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**Elementary Analysis of Lane-Emden Equation by Successive Differentiation Method**

Muhammad Khalid[1], Mariam Sultana[2], Fareeha Sami Khan[3]
[1,2,3]Department of Mathematical Sciences,
Federal Urdu University of Arts, Sciences & Technology, Karachi-75300, Pakistan.
Email: [1]khalidsiddiqui@fuuast.edu.pk, [2]marium.sultana@fuuast.edu.pk
[3]fareeha_samikhan@yahoo.com

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**Abstract.** This universe has veiled many secrets for mankind. On exploring them mathematically several differential equations have been created. Lane-Emden Equation (LEE) is among the most famous equations concerned with the study of stellar structure. The main purpose of this work is to show the effectiveness of a numerical method named Successive Differentiation Method (SDM) which can obtain analytical solutions for nonlinear singular differential equations. To establish the claim, the nonlinear singular boundary and initial value problem of Lane-Emden that governs the polytropic and isothermal gas spheres is considered here. Different illustrative types of Lane-Emden Equation have been solved numerically in this paper and these numerical results are compared with the numerical results of formerly solved numerical techniques. SDM proved to be the easiest method not only in accuracy but also in tranquil implementation of numerical techniques over the entire finite domain.

**AMS Subject Classification Codes: 34Bxx; 34B16; 85-08; 85A15; 65L05**
**Key Words:** Astrophysics, Lane-Emden Equation, Series Solution, Taylor Series, Differentiation.

## 1. Introduction

Stars and galaxies have always captivated mankind's interest. So to revivifying such thoughts stellar structures played a major role. The main objectives of studies going on stellar structures are 1) To understand the embodied physical variables completely 2) To describe the form of steady state that exists 3) To identify the physical phenomenon responsible for steady states and 4) To understand the processes involved in the recurrent loss of energy of stars.
Due to hydrostatic equilibrium conditions, stability in stellar structures is strictly maintained. These equilibrium conditions are described by the coupled field equations between gravitational field, matter and the gradients of gravitational potential. For stellar structures,

in Newtonian theory as polytropic equation of state (EOS) (pressure and density) is used to describe the thermodynamical relations, similarly the famous Lane-Emden Equation (LEE) describes the gravitational potential $\phi$. Lane-Emden is a pioneering work of two astrophysicists, Jonathan Homer Lane and Robert Emden. Recommended texts for thorough study related to stellar structure and their equilibriums, derivations of Lane-Emden Equation and discussion of their analytical and numerical solutions are [3,9]. It is a second order singular differential equation usually written as

$$\frac{d^2y}{dx^2} + \frac{\beta}{x}\frac{dy}{dx} + \delta g(x) f(y) = h(x); \quad 0 < x < 1,\, \beta > 0 \tag{1. 1}$$

with initial condition $y(0) = \alpha$, $y^{'}(0) = \gamma$ or boundary condition $y(0) = \alpha^{'}$, $y(1) = \gamma^{'}$. There are two commonly known kinds of Lane-Emden differential equation.

- **Case I: Lane-Emden of First Kind**
  If $\beta = 2$, $h(x) = 0$, $f(y) = y^n$, $\delta = 1$, $g(x) = 1$ then Eq. ( 1. 1 ) becomes the first kind of Lane-Emden Equation with polytropic index $n$.

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^n = 0; \quad 0 < x < 1, \tag{1. 2}$$

  with initial condition $y(0) = 1$, $y^{'}(0) = 0$. Solution of the first kind of Lane-Emden basically depends on the index $n$ known as polytropes. The values of $x$ on which $y(x)$ becomes zero or differential equation is known to be singular at origin basically represents the radius of polytrope. This kind has exact solutions only for index $n = 0, 1, 5$.
  At $n = 0$ the exact solution is $y(x) = 1 - \frac{x^2}{6}$ whereas for $x = \sqrt{6}$ the first zero occurs. This $y(x)$ solution shows the constant density compact sphere. At $n = 1$ the exact solution is $y(x) = \frac{\sin x}{x}$ where at $x = \pi$, second zero of LEE occurs. And for $n = 5$ the exact solution is $y(x) = (1 + \frac{x^2}{3})^{-0.5}$ and for $x = \infty$ third zero occurs.
  For remaining indices of $n$, numerical solutions can be calculated since no analytical solution exists. Such numerical solutions also have certain uses as approximated solution for $n = 1, 1.5$ and 3 describes the inter stellar structure of stars of class $M$, $S$, $C$, $K$ or having lower temperature than sun.

- **Case II: Lane-Emden of Second Kind**
  If $\beta = r$, $h(x) = 0$, $f(y) = e^{\frac{y}{1+\epsilon y}}$, $g(x) = 1$ then Eq. ( 1. 1 ) becomes the second kind of Lane-Emden Equation.

$$\frac{d^2y}{dx^2} + \frac{r}{x}\frac{dy}{dx} + \delta e^{\frac{y}{1+\epsilon y}} = 0; \quad 0 < x < 1,\, r \geq 0 \tag{1. 3}$$

Such equation is used to describe the, Landau Ginzburg critical phenomena, the thermal explosion in an enclosure and non isothermal zero order reaction in a catalytic pallet and

kinetics of combustion [4, 6, 15].
Moreover, different values of $r$ can describe the geometric shape of the catalyst particle formed during the reaction. For $\epsilon = 0$, $\delta = 1$ and $r = 0, 1, 2$ formed shapes are finitely thickened slab, a cylinder of infinite height and a sphere respectively. Analytical solutions exist only for first two values of $r$ but for sphere case only numerical solution can be obtained. For slab Eq. ( 1. 3 ) becomes

$$\frac{d^2y}{dx^2} + e^y = 0; \quad 0 < x < 1 \tag{1. 4}$$

with boundary condition $y(-1) = y(1) = 0$ and having $y(x) = 2\log\Big(cosh(\frac{x}{\sqrt{2}})\Big)$ as analytical solution . For infinite cylinder, Eq. ( 1. 3 ) becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + e^y = 0 \tag{1. 5}$$

with boundary condition $y^{'}(0) = 0$, $y(1) = 0$ and has analytical solution $y(x) = 2\ln\Big(1 + \frac{x^2}{8}\Big)$. For sphere Eq. ( 1. 3 ) becomes

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^y = 0 \tag{1. 6}$$

with boundary condition $y^{'}(0) = 0$, $y(1) = 0$ and has no analytical solution only numerical approximations can be obtained. For further detailed study on the effects of parameters $\delta, \epsilon, r$ see [26].
Lane-Emden Equation for sphere $r = 2$ is particularly used to describe the steady-state of the Bonnor-Ebert sphere and generally for self gravitating gas spheres and known as Bonnor-Ebert gas sphere equation. This equation also describes the equilibria of non-rotating polytropic fluids in a self-gravitating star. This particular equation has no exact solution.
Due to above uses not only astrophysicists have keen interest in this equation, also due to the singular and nonlinear nature of Lane-Emden Equation at origin, it has captured the interest of mathematicians as well. Finding its numerical solution was not an easy task so this equation became a classic example to test the efficiency of newly developed numerical methods. A brief literature review related to Lane-Emden is as follows.
Recently, Luma & Reem [20] proposed semi-analytic technique for second order nonlinear singular initial value problem and suggested to obtain Taylor series on boundary points, then forming the most suitable two point osculatory interpolating polynomial and solve it via integral equation to find the coefficients and constants. This technique is laborious and time consuming.
Sandile & Precious in [27] used Successive Linearization Method (SLM) to solve Lane-Emden Equation semi-analytically. Also Sandile & Precious [27] provided comparison of solutions obtained from SLM by results of Sinc-Collocation method and solutions given in [9] for polytropic indices $m = 2, 3, 4$. Hunter [11] used a bilinear Euler transformation to obtain a series solution of Lane-Emden Equation which converges for all $n$ of polytropes and this series also converges throughout the isothermal sphere by subtracting that singular component from Euler-transformed series. Liu [19] gave an approximate analytic solution

for both finite polytropic index $n$ and isothermal case at a level $< 1$ percent. Al-Jawary & Al-Qaissy [14] solved the integro-differential form of Lane-Emden using a new iterative method named DJM. Kenny et al. [16] obtained analytical approximants of the polytrope function, the radius and the mass of polytropes as a function of $n$ by solving scaled Lane-Emden Equation perturbatively as Eigen value problem. Iacono & De Felice [12] also constructed analytic approximate solutions to the Lane-Emden Equation.
Vanai & Aminataei in [30] utilized Pade series to obtain numerical solution of integro-differential form of Lane-Emden Equation. Also [35] used Pade Approximate series for finding the numerical solution of Lane-Emden Equation and obtained satisfactory results. Wazwaz et al. [24, 32] solved the Lane-Emden Equation of Volterra integral form numerically with initial and boundary conditions by utilizing the Adomian Decomposition Method (ADM). In this work the Adomian polynomials were obtained first and then the numerical series solution was acquired. Two dimensional Lane-Emden Equation was also solved by Adomian Decomposition Method see [25]. In [28] the analytical solution has been obtained by the decomposition method. Whereas certain modifications done in [28] and some new solutions and theories were developed in [1]. Wazwaz [31] came up with a new pattern of Adomian Decomposition Method to avoid the issue of singularity while solving Lane-Emden Equation. Hence obtained satisfactory approximated numerical result via this technique. In [33] Volterra type system of Lane-Emden Equation have been solved by Adomian Decomposition Method.
Several other authors [2,5,7,8,10,13,18,21–23,29] utilized various methods such as Differential Transform Method, Neural Network Method, New Galerkin Operational Matrices, Wavelet Galerkin Method, Pade Approximation, Legendre Wavelets Approximations, Cubic B-Spline Approximation, Fuzzy Modeling, Variational Approach respectively to obtain numerical series solution of Lane-Emden Equation.
Real life phenomenon mostly when expressed mathematically is usually articulated as first or second order differential equations (such as wave or heat experience etc) which imply that natural events have simple temperament. Therefore numerical or analytical methods implemented to acquire their solutions should also have an uncomplicated course of action. An effortless method of all is Successive Differentiation Method (SDM). SDM was first applied on Bratu type equations and proved it to be the easiest technique of solving Bratu type differential equation [34]. Also this method proved to be effective for solving any nonlinear partial differential equation impressively [17]. In this paper, Lane-Emden is numerically solved by using Successive Differentiation Method.

## 2. Successive Differentiation Method

Successive Differentiation Method is just like its name. In this method successive derivatives are obtained and then by using Maclaurin or Taylor series at $x = 0$ the solution of some differential equation is obtained in series form. Let an $n^{th}$ order differential equation be

$$F\Big(y^n(x), y^{n-1}(x), y^{n-2}(x), y^{n-3}(x), ..., y(x), x\Big) = 0 \tag{2. 7}$$

with initial condition as $y^n(0) = y_{n-1}$, $y^{n-1}(0) = y_{n-2}$, $y^{n-2}(0) = y_{n-3}$, ...., $y(0) = y_0$ or boundary condition as $y^n(a) = y_{n-1}$, $y^{n-1}(b) = y_{n-2}$, $y^{n-2}(c) = y_{n-3}$, ...., $y(1) = y_0$

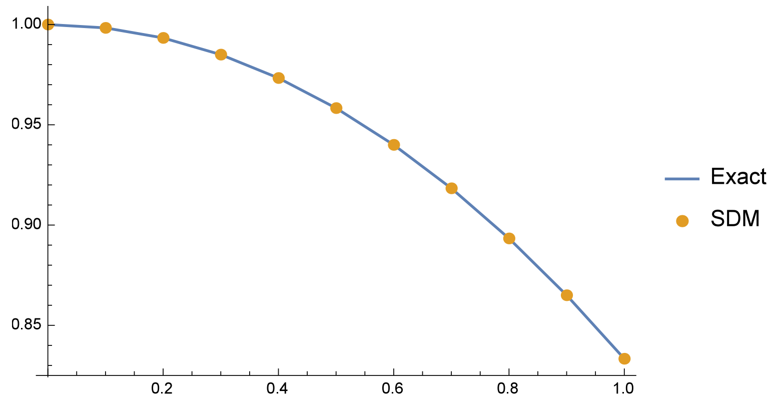


FIGURE 1. Graphical comparison between exact polytropic solution and series solution of case I at $n = 0$.

Then its derivatives can be written as

$$F\Big(y^{n+1}(x), y^{n}(x), y^{n-1}(x), y^{n-2}(x), ..., y^{'}(x)\Big) = 0 \qquad (2.\ 8)$$

$$F\Big(y^{n+2}(x), y^{n+1}(x), y^{n}(x), y^{n-1}(x), ..., y^{''}(x)\Big) = 0 \qquad (2.\ 9)$$

$$F\Big(y^{n+3}(x), y^{n+2}(x), y^{n+1}(x), y^{n}(x), ..., y^{'''}(x)\Big) = 0 \qquad (2.\ 10)$$

...

$$F\Big(y^{n+n}(x), y^{n+(n-1)}(x), y^{n+(n-2)}(x), y^{n-(n-3)}(x), ..., y^{n}(x)\Big) = 0 \qquad (2.\ 11)$$

**For Initial Value Problems**

By using initial conditions, values of further derivatives at x=0 can be obtained and then by using Maclaurin series, series solution of all kind of initial-value problems can be obtained from successive derivatives $y(0) = \alpha_0$, $y^{'}(0) = \alpha_1$, $y^{''}(0) = \alpha_2$, ..., $y^{n}(0) = \alpha_n$. Therefore, by using these evaluated terms in Maclaurin series, obtained series solution is given as

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{n}(0)}{n!} x^{n} = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_n x^n \qquad (2.\ 12)$$

**For Boundary Value Problems** Normally, it has the same procedure as SDM is applied on initial value problems but sometimes there arises a situation where only one initial condition or less number of initial conditions than highest order of derivative in the differential equation is given then, in that case it is difficult to find the values of derivatives such as $y^{''}(x)$,$y^{'''}(x)$,$y^{''''}(x)$ etc at $x = 0$ directly from the successive derivatives. This can be handled by using given initial condition and assuming the missing ones as $y(0) = \beta_1, ....$ To find these assumed constants, the calculated Maclaurin series is then compared with the boundary conditions and hence the value of unknown condition is found. Consider Eq. ( 2. 7 ) with initial condition as $y^{n}(0) = y_{n-1}$, $y^{n-1}(0) = y_{n-2}$, $y^{n-2}(0) = y_{n-3}$, ...., $y'(0) = y_1$ or boundary condition as $y^{n}(a) = y_{n-1}$, $y^{n-1}(b) = y_{n-2}$, $y^{n-2}(c) = y_{n-3}$,

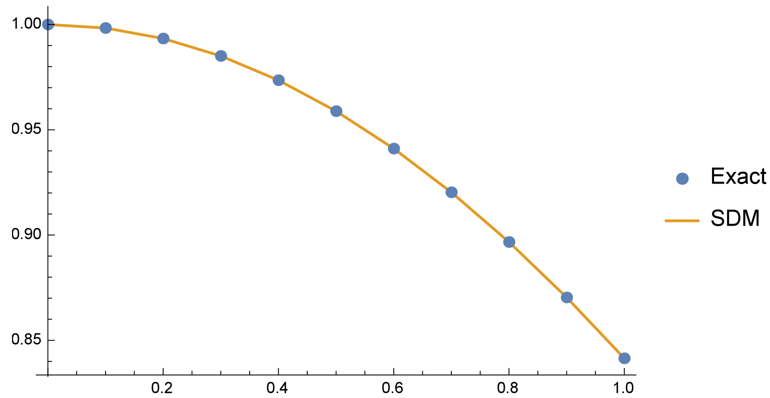


FIGURE 2. Graphical comparison between exact polytropic solution and series solution of case I at $n = 1$.

...., $y(1) = y_0$, so for applying SDM assume $y(0) = \beta_1$. Calculate the successive derivatives as in Eq. ( 2. 8 ) - Eq. ( 2. 11 ). Then calculate a Maclaurin series as

$$y(x) = \sum_{n=0}^{\infty} \frac{y^n(0)}{n!} x^n = \beta_1 + xy_1 + x^2 y_2, ... \qquad (2.\ 13)$$

apply boundary condition $y(1) = y_0$ in this series and get

$$y(1) = y_0 = \beta_1 + y_1 + y_2, ... \qquad (2.\ 14)$$

On comparison $\beta_1 = y_0$ is obtained and hence the series solution becomes

$$y(x) = \sum_{n=0}^{\infty} \frac{y^n(0)}{n!} x^n = y_0 + xy_1 + x^2 y_2, ... \qquad (2.\ 15)$$

## 3. SOLUTION OF LANE-EMDEN EQUATION OF FIRST KIND BY SDM:

Lane-Emden differential equation of first kind is solved as

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y^n = 0 \qquad (3.\ 16)$$

or

$$x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + xy^n = 0 \qquad (3.\ 17)$$

with initial condition $y(0) = 1$, $y^{'}(0) = 0$. For solving Lane-Emden Equation the trick to avoid the singularity or complexity caused by nonlinear term is to take one higher order derivative then given in initial conditions. In this example, the initial condition has first order derivative so take successive derivative of third order at $x = 0$ for finding $y^{''}(0)$ therefore the nonlinear terms automatically get vanished. Its successive derivatives can be taken as

$$y^n + nxy^{n-1} y^{'} + 3y^{''} + xy^{'''} = 0 \qquad (3.\ 18)$$

$$(n-1)nxy^{n-2}(y^{'})^2 + ny^{n-1}(2y^{'} + xy^{''}) + 4y^{'''} + xy^{(4)} = 0 \qquad (3.\ 19)$$

$$3(n-1)ny^{n-2}y^{'}(y^{'}+xy^{''})+n(n-1)(n-2)xy^{n-3}(y^{'})^{3}+$$
$$ny^{n-1}(3y^{''}+xy^{(4)})+5y^{(4)}+xy^{(5)}=0 \quad (3.\ 20)$$

On applying initial conditions $y^{''}(0)=-\frac{1}{3}$, $y^{'''}(0)=0$, $y^{(4)}(0)=\frac{n}{5}$ and so on. So upon putting values in Maclaurin series, the obtained series upto 15 terms is given as

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{n}(0)}{n!}x^{n} \quad (3.\ 21)$$
$$=1-\frac{x^2}{6}+\Big(\frac{nx^4}{120}\Big)+\frac{(5n-8n^2)x^6}{15120}+\frac{(70n-183n^2+122n^3)x^8}{3265920}+$$
$$\frac{(3150n-10805n^2+12642n^3-5032n^4)x^{10}}{1796256000}+$$
$$\frac{(138600n-574850n^2+915935n^3-663166n^4+183616)x^{12}}{84064780800}+... \quad (3.\ 22)$$

Since Lane-Emden Equation has analytical solutions for $n=0,1,5$. And through SDM it can be seen that the obtained series is accurately equal to the analytical solutions as
**Polytropic Solution 1:** Solution of dimensionless Lane-Emden radial coordinate at $n=0$ has exact solution of the form $y(x)=1-\frac{x^2}{6}$ which is exactly the solution obtained by putting $n=0$ in Eq. ( 3. 22 )
**Polytropic Solution 2:** Second polytropic solution has exact solution series for $n=1$ as

$$y(x)=1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+\frac{x^8}{362880}-\frac{x^{10}}{39916800}+$$
$$\frac{x^{12}}{6227020800}-\frac{x^{14}}{1307674368000}+... \quad (3.\ 23)$$

which is exactly the series obtained for $n=1$ in Eq. ( 3. 22 ).
**Polytropic Solution 3:** Third exact solution for Lane-Emden Equation for $n=5$ has series

$$y(x)=1-\frac{x^2}{6}+\frac{x^4}{24}-\frac{5x^6}{432}+\frac{35x^8}{10368}-\frac{7x^{10}}{6912}+$$
$$\frac{77x^{12}}{248832}-\frac{143x^{14}}{1492992}+... \quad (3.\ 24)$$

which is definitely equal to the series obtained from Eq. ( 3. 22 ) when n=5. All three solutions are compared graphically as well with exact solution given in Fig.1,2,3. Complete coherence can be seen in given figures.

## 4. LANE-EMDEN EQUATION OF SECOND KIND BY SDM

Similarly Lane-Emden differential equation of second kind $\frac{d^2y}{dx^2}+\frac{r}{x}\frac{dy}{dx}+\delta e^{\frac{y}{1+\epsilon y}}=0$ can be differentiated several times as

$$\frac{ry(x)}{x^2}+\frac{ry^{'}(x)}{x}+e^{\frac{y(x)}{1+\epsilon y(x)}}\delta\Big(-\frac{\epsilon y(x)y^{'}(x)}{(1+\epsilon y(x))^2}+\frac{y^{'}(x)}{1+\epsilon y(x)}\Big)+y^{'''}(x)=0 \quad (4.\ 25)$$

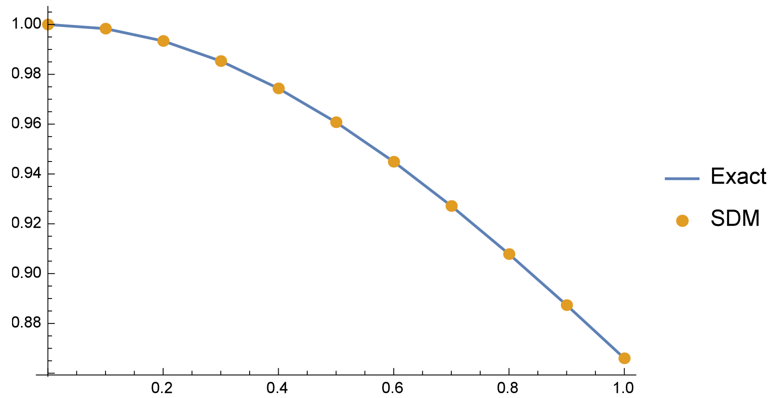


FIGURE 3. Graphical comparison between exact polytropic solution and series solution of case I at $n = 5$.

Again differentiate to obtain

$$\frac{2ry(x)}{x^3} - \frac{2ry^{'}(x)}{x^2} + \frac{ry^{''}(x)}{x} + \frac{e^{\frac{y(x)}{1+\epsilon y(x)}}\delta(y^{'}(x))^2}{(1+\epsilon y(x))^4} \tag{4. 26}$$

$$+ \frac{e^{\frac{y(x)}{1+\epsilon y(x)}}\delta(-2\epsilon(y^{'}(x))^2 + (1+\epsilon y(x))y^{''}(x))}{(1+\epsilon y(x))^3} + y^{''''}(x) = 0 \tag{4. 27}$$

hence more successive derivatives can be obtained on similar pattern. Approximated solutions of general form for $\delta = 1, \epsilon = 0$ is given below

$$y(x) = a - \frac{e^a x^2}{2(1+r)} + \frac{e^{2a}x^4}{8(1+r)(3+r)} - \frac{e^{3a}(2+r)x^6}{24(1+r)^2(3+r)(5+r)} \tag{4. 28}$$

$$+ \frac{e^{4a}(17+16r+3r^2)x^8}{192(1+r)^3(3+r)(5+r)(7+r)} + ... \tag{4. 29}$$

By applying boundary condition value of $a = 0$ is obtained for each case. Solution of three of its forms are given as

4.1. **For Slab.** The obtained solution through SDM is

$$y(x) = \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{180} - \frac{17x^8}{20160} + \frac{31x^{10}}{226800} - \frac{691x^{12}}{29937600} + \frac{5461x^{14}}{1362160800} - ... \tag{4. 30}$$

same series as exact solution i.e. $y(x) = 2\log\Big(\cosh(\frac{x}{\sqrt{2}})\Big)$.

4.2. **For Infinite Cylinder.** This equation is given with boundary conditions $y^{'}(0) = 0$, $y(1) = 0$ instead of initial conditions for that case an initial condition is supposed as

TABLE 1. Comparison of series solution of Bonnor-Ebert gas sphere equation obtained through SDM and other numerical methods

| x | SDM | DTM | ADM | HNN | SRM | ANN |
|---|---|---|---|---|---|---|
| 0.1 | -0.00166583386 | -0.0166583 | -0.0166583 | -0.0003607 | -0.0016658 | -0.0004162 |
| 0.2 | -0.00665336710 | -0.0066533 | -0.0066533 | -0.0014426 | -0.0066534 | -0.0016634 |
| 0.3 | -0.01493288327 | -0.0149329 | -0.0149329 | -0.0032459 | -0.0149329 | -0.0037411 |
| 0.4 | -0.02645547634 | -0.0264555 | -0.0264555 | -0.0057706 | -0.2645555 | -0.0066469 |
| 0.5 | -0.04115395729 | -0.0411540 | -0.0411540 | -0.0090167 | -0.0411539 | -0.0103767 |
| 0.6 | -0.05894407476 | -0.0589441 | -0.0589441 | -0.0129842 | -0.0589440 | -0.0149250 |
| 0.7 | -0.07972600424 | -0.0797260 | -0.0797260 | -0.0176730 | -0.0797259 | -0.0202850 |
| 0.8 | -0.10338605320 | -0.1033860 | -0.1033860 | -0.0230833 | -0.1033860 | -0.0264485 |
| 0.9 | -0.12979852450 | -0.1297980 | -0.1297980 | -0.0292149 | -0.1297980 | -0.0334065 |
| 1.0 | -0.15882767884 | -0.1588270 | -0.1588270 | -0.0360680 | -0.1558828 | -0.0411486 |

$y(0) = a$. Therefore the approximated solution by successively differentiating and then expanding by Maclaurin series is obtained as

$$y(x) = a - \frac{e^a x^2}{4} + \frac{e^{2a} x^4}{64} - \frac{e^{3a} x^6}{768} + \frac{e^{4a} x^8}{8192} - \frac{e^{5a} x^{10}}{81920} + \frac{e^{6a} x^{12}}{786432} - \frac{e^{7a} x^{14}}{7340032} + \ldots \quad (4.\ 31)$$

Then by using the boundary condition $y(1) = 0$ and solving for $a$ the obtained series through SDM is given as

$$y(x) = \frac{x^2}{4} - \frac{x^4}{64} + \frac{x^6}{768} - \frac{x^8}{8192} + \frac{x^{10}}{81920} - \frac{x^{12}}{786432} + \frac{x^{14}}{7340032} - \ldots \quad (4.\ 32)$$

which is accurately equal to the series solution obtained through exact solution $y(x) = 2\ln\left(1 + \frac{x^2}{8}\right)$ of this form of equation.

4.3. **Bonnor-Ebert Gas Sphere Equation.** Since no exact solution of Bonnor-Ebert gas sphere equation exists therefore in this work the obtained approximate solution $y(x) = -\frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{1890} + \frac{61x^8}{1632960} + \ldots$ is compared with the solutions of other numerical methods.

Table (1) gives the comparison of present method with other available methods. It can be observed that SDM is better option to solve such nonlinear singular differential equations as obtained results are very adequate and calculation effort is far less than other utilized methods.

## 5. CONCLUSION

In this work, SDM is utilized to obtain the analytic solutions of Lane-Emden type equations. Several cases have been discussed in this work to show the efficacy of SDM. All numerical methods and techniques utilize various iterative formulas or perturbations, matrices and approximations. Whereas method used in this work is rather simple and obtain the analytical solution series by successively differentiating the Lane-Emden differential equation, applying initial or boundary conditions and expanding it by Taylor or Maclaurin series. This method has less effort than any other contemporary methods yet available and this method gives pretty accurate series when compared to numerical or analytical solutions. No drawback has yet been observed for solving initial or boundary value problems. This method can solve any type of singular and nonlinear ordinary differential equations.

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