[\Lambda^{\sigma_{q+1}}(\varsigma_{q+1})]^{r-c_{q+1}}\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\kappa=1}^{q}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}}\left(\psi_q^{\sigma_1\cdots\sigma_q}(\mathbf{s_q},\varsigma_{q+1})\right)^r\right\}\Delta\varsigma_q\cdots\Delta\varsigma_1\Delta\varsigma_{q+1}.\end{aligned} \tag{3. 22}$$

Use induction hypothesis for $\psi_q^r(\mathbf{s_q},\varsigma_{q+1})$ with fixed $\varsigma_{q+1}\in\mathbb{T}_{q+1}$ instead for $\psi_q^r(\mathbf{s_q})$ to obtain

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_{q+1}}^{\infty}\prod_{\kappa=1}^{q+1}\frac{\lambda_\kappa(\varsigma_\kappa)}{(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{c_\kappa}}\left(\psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}})\right)^r\Delta\varsigma_{q+1}\cdots\Delta\varsigma_1\\
&\quad\ge\prod_{\kappa=1}^{q+1}\left(\frac{rL_\kappa^{1-c_\kappa}}{c_\kappa-1}\right)^r\int_{a_1}^{\infty}\cdots\int_{a_{q+1}}^{\infty}\prod_{\kappa=1}^{q+1}\lambda_\kappa(\varsigma_\kappa)(\Lambda_\kappa^{\sigma_\kappa}(\varsigma_\kappa))^{r-c_\kappa}g^r(\varsigma_{\mathbf{q+1}})\Delta\varsigma_{q+1}\cdots\Delta\varsigma_1.\end{aligned}$$

Hence by induction principle, statement is true for all positive integers n. □

4. Leindler-Type Inequalities for Functions of n Variables

Theorem 4.1. *Let $\iota = 1, 2, \ldots, n$ and $\mathbb{T}_i$ be time scales. Let $\lambda_\iota : \mathbb{T}_\iota \to \mathbb{R}^+$ are such that $\Lambda_\iota(\varsigma_\iota) := \int_{\varsigma_\iota}^{\infty} \lambda_\iota(s_\iota)\Delta s_\iota$ exist with $\Lambda_\iota(\infty) = 0$ and $g : \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}^+$ such that*

$$\psi_n(\varsigma_{\mathbf{n}}) := \int_{a_1}^{\varsigma_1} \cdots \int_{a_n}^{\varsigma_n} \prod_{\iota=1}^{n} (\lambda_\iota(s_\iota))\, g(\mathbf{s}_n)\Delta s_n \cdots \Delta s_1$$

exists, then for $a_\iota \in [0, \infty)_{\mathbb{T}_\iota}$;$0 < r < 1$ and $c_\iota \leq 0$, we have

$$\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \left[\frac{\lambda_\iota(\varsigma_\iota)}{(\Lambda_\iota(\varsigma_\iota))^{c_\iota}}\right] (\Psi_n^{\sigma_1\cdots\sigma_n}(\varsigma_{\mathbf{n}}))^r \Delta\varsigma_n \cdots \Delta\varsigma_1$$
$$\geq \prod_{\iota=1}^{n} \left[\left(\frac{r}{1-c_\iota}\right)^r\right] \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota - r}(\varsigma_\iota)}\right] g^r(\varsigma_{\mathbf{n}})\Delta\varsigma_n \cdots \Delta\varsigma_1. \quad (4.\ 23)$$

Proof. To prove the result we use the principle of mathematical induction. For $n = 1$ the statement is true by Theorem 1.3. Assume that (4. 23) holds for $1 \leq n \leq q$. To prove the result for $n = q + 1$, the left hand side of (4. 23) can be written as

$$\int_{a_1}^{\infty} \cdots \int_{a_q}^{\infty} \prod_{\iota=1}^{q} \left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota}(\varsigma_i)}\right] \left\{\int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}(\varsigma_{q+1})}} (\Psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}}))^r \Delta\varsigma_{q+1}\right\} \Delta\varsigma_q \cdots \Delta\varsigma_1. \quad (4.\ 24)$$

Denote

$$I_{q+1} = \int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}}(\varsigma_{q+1})} (\Psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}}))^r \Delta\varsigma_{q+1}. \quad (4.\ 25)$$

Use (1. 8) in (4. 25) with respect to $\varsigma_{q+1} \in \mathbb{T}_{q+1}$ for fix $(\varsigma_{\mathbf{q}}) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_q$ to obtain

$$[I_{q+1}]^{r+1-r} \geq \left(\frac{r}{1-c_{q+1}}\right) \int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})(\Psi_q^{\sigma_1\cdots\sigma_q}(\varsigma_{\mathbf{q}}, \varsigma_{q+1}))^r}{\Lambda_{q+1}^{c_{q+1}-r}(\varsigma_{q+1})} \Delta\varsigma_{q+1}. \quad (4.\ 26)$$

Substitute (4. 26) in (4. 24) and use (2. 11) q-times on the right hand side of the resultant inequality to get

$$\int_{a_1}^{\infty} \cdots \int_{a_q}^{\infty} \prod_{\iota=1}^{q} \left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota}(\varsigma_\iota)}\right] \times \left\{\int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}(\varsigma_{q+1})}} (\Psi_{q+1}^{\sigma_1\cdots\sigma_{q+1}}(\varsigma_{\mathbf{q+1}}))^r \Delta\varsigma_{q+1}\right\} \Delta\varsigma_q \cdots \Delta\varsigma_1$$
$$\geq \left(\frac{r}{1-c_{q+1}}\right)^r$$
$$\int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}-r}(\varsigma_{q+1})} \times \left\{\int_{a_1}^{\infty} \cdots \int_{a_q}^{\infty} \prod_{\iota=1}^{q} \left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota}(\varsigma_\iota)}\right] (\Psi_q^{\sigma_1\cdots\sigma_q}(\varsigma_{\mathbf{q}}, \varsigma_{q+1}))^r \Delta\varsigma_q \cdots \Delta\varsigma_1\right\} \Delta\varsigma_{q+1}.$$

Use induction hypothesis for fixed $\varsigma_{q+1} \in \mathbb{T}_{q+1}$ on $(\Psi_q^{\sigma_1 \cdots \sigma_q}(\varsigma_{\mathbf{q}}, \varsigma_{q+1}))^r$ instead of $\Psi_q^{\sigma_1 \cdots \sigma_q}(\varsigma_{\mathbf{q}})$ to obtain

$$\int_{a_1}^{\infty} \cdots \int_{a_{q+1}}^{\infty} \prod_{\iota=1}^{q+1} \left[\frac{\lambda_\iota(\varsigma_\iota)}{(\Lambda_\iota(\varsigma_\iota))^{c_\iota}} \right] (\Psi_{q+1}^{\sigma_1 \cdots \sigma_{q+1}}(\varsigma_{\mathbf{q+1}}))^r \Delta\varsigma_{q+1} \cdots \Delta\varsigma_1$$
$$\geq \prod_{\iota=1}^{q+1} \left[\left(\frac{r}{1-c_\iota} \right)^r \right] \int_{a_1}^{\infty} \cdots \int_{a_{q+1}}^{\infty} \prod_{\iota=1}^{q+1} \left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota - r}(\varsigma_\iota)} \right] g^r(\varsigma_{\mathbf{q+1}}) \Delta\varsigma_{q+1} \cdots \Delta\varsigma_1.$$

Hence by the principle of mathematical induction inequality (4. 23) is true for all positive integers n. □

Theorem 4.2. *Let $\iota = 1, 2, \ldots, n$ and $\mathbb{T}_\iota$ be time scales. Let $\lambda_\iota : \mathbb{T}_\iota \to \mathbb{R}^+$ is such that $\Lambda_\iota(t_\iota) := \int_{t_\iota}^{\infty} \lambda_\iota(s_\iota)\Delta s_\iota$ exist with $\Lambda_\iota(\infty) = 0$ and $K_\iota = \inf_{\varsigma_\iota \in \mathbb{T}_\iota} \frac{\Lambda_\iota(\sigma_\iota(\varsigma_\iota))}{\Lambda_\iota(\varsigma_\iota)} > 0$ and $g : \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}^+$ is such that*

$$\bar{\Psi}_n(\varsigma_{\mathbf{n}}) := \int_{\varsigma_1}^{\infty} \cdots \int_{\varsigma_n}^{\infty} \prod_{\iota=1}^{n} (\lambda_\iota(s_\iota))\, g(\mathbf{s_n}) \Delta s_n \cdots \Delta s_1$$

exists, then for $a_\iota \in [0, \infty)_{\mathbb{T}_\iota}$; $0 < r < 1 < c_\iota$, we have

$$\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota}(\varsigma_\iota)} \bar{\Psi}_n^r(\varsigma_{\mathbf{n}}) \Delta\varsigma_n \cdots \Delta\varsigma_1$$
$$\geq \prod_{\iota=1}^{n} \left[\left(\frac{rK_\iota^{c_\iota}}{c_\iota - 1} \right)^r \right] \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota - r}(\varsigma_\iota)} g^r(\varsigma_{\mathbf{n}}) \Delta\varsigma_n \cdots \Delta\varsigma_1. \quad (4.\ 27)$$

Proof. To prove the result we use the principle of mathematical induction. For $n = 1$. the statement is true by Theorem 1.4. Assume that (4. 27) holds for $1 \leq n \leq q$. To prove the result for $n = q + 1$, the left hand side of (4. 27) can be written as

$$\int_{a_1}^{\infty} \cdots \int_{a_q}^{\infty} \prod_{\iota=1}^{q} \frac{\lambda_\iota(\varsigma_\iota)}{(\Lambda_i(\varsigma_\iota))^{c_\iota}} \times \left\{ \int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}}(\varsigma_{q+1})} \bar{\Psi}_{q+1}^r(\varsigma_{\mathbf{q+1}}) \Delta\varsigma_{q+1} \right\} \Delta\varsigma_q \cdots \Delta\varsigma_1. \quad (4.\ 28)$$

Denote

$$I_{q+1} = \int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{p+1}}(\varsigma_{q+1})} \bar{\Psi}_{q+1}^r(\varsigma_{\mathbf{q+1}}) \Delta\varsigma_{q+1}. \quad (4.\ 29)$$

Use (1. 10) in (4. 29) with respect to $\varsigma_{q+1} \in \mathbb{T}_{q+1}$ for fix $(\varsigma_{\mathbf{q}}) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_q$ to obtain

$$I_{q+1} \geq \left(\frac{rK_{q+1}^{c_{q+1}}}{c_{q+1} - 1} \right)^r \int_{a_{q+1}}^{\infty} \frac{\lambda_{q+1}(\varsigma_{q+1}) \bar{\Psi}_q^r(\varsigma_{\mathbf{q}}, \varsigma_{q+1})}{[\Lambda_{q+1}(\varsigma_{q+1})]^{c_{q+1} - r}} \Delta\varsigma_{q+1}. \quad (4.\ 30)$$

Substitute (4. 30) in (4. 28) and use (2. 11) q-times on the right hand side of resultant inequality to get

$$\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\iota=1}^{q}\frac{\lambda_\iota(\varsigma_\iota)}{(\Lambda_\iota(\varsigma_\iota))^{c_\iota}}\left\{\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}}(\varsigma_{q+1})}\bar{\Psi}_{q+1}^{r}(\varsigma_{\mathbf{q+1}})\Delta\varsigma_{q+1}\right\}\Delta\varsigma_q\cdots\Delta\varsigma_1$$
$$\geq\left(\frac{rK_{q+1}^{c_{q+1}}}{c_{q+1}-1}\right)^{r}\int_{a_{q+1}}^{\infty}\frac{\lambda_{q+1}(\varsigma_{q+1})}{\Lambda_{q+1}^{c_{q+1}-r}(\varsigma_{q+1})}\left\{\int_{a_1}^{\infty}\cdots\int_{a_q}^{\infty}\prod_{\iota=1}^{q}\left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota}(\varsigma_\iota)}\right]\bar{\Psi}_q^{r}(\varsigma_{\mathbf{q+1}})\Delta\varsigma_q\cdots\Delta\varsigma_1\right\}\Delta\varsigma_{q+1}.$$

Use induction hypothesis for fixed $\varsigma_{q+1}\in\mathbb{T}_{q+1}$ on $\bar{\Psi}_q(\varsigma_{\mathbf{q+1}})$ instead of $\bar{\Psi}_q(\varsigma_{\mathbf{q}})$ to get

$$\int_{a_1}^{\infty}\cdots\int_{a_{q+1}}^{\infty}\left[\prod_{\iota=1}^{q+1}\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota(\varsigma_\iota))^{c_\iota}}\right]\bar{\Psi}_{q+1}^{r}(\varsigma_{+\mathbf{1}})\Delta\varsigma_{q+1}\cdots\ \Delta\varsigma_1$$
$$\geq\prod_{\iota=1}^{q+1}\left[\left(\frac{rK_\iota^{c_\iota}}{c_\iota-1}\right)^{r}\right]\int_{a_1}^{\infty}\cdots\int_{a_{q+1}}^{\infty}\prod_{\iota=1}^{q+1}\left[\frac{\lambda_\iota(\varsigma_\iota)}{\Lambda_\iota^{c_\iota-r}(\varsigma_\iota)}\right]g^{r}(\varsigma_{\mathbf{q+1}})\Delta\varsigma_{q+1}\cdots\Delta\varsigma_1.$$

Hence by the principle of mathematical induction inequality (4. 27) is true for all positive integers n. □

Remark 4.3. *It is possible to provide examples similar to Example 3.2–Example 3.4 for Theorem 3.5, Theorem 4.1 and Theorem 4.2 by using special time scales.*

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