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Monogenity of Biquadratic Fields Related to Dedekind-Hasse's Problem

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Abstract. The aim of this paper is to determine the monogenity of imaginary, and real biquadratic fields K over the field $\boldsymbol{Q}$ of rational numbers and the relative monogenity of K over its quadratic subfield k. To characterize such phenomena it is necessary to determine an integral basis of the field K and to evaluate the relative norm of the different $\mathfrak{d}(\xi)$ with respect to K/k of an integer ξ in K. Here $\mathfrak{d}(\xi)$ is defined by $\prod_{\rho\in G\setminus\{\iota\}}(\xi-\xi^{\rho})$, where $\xi-\xi^{\rho}$ denotes the partial different of an integer ξ in K, and G and ι denote the Galois group of $K/\boldsymbol{Q}$ and the identity embedding of K, respectively. For the succinct proof of non-monogenity, we consider a single linear Diophantine equation consisted of the partial differents with unit coefficients.

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1. Introduction

In the 1960's Hasse proposed to characterize an algebraic number field K whose ring Z_K of integers has a power integral basis or not. Let p be a prime number and ζ_{p^e} be a primitive p^eth root of unity, which is a root of an irreducible cyclotomic polynomial $\Phi_{p^e}(x) = (x^{p^e} - 1)/(x^{p^{e-1}} - 1)$ over $\boldsymbol{Q}$ with $\zeta_2 = -1$, $\zeta_3 = (-1 + \sqrt{-3})/2$, $\zeta_4 = \sqrt{-1}$ and $\zeta_{p^e} = \exp(2\pi i/p^e), p \geqq 2, e \geqq 1$, Then for the Eisenstein field $k_3 = \boldsymbol{Q}(\zeta_3) = \boldsymbol{Q}(\sqrt{-3})$, the Gauß field $k_4 = \boldsymbol{Q}(\sqrt{-1})$ and a cyclotomic field $k_{p^e} = \boldsymbol{Q}(\zeta_{p_e})$, it is known that $Z_{k_3} = \boldsymbol{Z}[1, \zeta_3]$, $Z_{k_4} = \boldsymbol{Z}[1, \sqrt{-1}]$ and $\boldsymbol{Z}_{k_{p^e}} = \boldsymbol{Z}[1, \zeta, \cdots, \zeta^{p^{e-1}(p-1)-1}]$ with $\zeta = \zeta_{p^e}$ as a $\boldsymbol{Z}-$free module of rank $p^{e-1}(p-1)$ [15]. Each of the fields is called monogenic. For an algebraic number field K over the rationals $\boldsymbol{Q}$, Z_K denotes the ring of integers in K. Let $\boldsymbol{Q} \subset F \subset K$ be an algebraic number field tower. It is said that a field K is relatively monogenic in the relative field extension K/F of degree n or equivalently, Z_K has a power integral basis of rank n over Z_F if for a suitable integer $\alpha \in Z_K, Z_K$ coincides with the Z_F-module $Z_F[\alpha] = Z_F \cdot 1 + Z_F \cdot \alpha + \cdots + Z_F \cdot \alpha^{n-1}$ of rank n over Z_F. In the case of $F = \boldsymbol{Q}$, we say that K is monogenic or Z_K has a power integral basis [4]. Then to determine the monogenity of $Z_K = Z_F[\alpha]$ with a suitable integer α or $Z_K \neq Z_F[\beta]$ for any integer β in F is called Dedekind-Hasse's problem. Let d_F and $d_F(\alpha)$ denote the field discriminant and the discriminant of a number α in F and the Index $\mathrm{Ind}_F(\alpha)$ of a number α is defined by $\sqrt{\frac{|d_F(\alpha)|}{|d_F|}}$. It is known that Dedekind's example of a cubic field $K = \boldsymbol{Q}(\theta)$ is non-monogenic, where θ satisfies a cubic irreducible equation; $x^3 - x^2 - 2x - 8 = 0$ with the discriminant $d_K(\theta) = \{(\theta - \theta^{\sigma})(\theta - \theta^{\sigma^2})(\theta^{\sigma} - \theta^{\sigma^2})\}^2 = 2^2 \cdot (-503)$ of θ and a non-trivial conjugate map σ on K [3]. In fact, $\{1, \theta, \eta\}$ with $\eta = \theta(\theta - 1)/2$ is an integral basis of K and it holds that ${}^t(1\ \theta\ \theta^2) = ({}^t(1\ 0\ 0)\ {}^t(0\ 1\ 0)\ {}^t(0\ 1\ 2)) \cdot {}^t\ (1\ \theta\ \eta) = A\boldsymbol{\eta}$, where $\boldsymbol{\eta} = {}^t\ (1\ \theta\ \eta)$ and ${}^tB$ denotes the transposed of a matrix B, and hence $d_K(\theta) = (\det A)^2 d_K$. Here the field discriminant d_K is defined by $\det({}^t\boldsymbol{\eta}\ {}^t\boldsymbol{\eta}^{\sigma}\ {}^t\boldsymbol{\eta}^{\sigma^2})^2$ and $\mathrm{Ind}_K(\theta) = |\det A| = 2$. Then it follows that the ring $\boldsymbol{Z}[1, \theta, \theta^2]$ is a proper subring of Z_K. Moreover for any integer $\xi = x + y\theta + z\eta$, we know that $\mathrm{Ind}_K(\xi) \equiv 0 \pmod 2$, namely Z_K has no power integral basis. In this paper we consider the problem on a family of imaginary, and real biquadratic fields $K = \boldsymbol{Q}(\sqrt{DM}, \sqrt{DN})$, where DMN is a square free integer with $1 \leqq |D|, 1 < N, M$ as an analogue of a work by Y. Motoda [9].

Theorem 1.1. *Let K be a biquadratic field $\boldsymbol{Q}(\sqrt{DM}, \sqrt{DN})$, where DMN is square free with $DM \equiv DN \equiv 3, MN \equiv 1 \pmod 4$ and $1 \leqq |D|, 1 < N, 1 < M$. Then K has an integral basis $\boldsymbol{Z}[1, \sqrt{DM}, \frac{1+\sqrt{MN}}{2}, \frac{\sqrt{DM}+\sqrt{DN}}{2}]$ with the field discriminant $d_K = 2^4 D^2 M^2 N^2$ and Z_K has a relative power integral basis $Z_k[1, \sqrt{DM}]$ over Z_k with a quadratic subfield $k = \boldsymbol{Q}(\omega)$ and $\omega = \frac{1+\sqrt{MN}}{2}$. But, if $4D \pm M \pm N \neq 0$ holds, then Z_K has no power integral basis.*

Corollary 1.2. *There exist infinitely many non-monogenic biquadratic fields.*

Corollary 1.3. *Using the same notation as in Theorem 1.1, there exist monogenic biquadratic fields for $D = \pm 1$, $M - N = \pm 4$.*

Our theorem gives a negative solution to the problem 6 in [11]. An explicit integral basis of any biquadratic field K is shown in K. S. Williams using evaluation modulo powers of 2 without the process of a relative extension $K/k/\boldsymbol{Q}$ for a quadratic subfield k of K [16]. On the family of imaginary biquadratic fields K with $D < 0$ a complete classification of monogenity has been given by G. Nyul using the evaluation of the full norm of the different $\mathfrak{d}_K(\xi)$ for any element $\xi \in K$ [12]. On the contrary, based on the works of M.-N. Gras, F. Tanoè, it is shown that there exist infinitely many real monogenic biquadratic fields not depending on Dirichlet's theorem on arithmetic progression [9, 4]. We prove our theorem by the consideration of the relative norm with respect to K/k of partial differents $\xi - \xi^{\rho}$ of the different $\mathfrak{d}(\xi)$ of an integer ξ, and a single linear Diophantine equation consisted of three relative norms of the partial differents with unit coefficients. Here $\mathfrak{d}(\xi)$ is defined by $\prod_{\rho \in G \setminus \{\iota\}}(\xi - \xi^{\rho})$ with Galois group G of $K/\boldsymbol{Q}$ and the identity embedding ι of K for a family of certain biquadratic fields. Related works are found in [1, 2, 5, 6, 8, 10, 13, 14].

2. Integral bases

Let K be a biquadratic field $\boldsymbol{Q}(\sqrt{DM}, \sqrt{DN})$ with a square free DMN, $1 \leqq |D|, 1 < N$, $1 < M$ and $DM \equiv DN \equiv 3 \pmod 4$, $MN \equiv 1 \pmod 4$. Let k be a quadratic subfield $\boldsymbol{Q}(\sqrt{DM})$. Then it holds that $K = k[1, \omega] = \boldsymbol{Q}[1, \sqrt{DM}, \omega, \gamma_0]$ with $\omega = \frac{1+\sqrt{MN}}{2}$ and $\gamma_0 = \sqrt{DM}\omega = \frac{\sqrt{DM}+M\sqrt{DN}}{2}$.
Let $k = \boldsymbol{Q}(\sqrt{DM}), k_1 = \boldsymbol{Q}(\sqrt{MN})$ and $k_2 = \boldsymbol{Q}(\sqrt{DN})$ be the quadratic subfields of K. Let $G = \mathrm{Gal}(K/\boldsymbol{Q})$ be the Galois group of K over $\boldsymbol{Q}$ generated by embeddings σ and τ. Let $H_k =< \sigma >, H_{k_1} =< \tau >$ and $H_{k_2} =< \sigma\tau >$ be the Galois subgroups corresponding to subfields k, k_1 and k_2 of K, respectively. Then it holds that

$$\begin{aligned} \sigma &: \sqrt{DM} \mapsto \sqrt{DM}, \quad \sqrt{MN} \mapsto -\sqrt{MN}, \quad \sqrt{DN} \mapsto -\sqrt{DN}, \\ \tau &: \sqrt{DM} \mapsto -\sqrt{DM}, \quad \sqrt{MN} \mapsto \sqrt{MN}, \quad \sqrt{DN} \mapsto -\sqrt{DN}, \\ \sigma\tau &: \sqrt{DM} \mapsto -\sqrt{DM}, \quad \sqrt{MN} \mapsto -\sqrt{MN}, \quad \sqrt{DN} \mapsto \sqrt{DN}. \end{aligned}$$

First we show that an integral basis of Z_K is explicitely determined. For an integer $\xi \in Z_K$ there exist coefficients $a, b, c, d \in \boldsymbol{Q}$ such that $\xi = a + b\sqrt{DM} + c\omega + d\gamma_0$. If $c = d = 0$ holds, then $\xi = a + b\sqrt{DM} \in Z_K \cap k$, and hence $a, b \in \boldsymbol{Z}$ holds by $Z_K \cap k = Z_k = \boldsymbol{Z}[1, \sqrt{DM}]$. Put $\xi_1 = \xi - a - b\sqrt{DM}$ with $a, b \in \boldsymbol{Z}$. Then $\xi_1 = c\omega + d\gamma_0 \in Z_K$ holds. If we choose $d = 0$, then $c \in \boldsymbol{Z}$. Put $\xi_2 = \xi_1 - c\omega$ with $c \in \boldsymbol{Z}$. By $\xi_2 = d\frac{\sqrt{DM}+M\sqrt{DN}}{2} = d\frac{M-1}{2}\sqrt{DN} + d\frac{\sqrt{DM}+\sqrt{DN}}{2}$, $\xi_2 - d\frac{M-1}{2}\sqrt{DN} = d\frac{\sqrt{DM}+\sqrt{DN}}{2}$, which is denoted by ξ_3 should belong to Z_K as $d\frac{M-1}{2} \in \boldsymbol{Z}$. Put $\gamma = \frac{\sqrt{DM}+\sqrt{DN}}{2}$. Thus by $T_{K/k}(\xi_3) = d\gamma + d\gamma^{\sigma} = d\sqrt{DM} \in Z_k, d \in \boldsymbol{Z}$ is deduced. Here, $T_{K/k}(\cdot)$ means the relative trace with respect to K/k. Put $Z'_K = \boldsymbol{Z}[1, \sqrt{DM}, \omega, \gamma]$. Therefore if $\xi \in Z_K$, it holds that $\xi \in Z'_K$, namely $Z_K \subsetneqq Z'_K$.
Conversely for any $\xi = s + t\sqrt{DM} + u\omega + v\gamma \in Z'_K$ with $s, t, u, v \in \boldsymbol{Z}$, we have $T_{K/k}(\xi) = \xi + \xi^{\sigma}$ $= 2s + 2t\sqrt{DM} + u + v\sqrt{DM} \in Z_k$ and $N_{K/k}(\xi) = \xi\xi^{\sigma} \in Z_k$, namely, $4N_{K/k}(\xi) = 2\xi \cdot 2\xi^{\sigma}$ $= (2s + u + (2t + v)\sqrt{DM})^2 - (u\sqrt{MN} + v\sqrt{DN})^2 \equiv (u^2 + v^2DM + 2uv\sqrt{DM}) - (u^2MN + 2uvN\sqrt{DM} + v^2DN) \equiv 0 \pmod{4Z_k}$. Here, $N_{K/k}(\cdot)$ means the relative norm with respect to K/k. In fact, because of $u^2(1 - MN) + v^2D(-M + N) \equiv 0 \pmod 4$ and $2uv(1 - N) \equiv 0 \pmod 4$, we obtain $\xi \in K \cap \tilde{\boldsymbol{Z}} = Z_K$. Here $\tilde{\boldsymbol{Z}}$ denotes the ring of integral closure over $\boldsymbol{Q}$. Thus $Z'_K \subsetneqq Z_K$ holds. Then for a biquadratic field K, Z_K coincides with $\boldsymbol{Z}[1, \sqrt{DM}, \frac{1+\sqrt{MN}}{2}, \frac{\sqrt{DM}+\sqrt{DN}}{2}]$. □

3. Relative monogenity of a biquadratic field over a quadratic subfield

Assume that $Z_K = Z_{k_1}[1, \eta]$ over Z_{k_1} for $Z_{k_1} = \boldsymbol{Z}[1, \omega]$ and $\eta = a + b\sqrt{DM}$ with $a, b \in \boldsymbol{Q}$. Thus $Z_K = \boldsymbol{Z}[1, \frac{1+\sqrt{MN}}{2}][1, a + b\sqrt{DM}]$

$= \boldsymbol{Z}[1, \frac{1+\sqrt{MN}}{2}, a+b\sqrt{DM}, a(\frac{1+\sqrt{MN}}{2}) + b(\frac{\sqrt{DM}+\sqrt{DN}}{2})]$ be a free module of rank 4 over $\boldsymbol{Q}$. Then we show that Z_K has a relative integral basis over Z_{k_1}. Let $d_K(\alpha,\beta,\gamma,\delta)$ be the discriminant $\det\,({}^t\boldsymbol{\alpha}, {}^t\boldsymbol{\beta}, {}^t\boldsymbol{\gamma}, {}^t\boldsymbol{\delta})^2$ with a column vector $\boldsymbol{\mu} = (\mu, \mu^{\sigma}, \mu^{\tau}, \mu^{\sigma\tau})$.
Then by $d_K(\alpha,\beta,\gamma,a\beta+b\delta) = d_K(\alpha,\beta,\gamma,b\delta)$, it follows that

$$\begin{aligned} & d_K(1,\omega,a+b\sqrt{DM}, a\omega + b\tfrac{\sqrt{DM}+\sqrt{DN}}{2}) \\ &= d_K(1,\omega,b\sqrt{DM}, b\tfrac{\sqrt{DM}+\sqrt{DN}}{2}) \\ &= b^{2+2} d_K(1,\omega,\sqrt{DM}, \tfrac{\sqrt{DM}+\sqrt{DN}}{2}) \\ &= b^4 d_{k_1}(1,\omega)(2^2 D\sqrt{MN})^2 \\ &= b^4 MN\cdot 2^4 D^2 MN \\ &= b^4\cdot 2^4 D^2 M^2 N^2. \end{aligned}$$

Thus for $\eta = a+b\sqrt{DM}$ with $a=0, b=1$, Z_K has a relative power integral basis $\{1,\eta\}$ over Z_{k_1}. □

4. MONOGENITY OF A BIQUADRATIC FIELD

Let K be an imaginary, or real biquadratic field $\boldsymbol{Q}(\sqrt{DM}, \sqrt{DN})$ with positive square free relatively prime integers $|D| \geqq 1, N>1, M>1$ and $DM \equiv DN \equiv 3, MN \equiv 1 \pmod 4$. Let $k = \boldsymbol{Q}(\sqrt{DM})$ and $k_2 = \boldsymbol{Q}(\sqrt{DN})$ be quadratic subfields of K and $k_1 = \boldsymbol{Q}(\sqrt{MN})$ be a real one. Let $G(K/\boldsymbol{Q})$ be the Galois group G of K over $\boldsymbol{Q}$ generated by embeddings σ and τ. Let the subfields k, k_1 and k_2 of K have corresponding Galois subgroups $H_k = <\sigma>$, $H_{k_1} = <\tau>$ and $H_{k_2} = <\sigma\tau>$ in G, respectively. Let X denote the character group corresponding to $G(K/\boldsymbol{Q})$ generated by χ and λ, which denote primitive characters of order 2 defined by $\chi(\sigma)=-1, \chi(\tau)=1$ and $\lambda(\sigma)=\lambda(\tau)=-1$. By virtue of Hasse's conductor-discriminant formula, the field discriminant d_K of K coincides with

$$\prod_{\psi\in X} f_\psi = f_{\chi^0}\cdot f_\chi \cdot f_\lambda \cdot f_{\chi\lambda} = 1\cdot 2^2|DM|\cdot 2^2|DN|\cdot MN = 2^4\cdot D^2\cdot M^2\cdot N^2,$$

where f_ψ denote the conductor corresponding to a character ψ of X with the principal character χ^0 [7, 15]. Assume that the field K has a power integral basis for some suitable integer $\xi = a+b\sqrt{DM}+c\frac{1+\sqrt{MN}}{2}+d\frac{\sqrt{DM}+\sqrt{DN}}{2}$ in K such that

$$Z_K = \boldsymbol{Z}[\xi] = \boldsymbol{Z}[1,\xi,\xi^2,\xi^3].$$

For an algebraic number field tower $\boldsymbol{Q}\subset F\subset L$ with the Galois group $G = G(L/\boldsymbol{Q})$, the field different $\mathfrak{d}_L$ is defined as an ideal

$$(\beta-\beta^\rho; \forall\beta\in Z_L, \forall\rho\in G(L/\boldsymbol{Q}))$$

of L, and the relative field different $\mathfrak{d}_{L/F}$ as an ideal

$$(\gamma-\gamma^\rho; \forall\gamma\in Z_L, \forall\rho\in G(L/F))$$

of L/F. By the assumption $Z_K = \boldsymbol{Z}[\xi]$, it holds that

$$(d_K(\xi)) = (N_K(\mathfrak{d}_K(\xi))) = (\mathrm{N}_K(\mathfrak{d}_K)) = (d_K),$$

where (α) means the principal ideal generated by a number α in K and $N_F(\alpha)$, $\mathrm{N}_F(\mathfrak{a})$ are the norms of α and of $\mathfrak{a}$ with respect to $F/\boldsymbol{Q}$, respectively. Hence for the biquadratic field K, the different $\mathfrak{d}_K(\xi)$ of an element $\xi\in Z_K$ is given by $(\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau})$. Thus it holds that

$N_K(\mathfrak{d}_K(\xi)) = N_K((\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau})) = N_k(N_{K/k}((\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau})))$
$= N_k(((\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau}))((\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau}))^\sigma)$
$= ((\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau})\ ((\xi^\sigma-\xi)(\xi^\sigma-\xi^{\sigma\tau})(\xi^\sigma-\xi^\tau))$
$\cdot((\xi-\xi^\sigma)(\xi-\xi^\tau)(\xi-\xi^{\sigma\tau}))\ ((\xi^\sigma-\xi)(\xi^\sigma-\xi^{\sigma\tau})(\xi^\sigma-\xi^\tau))^\tau$
$= ((\xi-\xi^\sigma)(\xi-\xi^\sigma)^\tau(\xi-\xi^\tau)(\xi-\xi^\tau)^\sigma(\xi-\xi^{\sigma\tau})(\xi-\xi^{\sigma\tau})^\sigma)^2$
and hence

$$d_K(\xi) = (N_{K/k_1}(\xi-\xi^\sigma)N_{K/k}(\xi-\xi^\tau)N_{K/k_2}(\xi-\xi^{\sigma\tau}))^2.$$

Here we have $(\xi-\xi^\sigma)(\xi-\xi^\sigma)^\tau \in F_{<\sigma,\tau>} = k\cap k_1 = \boldsymbol{Q}$, $(\xi-\xi^\tau)(\xi-\xi^\tau)^\sigma \in F_{<\sigma,\tau>} = k\cap k_1 = \boldsymbol{Q}$
and $(\xi-\xi^{\sigma\tau})\ (\xi-\xi^{\sigma\tau})^\sigma \in F_{<\sigma,\tau>} = k\cap k_2 = \boldsymbol{Q}$.
Now, for the candidate ξ of power integral basis in Z_K with

$$\xi = a + b\sqrt{DM} + c\frac{1+\sqrt{MN}}{2} + d\frac{\sqrt{DM}+\sqrt{DN}}{2},$$

we calculate the relative differents from the biquadratic field $K = \boldsymbol{Q}(\sqrt{DM},\sqrt{DN})$ to a suitable quadratic subfield as follows;

$\mathfrak{d}_{K/k}(\xi) = \xi-\xi^\sigma = c\sqrt{MN} + d\sqrt{DN} = \sqrt{N}(c\sqrt{M}+d\sqrt{D})$,
$\mathfrak{d}_{K/k_1}(\xi) = \xi-\xi^\tau = (2b+d)\sqrt{DM} + d\sqrt{DN}$,
$\mathfrak{d}_{K/k_2}(\xi) = \xi-\xi^{\sigma\tau} = (2b+d)\sqrt{DM} + c\sqrt{MN}$.

Then the relative norm N_{K/k_1} of the relative different $\mathfrak{d}_{K/k_1}(\xi)$ is given by

$$|N_{K/k}(\xi-\xi^\sigma)| = |(\xi-\xi^\sigma)(\xi-\xi^\sigma)^\tau| = N|(c^2M-d^2D)| \tag{4.1}$$

with any c and d. Next, we have

$$|N_{K/k}(\xi-\xi^\tau)| = |(\xi-\xi^\tau)(\xi-\xi^\tau)^\sigma| = |-(2b+d)^2(DM)+d^2DN| \equiv 0 \,(\mathrm{mod}\ 4D) \tag{4.2}$$

with any b and d. Finally, it holds that
$|N_{K/k_2}(\xi-\xi^{\sigma\tau})| = |(\xi-\xi^{\sigma\tau})(\xi-\xi^{\sigma\tau})^\sigma|$
$= |((2b+d)\sqrt{DM}+c\sqrt{MN})((2b+d)\sqrt{DM}+c\sqrt{MN})^\sigma| = |(2b+d)^2(DM)-c^2MN|$
$= M|(2b+d)^2D-c^2N|$ (4.3)

with any b, c and d. By the assuption $Z_K = \boldsymbol{Z}[\xi]$, from equations (4.1), (4.2) and (4.3), each norm of the partial factor $\xi-\xi^\sigma$, $\xi-\xi^\tau$ and $\xi-\xi^{\sigma\tau}$ should be equal to N, $4D$ and M modulo a unit, respectively. In fact we obtain the identity relation;

$$0 = (\xi-\xi^\sigma)(\xi-\xi^\sigma)^\tau - (\xi-\xi^\tau)(\xi-\xi^\tau)^\sigma + (\xi-\xi^{\sigma\tau})(\xi-\xi^{\sigma\tau})^\sigma$$

and hence

$$0 = N(c^2M-d^2D) - 4D\tfrac{-(2b+d)^2(M)+d^2N}{4} + M((2b+d)^2D-c^2N). \tag{4.4}$$

From (4.4) since each of the coefficients of $N, 4D, M$ is a unit in $\boldsymbol{Z}$ we obtain the linear Diophantine equation;

$$0 = N \pm 4D \pm M,$$

which contradicts to the assumption. Thus we have

$$|d_K(\xi)| > (N\cdot 2^2D\cdot M)^2 = 2^4D^2M^2N^2 = d_K.$$

Therefore $\mathrm{Ind}_K(\xi) > 1$ holds for $\mathrm{Ind}_K(\xi) = \sqrt{\frac{|d_K(\xi)|}{|d_K|}}$, which shows that Z_K does not have any power integral basis and hence K is non-monogenic. □

Proof of Corollary 1.2. Put $N = 8D_0t + M_0$ with a valuable t $(1 \leqq t)$ for $D = D_0 > 0$, $M = M_0 > 0$ and
$(8D_0, M_0) = 1$. Then there exist infinitely many prime numbers N by Dirichlet's theorem. □

Proof of Corollary 1.3. Let $D = \pm 1$, $N - M = 4D$, and $b = c = 0$, $d = 1$. Then by (4.1), (4.2) and (4.3), we obtain that the product is equal to $(M \cdot 4(\pm 1) \cdot N)^2 = 2^4 D^2 M^2 N^2$. □

Remark 4.1. By the next work it will be investigated on monogenity for a complete classification of the real biqadratic fields $\boldsymbol{Q}(\sqrt{DM}, \sqrt{DN})$ such that
(i) $D \equiv M \equiv N \equiv 1 \text{ or } 3 \pmod 4$
(ii) $DM \equiv DN \equiv 2 \pmod 4$ and $MN \equiv 3 \pmod 4$
and
(iii) $DM \equiv DN \equiv 2 \pmod 4$ and $MN \equiv 1 \pmod 4$.

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