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The Good Property of Two-Generated Ideals in Integral Domains

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Abstract. In this paper, we introduce and study a class of integral domains D characterized by the property that whenever $r, s \in D - \{0\}$ and the ideal (r^k, s^k) is principal for some $k \in \mathbb{N}$, then the ideal (r, s) is principal. We call them Good domains. We show that a Good domain D is a root closed domain and the converse is true in different cases as follows: (1) D is quasi-local, (2) $Pic(D) = 0$, (3) $u^{1/k} \in D$ for all $u \in D$ and $k \in \mathbb{N}$, (4) D is t-local. We also show that a quasi-local domain D with the property that $(r, s)^k = (r^k, s^k)$ for all $r, s \in D - \{0\}$ and $k \in \mathbb{N}$, is a Good domain, that a Prüfer Good domain with torsion Picard group is a B$\acute{e}$zout domain, and that the integral closure of a domain in an algebraically closed field is a Good domain.

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1. Introduction

In [6], Judith D. Sally showed that for J any ideal of a quasi-local ring R, if for some integer $q \geq 1$, $v(J^q) = 1$, then either $v(J^k) = 1$ for all positive integers k or J consists of zero divisors. If for some integer $q > 1$, $v(J^q) = 2$, then either $v(J^k) = 2$ for all positive integers k or J consists of zero divisors, where *v(J)* denotes the minimal number, which may be infinite, of generators of J.

In [2], Gerhard Angermuller introduced n-root closed domains. He called a domain D with quotient field K n-root closed if whenever $x \in K$ with $x^n \in D$ for an integer $n \geq 1$, then $x \in D$; D is called root closed if D is n-root closed, for all $n > 1$. Obviously, any integrally closed domain is a root closed domain. The converse is not true. He showed that if q is a square-free integer, then $Z[\sqrt{q}]$ is non-integrally closed and root closed iff $q \equiv 1$ (mod 8).

In [1], D.D. Anderson and M. Zafrullah introduced almost Bézout domains and almost Prüfer domains. They called D an almost Bézout domain (AB-domain) if for every $r, s \in D - \{0\}$, there exists a positive integer $k = k(r, s)$ such that (r^k, s^k) is principal, and D is an almost Prüfer domain (AP-domain) if for every $r, s \in D - \{0\}$, there exists a positive integer $k = k(r, s)$ such that (r^k, s^k) is invertible. They showed that D is an almost Bézout domain iff D is an almost Prüfer domain with torsion class group. They also showed that an integrally closed AB-domain (respectively, almost AP-domain) is a Prüfer domain with torsion class group (respectively, Prüfer domain).

In this paper, we study the following good property of two-generated ideals in integral domains. We call an integral domain D a *Good domain* if whenever $r, s \in D - \{0\}$ and the ideal (r^k, s^k) is principal for some $k \in \mathbb{N}$, then the ideal (r, s) is principal.

We show that a Good domain is a root closed domain (Proposition 3). The converse holds in some cases: (1) D is quasi-local (Proposition 5), (2) $Pic(D) = 0$ (Corollary 6), (3) $u^{1/k} \in D$ for all $u \in D$ and $k \in \mathbb{N}$ (Corollary 9), (4) D is t-local (Proposition 10). If D is a root closed domain with $(r^k, s^k) = (u^k)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$, then $(r, s) = (u)$ (Proposition 8). A quasi-local domain with the property that $(r, s)^k$=(r^k, s^k) for all $r, s \in D$ and for all $k \in \mathbb{N}$, is a Good domain (Proposition 11). An almost Bézout Good domain is a Bézout domain (Proposition 13). A Prüfer Good domain with torsion Picard group is a Bézout domain (Proposition 14). The integral closure of a domain in an algebraically closed field is a Good domain (Proposition 15).

For the reader's convenience, we give a working introduction here for the notions involved. Let D be an integral domain with quotient field K, and let $F(D)$ denote the set of nonzero fractional ideals of D.

A function $A \mapsto A^* : F(D) \to F(D)$ is called a *star operation* on D if $*$ satisfies the following three conditions for all $0 \neq x \in K$ and for all $A, B \in F(D)$: (1) $D^* = D$ and $(xA)^* = xA^*$, (2) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$, (3) $(A^*)^* = A^*$. An ideal $A \in F(D)$ is called a $*$-ideal if $A^* = A$. For all $A, B \in F(D)$, we have $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations define the so-called $*$*-multiplication*. If $\{A_\alpha\}$ is a subset of $F(D)$ such that $\cap A_\alpha \neq 0$, then $\cap A_\alpha^*$ is a $*$-ideal. Also, if $\{A_\alpha\}$ is a subset of $F(D)$ such that $\sum A_\alpha$ is a fractional ideal, then $(\sum A_\alpha)^* = (\sum A_\alpha^*)^*$. The function $*_f : F(D) \to F(D)$ given by $A^{*_f} = \cup B^*$, where B ranges over all nonzero finitely generated sub-ideals of A, is also a star operation; $*$ is said to be a star operation of *finite character* if $* = *_f$. Clearly $(*_f)_f = *_f$. Let $Max_*(D)$ denote the set of maximal $*$-ideals, that is, ideals maximal among proper integral $*$-ideals of D. If $*$ is of finite character, then every proper $*$-ideal is contained in some maximal $*$-ideal and every maximal $*$-ideal is a prime ideal. A $*$-ideal A is of *finite type* if $A = (x_1, ..., x_n)^*$ for some $x_1, ..., x_n \in A$. An ideal $A \in F(D)$ is said to be $*$*-invertible* if $(AA^{-1})^* = D$, where $A^{-1} = (D : A) = \{x \in K \mid xA \subseteq D\}$. If $*$ is of finite character, then A is $*$-invertible if and only if AA^{-1} is not contained in any maximal $*$-ideal of D; in this case $A^* = (x_1, ..., x_n)^*$ for some $x_1, ..., x_n \in A$. Some well-known star operations are: the d*-operation* (given by $A \mapsto A$), the v*-operation* (given by $A \mapsto A_v = (A^{-1})^{-1}$) and the t*-operation* (defined by $t = v_f$). Call A a *v-ideal* if $A = A_v$ and a *t-ideal* if $A = A_t$. For every $A \in F(D)$, we have $A \subseteq A_t \subseteq A_v$; so a v-ideal is a t-ideal. The fractional ideal A is invertible (resp., t-invertible) if $AB = D$ (resp., $(AB)_t = D$) for some fractional ideal B. The *Picard group* of D, $Pic(D)$, is the multiplicative group of invertible fractional ideals of D modulo the subgroup of principal ideals. The *t-class group* of D, $Cl_t(D)$, is the group of all t-invertible fractional t-ideals of D under t-multiplication (i.e., the operation sending a pair of t-ideals A, B of D to $(AB)_t$) modulo the subgroup of principal ideals [3]. $Pic(D)$ is a subgroup of $Cl_t(D)$.

Throughout this paper, we denote the integral closure of a domain D by D' and the quotient field of a domain D by K. Our standard reference for any undefined notation or terminology is [4].

2. Good domain

Remark 1. Let D be a Dedekind domain with torsion class group, which is not a PID. Then there exists a two-generated ideal (r,s) of D that is not principal, but the ideal $(r,s)^k$ of D is principal for some $k \in \mathbb{N}$. Now in a Prüfer domain, $(r,s)^k = (r^k, s^k)$ for all $r, s \in D$ [4, Theorem 24.3]. For example, let $D = \mathbb{Z}[\sqrt{-5}]$. The ring $\mathbb{Z}[\sqrt{-5}]$ is known to be a non-PID Dedekind domain such as $Cl(\mathbb{Z}[\sqrt{-5}])$=$\mathbb{Z}/2\mathbb{Z}$. Since $Cl(\mathbb{Z}[\sqrt{-5}]) \neq 0$, there is a prime ideal P of $\mathbb{Z}[\sqrt{-5}]$ which is not principal. Now every ideal of a Dedekind domain which is not principal is always generated by two elements [4, Theorem 38.5]. So we can take $P = (r,s)$. Since $|Cl(\mathbb{Z}[\sqrt{-5}])| = 2$, we must have $P^2 = (r^2, s^2)$ principal. For instance, take $r = 2$ and $s = 1 + \sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$. Then $(2, 1+\sqrt{-5})$ is non-principal and $(2^2, (1+\sqrt{-5})^2)$ =(2) in $\mathbb{Z}[\sqrt{-5}]$.

Example 2. Let F be a field with characteristic $m \neq 0$, L be a purely inseparable extension of F such that $L^m \subset F$ and X be an indeterminate over L. Define $D = F + XL[X] = \{a_0 + \sum_{i=1}^{n} a_i X^i : a_0 \in F \text{ and } a_i \in L\}$. By [7, Example 2.13], it is clear that D is a non-integrally closed AB-domain. Now let $l_1, l_2 \in L/F$ such that $l_1/l_2 \notin F$. Then $(l_1X, l_2X)D$ is non-principal, but $(l_1^m X^m, l_2^m X^m) = (X^m)$ in $F[X]$ and so in D.

Proposition 3. *A Good domain is a root closed domain.*

Proof. Let D be a Good domain and $x \in K - \{0\}$ with $x^k \in D$ for some $k > 1$. Say $x = r/s$, where $r, s \in D - \{0\}$; so $x^k = r^k/s^k$. Now $r^k/s^k \in D$ implies that $s^k | r^k$, which gives (r^k, s^k) is principal and so gives (r,s) is principal. We claim (r,s)=(s). Suppose not and let (r,s)=(d). Then $r = ad$ and $s = bd$, where (a,b)=D. Since $s^k | r^k$, we have $b^k d^k | a^k d^k$ implies that $b^k | a^k$. But this can happen only if b^k, and hence b is a unit. Thus $(d) = (s)$ and $(r,s) = (s)$ implies that $s | r$, so $x = r/s \in D$. Hence D is root closed. ☐

Remark 4. *The converse of Proposition 3 is false. Indeed,* $\mathbb{Z}[\sqrt{-5}]$ *is an integrally closed domain and so is also root closed* [2]*. Note that the ideal* $(2, 1+\sqrt{-5})$ *is not principal but* $(2^2, (1+\sqrt{-5})^2)$=$(4, -4+2\sqrt{-5})$=$(4, -4+2\sqrt{-5}+4)$=$(4, 2\sqrt{-5})$=$2(2, \sqrt{-5})$ *=2D=*(2)*. Hence* D *is not a Good domain.*

Proposition 5. *A root closed quasi-local domain is a Good domain.*

Proof. Let D be a root closed quasi-local domain, and let $(r^k, s^k) = (u)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Since D is quasi-local, so $(u) = (r^k)$ or $(u) = (s^k)$ implies that $r^k \mid s^k$ or $s^k \mid r^k$. As D is also root closed, so $r \mid s$ or $s \mid r$. Finally, $(r,s) = (r)$ or $(r,s) = (s)$. Hence D is a Good domain. ☐

Corollary 6. *Let* D *be a root closed domain and* $r, s \in D - \{0\}$*. If the ideal* (r^k, s^k) *is principal for some* $k \in \mathbb{N}$*, then the ideal* (r,s) *is invertible. In particular, a root closed domain* D *with* $Pic(D) = 0$ *is a Good domain.*

Proof. Since D is root closed, D_M is also root closed for all $M \in Max(D)$ [2, Lemma 2]. Now (r^k, s^k) is principal implies that $(r^k, s^k)D_M$ is principal. By Proposition 5, $(r, s)D_M$ is principal. Thus (r, s) is locally principal, and hence invertible [5, Theorem 62]. □

Remark 7. Corollary 6 can be used to give an example of a Good domain which is not quasi-local. Indeed, if S is a subfield of another field L, then the t-class group of a domain $S + XL[X]$ is zero [3, Example 1.10]. As the Picard group is a subgroup of the t-class group, we have $Pic(S + XL[X]) = 0$. Now it is well known that $S + XL[X]$ is not a quasi-local domain. So, if $S + XL[X]$ is root closed then by Corollary 6, $S + XL[X]$ is a Good domain. For instance, let L be an algebraic closure of $\mathbb{Q}$ and let S be the subfield of L consisting of all elements θ of L such that the minimal polynomial for θ over $\mathbb{Q}$ is solvable by radicals over $\mathbb{Q}$. Define $D = S + XL[X]$. Then by [4, Exercise 6, Page 184], D is a root closed domain with $Pic(D) = 0$, which is not a quasi-local domain. Hence by Corollary 6, D is a Good domain.

Proposition 8. *If D is a root closed domain with $(r^k, s^k) = (u^k)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$, then $(r, s) = (u)$.*

Proof. Let $(r^k, s^k) = (u^k)$ for some $r, s, u \in D-\{0\}$ and $k \in \mathbb{N}$ implies that $u^k|r^k, u^k|s^k$. Since D is root closed, so $u|r, u|s$. Then $r = au$, $s = bu$ for some $a, b \in D$, where $(a, b) = D$. Hence $(r, s) = (u)$. □

Corollary 9. *If D is a root closed domain in which $u^{1/k} \in D$ for all $u \in D$ and $k \in \mathbb{N}$, then D is a Good domain.*

Proof. Let $(r^k, s^k) = (u) = ((u^{1/k})^k)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Then by Proposition 8, $(r, s) = (u^{1/k})$. Hence D is a Good domain. □

Recall from [1] that a domain D is called a *t-local domain* if D has a unique maximal t-ideal, equivalently, if D has a unique maximal ideal M which is also a t-ideal.

Recall from [7] that $r, s \in D - \{0\}$ are called v-coprime if $(r, s)_v = D$.

Proposition 10. *Let D be a t-local domain. Then the following assertions hold:*
(1) *Any two nonzero nonunit elements of D are not v-coprime.*
(2) *If $(r, s)_v = (u)$ for some $r, s, u \in D - \{0\}$, then either $(u) = (r)$ or $(u) = (s)$.*
(3) *If D is a root closed domain, then D is also a Good domain.*

Proof. (1) Let D be a t-local domain with maximal ideal M, and let $r, s \in D - \{0\}$ be nonunits. Then $(r, s) \subseteq M$ implies that $(r, s)_v \subseteq M$. Hence r, s are not v-coprime.

(2) Let $(r, s)_v = (u)$ for some $r, s, u \in D - \{0\}$. Then $(r/u, s/u)_v = D$ implies by (1) that r/u or s/u is a unit. Therefore, $(r/u) = D$ or $(s/u) = D$ gives $(u) = (r)$ or $(u) = (s)$.

(3) Let $(r^k, s^k) = (u)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Then by (2), $(u) = (r^k)$ or $(u) = (s^k)$ gives $r^k \mid s^k$ or $s^k \mid r^k$. Since D is root closed; so $r \mid s$ or $s \mid r$ implies that $(r, s) = (r)$ or $(r, s) = (s)$. Hence D is a Good domain. □

Proposition 11. *If D is a quasi-local domain with property $P : (r, s)^k = (r^k, s^k)$ for all $r, s \in D - \{0\}$ and for all $k \in \mathbb{N}$, then D is a Good domain.*

Proof. Let $(r^k, s^k) = (u)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Then by property P, $(r, s)^k = (r^k, s^k) = (u)$ implies that (r, s) is principal [6]. Hence D is a Good domain. □

Corollary 12. *If D is a domain with property $P : (r,s)^k = (r^k, s^k)$ for all $r, s \in D - \{0\}$ and for all $k \in \mathbb{N}$, then D is root closed.*

Proof. Suppose D has property P. Then D locally also has property P. Therefore, by Proposition 11, Proposition 3, and [2, Lemma 2], D is root closed. □

Recall from [1] that a domain D is called an *almost Bézout domain (AB-domain)* if for each pair $r, s \in D - \{0\}$, there exists a positive integer $k = k(r,s)$ such that (r^k, s^k) is principal.

Proposition 13. *A domain D is a Bézout domain if and only if it is an almost Bézout and a Good domain.*

Proof. Clearly a Bézout domain is an almost Bézout and a Good domain. Conversely, let $r, s \in D - \{0\}$. Since D is an almost Bézout domain, there exists a positive integer $k = k(r,s)$ such that (r^k, s^k) is principal. As D is also a Good domain, we get that (r,s) is principal. Hence D is a Bézout domain. □

Proposition 14. *If D is a Prüfer Good domain with torsion Picard group, then D is a Bézout domain.*

Proof. Let $r, s \in D - \{0\}$. Since D is a Prüfer domain with torsion Picard group, there exists $k \in \mathbb{N}$ such that $(r,s)^k = (r^k, s^k) = (u)$ for some $u \in D$. As D is also a Good domain, we get that (r,s) is principal. Hence D is a Bézout domain. □

Proposition 15. *The integral closure of a domain D in an algebraically closed field is a Good domain.*

Proof. Let $E = D'_L$, where L is an algebraically closed field containing the quotient field of D, and let $(e^k, f^k) = (g)$ for some $e, f, g \in E - \{0\}$ and $k \in \mathbb{N}$. Take $h = g^{1/k}$. Since E being integrally closed is also root closed and $h^k \in E$, then $h \in E$. We have $(e^k, f^k) = (h^k)$ implies that $h^k \mid e^k, h^k \mid f^k$ in E. As E is integrally closed, we have $h \mid e, h \mid f$ in E. Say $e = xh$ and $f = yh$ for some $x, y \in E$ with $(x,y) = D$, this gives $(xh, yh) = hD = (h)$, which implies that $(e,f) = (h)$. Hence E is a Good domain. □

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