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Multipolar Interval-Valued Fuzzy Set with Application of Similarity Measures and multi-person TOPSIS technique

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Abstract. A Similarity measure in the fuzzy structure plays a very considerable role in manipulating hurdles that apprehend vague data, but unable to deal with the ambiguous and variability of the problems having multipolar interval-valued data. In this research article, a certain distance between two multipolar interval-valued fuzzy sets (mIVF sets) has been defined. A new similarity measure $(Sim.M)$ for mIVF based on distances has been introduced, also some of the basic operations on the structure has been defined such as union, intersection, and complement. MCDM is performed for mIVF information that measure the similarity measure based on distance measure for the best alternative. An application is given that the proposed $Sim.M$ for mIVF set is capable of recognition the nature and structure of different entities which belongs to the same family. Furthermore, a multiperson TOPSIS technique is developed for the structure of mIVF with an algorithm for the selection of the best alternative.

Key Words: Multipolar Interval-valued Fuzzy Set (mIVF set), Operators and Properties, Distance measure and similarity measure $(Sim.M)$, Pattern Recognition, multi-person TOPSIS technique.

1. INTRODUCTION

The scientific models of crisp set logic and theory are not sufficient to carry out the problems that contains hesitancy. Zadeh [41] defined fuzzy set logic as the generalisation crisp set theory, emphasizing the complexities of these structure with the membership approach. Atanassov [10] defined an intuitionistic fuzzy set by separately assigning membership and non-membership function. Zadeh [42], Grattan [17], and Jahn [20] individualistically established the interval-valued fuzzy set (IVF)in the same year. IVF is defined by an association function interval-valued about membership. An Interval-valued fuzzy set is an unusual case of L-fuzzy sets in the Goguen sense [18], so as a mathematical object, it is not of exact interest.

Zhang [26] introduced the novel idea to deal the vagueness data in two different opinion as bipolar fuzzy sets. Chen [12] presented the concept of multipolar fuzzy sets as a simplified type of fuzzy bipolar set. Akram ([2]-[8],[1]) introduced mpolar fuzzy graph, characterization of mpolar fuzzy graph and metrics in mpolar fuzzy graph. Riaz([28]-[29]) proposed the idea of bipolar fuzzy soft set topology and fuzzy parameterized fuzzy soft set metric spaces in decision making for the world problem. Rashid ([27]-[11]) inherited the experimental verification using fuzzy logic modeling on used foundry sand, also measure the similarity between intuitionistic fuzzy through inclusion.

Chen ([15]-[13]) gave the idea of Similarity Measure of unknown sets, but some problems had been unable to be contained by it. To overcome with this problem, Kim and Hong [19] proposed several new updated steps. The Interval-valued fuzzy set was firstly proposed by Gorzalczany [16] in 1987, also defines their basic operations including union, intersection, and complement. Kharal [21] presented some similarity measure based on set-theoretical operations. Akram [9] implemented a multipolar fuzzy set and multipolar fuzzy soft software similarity measures by application of pattern recognition and medical diagnosis. riaz et al. [44] studied the TOPSIS approach in exploring most effectual method for curing from COVID-19 in Pythagorean m-polar fuzzy topological structure. Young [39] proposed a new method for determining the best location for a plantation in terms of the linguistic structure, using graded mean reparation and fuzzy logic. Olson's work [24] discusses the multiple criteria for identification of solution in TOPSIS and take its comparison with SMART and centroid weighting schemes. Chen [14] describes the TOPSIS to the fuzzy environment in the triangular sense in 2000. Hsu [38] explained the strategy of order choice by similarities using MADM in a TOPSIS-based group decision setting. Furthermore, Similarity measure and a TOPSIS technique is used by Saeed ([30]-[33]). Application of fuzzy numbers in mobile selection in metros like Lahore is proposed by Saqlain [37]. Application of similarity measure on multipolar structure is discussed by Saeed et al. [34] in medical diagnosis and decision-making. Mehmood et al. ([35],[36]) applied the distance based similarity measure in the spread recognition of COVID-19 in Pakistan with the correspondence to top 16 affected countries and extented the fuzzyy TOPSS to intuinistic fuzzy environment under linguistic variable and triangular numbers. Khalifa et al. [22] discussed new decision making technique with the application in mass media. Petrovic and Kankaras [25] developed a hybridized interval type-2 fuzzy sets DEMATEL,AHP and TOPSIS multicriteria decision making approach for the selection and evaluation of criteria for determination of air traffic control radar position. Yorulmaz et al. [40] discussed the robust mahalanobis distance based TOPSIS to evaluate the economic development of 81 provinces of Turkey. Zolfani et al. [43] studied VIKOR and TOPSIS focused reanalysis of the MADM methods based on logarithmic normalization.

1.1. **Motivation.** This article extends the idea of mF set and interval-valued fuzzy set into mIVF set, the reason behind this extension is the existence of the multipolar interval-valued fuzzy data because many real world problems are in the form of m-attributes expressed in interval valued rather than single-valued form. Existing theories like fuzzy set discussed by Zadeh in [41], bipolar fuzzy set discussed by Wen-Ran in [26], multipolar fuzzy set discussed by Akram et al. in [9] and Interval-valued discussed by Jahn in [20] are inadequate for this purpose as they don't utilize data comprehensively in the form multi-interval-valued fuzzy. The persistence of this article is to the association of the interval-valued fuzzy set (IVF set) and mF set to acquire a new fuzzy set model: multipolar Interval-valued fuzzy set. The goal of this research is to develop a suitable framework mIVF and investigates the distance and similarity measures on mIVF. Furthermore, for this new developed structure mIVF, a multi-person TOPSIS technique has been presented to get a better and accurate results with the evaluation of multi experts in the related field of any decision analysis.

1.2. **Structure of Paper.** To smooth our discussion, at first in Section 2, we extant some fundamentals related to this article. In Section 3, the notion of multipolar interval-valued fuzzy (mIVF) set and its basic operations are defined. In Section 4, some properties based on operators are studied. Section 5 discusses the distance measure formulas for the mIVF set. In Section 6, Similarity Measure based on distance has been introduced. Section 7 has application of similarity measure on the mIVF set in pattern recognition of a watch brand with the algorithm. In section 8, a TOPSIS technique is developed for the multiperson decision analysis on the structure of mIVF set. A hypothetical data numerical example is considered for the evaluation of proposed multiperson TOPSIS technique. After that, the article concludes with an analysis and and future work.

2. Preliminaries

This section discusses some basic definitions related to the mIVF Set.

Definition 2.1 (Fuzzy Set)**.** *[1] A Fuzzy Set Q over universal set Y is defined as*

$$Q = \{(y, \mu_Q(y)) : y \in Y\}$$

where the mapping $\mu_Q : Y \to [0,1]$,

Definition 2.2 (mF Set)**.** *[3] An mF set over universal set Y is a mapping $T = (z_1 \circ T(y), z_2 \circ T(y), \cdots, z_m \circ T(y)) : Y \to [0,1]^m$ where the i-th projection mapping is defined $z_i \circ T : [0,1]^m \to [0,1]$.*

Definition 2.3 (Interval Valued Fuzzy Set)**.** *[15] An Interval-valued Fuzzy set Q over universal set Y is defined as*

$$Q = \{(y, I_Q(y)) : y \in Y\}$$

where,

$$I_Q(y) \subseteq [0,1]$$

3. Multipolar Interval-Valued Fuzzy Set and Some Operations defined on mIVF sets

Definition 3.1 (Multipolar Interval-Valued Fuzzy Set)**.** *An $mIVF$ set Q over nonempty universal set Y, is a function*

$$Q = \{y, (p_1 \circ I_Q(y), p_2 \circ I_Q(y), \cdots, p_m \circ I_Q(y))\},$$

where i-th projection mapping is

$$p_i \circ I_Q(y) \subseteq [0,1]$$
$$\text{for all } i = 1, 2, \cdots, m$$

Example 3.1 $2-$polar IVF set over the universal set
$Y = \{y_1, y_2\}$ is expresses as

$$R = \{(y_1, [0.2, 0.3], [0, 2, 0.4]), (y_2, [0.4, 0.45], [0.5, 0.6])\}$$

Definition 3.2 (mIVF Subset). *Let Q and R be two mIVF set over universal set Y and defined as*

$$Q = \{y, p_i \circ\ I_Q(y) : y \in Y\} \text{ and } R = \{y, p_i \circ\ I_R(y) : y \in Y\}$$
$$\text{for all } i = 1, 2, \cdots, m$$

where,

$$p_i \circ\ I_Q(y) = [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)],$$
$$p_i \circ\ I_R(y) = [p_i \circ I_R^L(y), p_i \circ I_R^U(y)]$$
$$\text{for all } i = 1, 2, \cdots, m$$

Q is subset of R if $p_i \circ\ I_Q(y)$ is contained in $p_i \circ\ I_R(y)$. i.e.

$$p_i \circ I_Q^L(y) \geq p_i \circ I_R^L(y) \text{ and } p_i \circ I_Q^U(y) \leq p_i \circ I_R^U(y)$$

$$\text{for all } i = 1, 2, \cdots, m$$

Example 3.2 Q and R are $2-$polar IVF sets over universal set Y.

$$Q = \{(y_1, [0.2, 0.3], [0, 4, 0.5]), (y_2, [0.3, 0.6], [0.4, 0.5])\}$$
$$R = \{(y_1, [0.1, 0.4], [0, 4, 0.6]), (y_2, [0.2, 0.6], [0.4, 0.5])\}$$

here Q is subset of R.

Definition 3.3 (mIVF Equal set). *Let $Q = \{y, p_i \circ I_Q(y) : y \in Y\}$ and $R = \{y, p_i \circ I_R(y) : y \in Y\}$ be the two mIVF sets over the universal set Y are equal if*

$$Q \hat{\subseteq} R \text{ such that } p_i \circ I_Q(y) \text{ is contained in } p_i \circ I_R(y)$$

and

$$R \hat{\subseteq} Q \text{ such that } p_i \circ I_R(y) \text{ is contained in } p_i \circ I_Q(y)$$

then

$$Q = R$$

Definition 3.4 (mIVF Null Set). *A set Q is said to be mIVF null set, denoted by ϕ if*

$$Q = \{(y, p_i \circ I_Q(y) = [1, 0]) : y \in Y\} = \phi$$
$$\text{for all } i = 1, 2, \cdots, m$$

Definition 3.5 (mIVF Absolute Set). *A set R is said to be mIVF absolute set, denoted by Y if*

$$R = \{(y, p_i \circ I_R(y) = [0, 1]) : y \in Y\} = Y$$
$$\text{for all } i = 1, 2, \cdots, m$$

Definition 3.6 (Union of mIVF Sets). *Let $Q = \{y, p_i \circ I_Q(y) : y \in Y\}$ and $R = \{y, p_i \circ I_R(y) : y \in Y\}$ be two mIVF sets over the universal set Y where,*

$$p_i \circ I_Q(y) = [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)],$$
$$p_i \circ I_R(y) = [p_i \circ I_R^L(y), p_i \circ I_R^U(y)]$$
$$\text{for all } i = 1, 2, \cdots, m$$

then their union is defined as,

$$Q\hat{\cup}R = \{y, [min(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), max(p_i \circ I_Q^U(y), p_i \circ I_R^U(y))]\}$$
$$\textit{for all } i = 1, 2, \cdots, m$$

Definition 3.7 (Intersection of mIVF Sets)**.** *Let Q and R be are two mIVF set over Y defined as*

$$Q = \{y, p_i \circ I_Q(y) : y \in Y\} \textit{ and } R = \{y, p_i \circ I_R(y) : y \in Y\}$$
$$\textit{for all } i = 1, 2, \cdots, m$$

where,

$$p_i \circ I_Q(y) = [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]$$
$$p_i \circ I_R(y) = [p_i \circ I_R^L(y), p_i \circ I_R^U(y)]$$

$$\textit{for all } i = 1, 2, \cdots, m$$

then, their intersection is given by

$$Q\hat{\cap}R = \{y, [max(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), min(p_i \circ I_Q^U(y), p_i \circ I_R^U(y))]\}$$

$$\textit{for all } i = 1, 2, \cdots, m$$

Definition 3.8 (Complement)**.** *Complement of a mIVF set $Q = \{(y, p_i \circ I_Q(y)) : y \in Y\}$ can be defined as*

$$Q^c = \{y, ([0,1] - p_i \circ I_Q(y)) : y \in Y\}$$

$$\textit{for all } i = 1, 2, \cdots, m$$

Example 3.3 If

$$Q = \{(y_1, [0.2, 0.3], [0,4, 0.5]), (y_2, [0.3, 0.6], [0.4, 0.5])\} \text{ and}$$
$$R = \{(y_1, [0.3, 0.5], [0,2, 0.4]), (y_2, [0.3, 0.5], [0.2, 0.5])\}$$

be the $2-$polar interval valued fuzzy set over universal set $Y = \{y_1, y_2\}$, then their union, intersection, and complement will be respectively,

$$Q\hat{\cup}R = \{(y_1, [0.2, 0.5], [0,2, 0.5]), (y_2, [0.3, 0.6], [0.2, 0.5])\},$$
$$Q\hat{\cap}R = \{(y_1, [0.3, 0.3], [0.4, 0.4]), (y_2, [0.3, 0.5], [0.4, 0.5])\},$$
$$Q^c = \{(y_1, [0, 0.2) \cup (0.3, 1], [0, 0.4) \cup (0.5, 1]), (y_2, [0, 0.3) \cup (0.6, 1], [0, 0.4) \cup (0.5, 1])\}.$$

4. PROPERTIES OF mIVF SETS OPERATIONS

Some of the major properties of set-theoretic operators defined on mIVF sets are discussed below.

Idempotent Properties 4.1 Idempotent properties hold true for an mIVF set Q over universal set Y, can be defined as

$$Q\hat{\cup}Q = \{y, [min(p_i \circ I_Q^L(y), p_i \circ I_Q^L(y)), max(p_i \circ I_Q^U(y), p_i \circ I_Q^U(y))]\} = Q$$

and

$$Q\hat{\cap}Q = \{y, [max(p_i \circ I_Q^L(y), p_i \circ I_Q^L(y)), min(p_i \circ I_Q^U(y), p_i \circ I_Q^U(y))]\} = Q$$
$$for\ all\ i = 1, 2, \cdots, m$$

Identity Properties 4.2 Identity properties also hold true for an mIVF set Q over universal set Y as

$$Q\hat{\cup}\phi = \{y, [min(p_i \circ I_Q^L(y), 1), max(p_i \circ I_Q^U(y), 0)]\} = Q$$

and

$$Q\hat{\cap}Y = \{y, [max(p_i \circ I_Q^L(y), 0), min(p_i \circ I_Q^U(y), 1)]\} = Q$$

$$\textit{for all } i = 1, 2, \cdots, m$$

Domination Properties 4.3 Domination properties for a mIVF set Q over universal set Y is given as

$$Q\hat{\cup}Y = \{y, [min(p_i \circ I_Q^L(y), 0), max(p_i \circ I_Q^U(y), 1)]\} = Y$$

and

$$Q\hat{\cap}\phi = \{y, [max(p_i \circ I_Q^L(y), 1), min(p_i \circ I_Q^U(y), 0)]\} = \phi$$
$$for\ all\ i = 1, 2, \cdots, m\ and\ y \in Y$$

Complementation Properties 4.4 The Complementation properties of absolute mIVF set Y and null mIVF set ϕ hold and given follows,

$$\phi^c = Y \text{ and}$$
$$Y^c = \phi$$

Double Complementation Property 4.5 Double complentation property holds for mIVF set Q over universal set Y

$$(Q^c)^c = \{y, ([0,1] - p_i \circ I_Q(y))\}^c = \{y, ([0,1] - [0,1] + p_i \circ I_Q(y))\} = Q$$

$$\textit{for all } i = 1, 2, \cdots, m$$

Exclusion and Contradiction Property 4.6 The Exclusion and Contradiction Property for mIVF set Q over universal set Y holds and given as.

$$Q\hat{\cup}Q^c = \{y, [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cup}\{y, ([0,1] - [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)])\}$$
$$= \{y, [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cup}\{y, [0, p_i \circ I_A^L(y)) \cup (p_i \circ I_A^U(y), 1]\} = Y$$

and

$$Q\hat{\cap}Q^c = \{y, [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cap}\{y, ([0,1] - [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)])\}$$
$$= \{y, [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cap}\{y, [0, p_i \circ I_Q^L(y)) \cap (p_i \circ I_Q^U(y), 1] = \phi$$

$$\textit{for all } i = 1, 2, \cdots, m$$

Commutative Properties 4.7 Let Q and R be two mIVF then following holds true.

(1) $Q\hat{\cup}R = R\hat{\cup}Q$

(2) $Q\hat{\cap}R = R\hat{\cap}Q$

$(1) Proof: -$

$$L.H.S = Q\hat{\cup}R$$
$$= \{y, [min(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), max(p_i \circ I_Q^U(y), p_i \circ I_R^U(y))]\}$$
$$= \{y, [min(p_i \circ I_R^L(y), p_i \circ I_Q^L(y), max(p_i \circ I_R^U(y), p_i \circ I_Q^U(y))]\}$$
$$= R\hat{\cup}Q = R.H.S$$
$$for\ all\ i = 1, 2, \cdots, m$$

$(2) Proof :-$

$$L.H.S = Q\hat{\cap}R$$
$$= \{y, [max(p_i \circ I^L_Q(y), p_i \circ I^L_R(y)), min(p_i \circ I^U_Q(y), p_i \circ I^U_R(y))]\}$$
$$= \{y, [max(p_i \circ I^L_R(y), p_i \circ I^L_Q(y), min(p_i \circ I^U_R(y), p_i \circ I^U_Q(y))]\}$$
$$= R\hat{\cap}Q = R.H.S$$

$$for\ all\ i = 1, 2, \cdots, m$$

Associative Properties 4.8 Associative properties hold for mIVF sets Q, R and S over universal set Y as
(1) $Q\hat{\cup}(R\hat{\cup}S) = (Q\hat{\cup}R)\hat{\cup}S$
(2) $Q\hat{\cap}(R\hat{\cap}S) = (Q\hat{\cap}R)\hat{\cap}S$

$(1) Proof :-$

$$L.H.S = Q\hat{\cup}(R\hat{\cup}S)$$
$$= Q\hat{\cup}\{y, [min(p_i \circ I^L_R(y), p_i \circ I^L_S(y)), max(p_i \circ I^U_R(y), p_i \circ I^U_S(y))]\}$$
$$= \{y, [min(p_i \circ I^L_Q(y), p_i \circ I^L_R(y), p_i \circ I^L_S(y)), max(p_i \circ I^U_Q(y), p_i \circ I^U_R(y), p_i \circ I^U_S(y))]\}$$
$$= \{y, [(min(p_i \circ I^L_Q(y), p_i \circ I^L_R(y)), max(p_i \circ I^U_Q(y), p_i \circ I^U_R(y))]\}\hat{\cup}S$$
$$= (Q\hat{\cup}R)\hat{\cup}S = R.H.S$$
$$for\ all\ i = 1, 2, \cdots, m$$

$(2) Proof :-$

$$L.H.S = Q\hat{\cap}(R\hat{\cap}S)$$
$$= Q\hat{\cap}\{y, [max(p_i \circ I^L_R(y), p_i \circ I^L_S(y)), min(p_i \circ I^U_R(y), p_i \circ I^U_S(y))]\}$$
$$= \{y, [max(p_i \circ I^L_Q(y), p_i \circ I^L_R(y), p_i \circ I^L_S(y)), min(p_i \circ I^U_Q(y), p_i \circ I^U_R(y), p_i \circ I^U_S(y))]\}$$
$$= \{y, [(max(p_i \circ I^L_Q(y), p_i \circ I^L_R(y)), min(p_i \circ I^U_Q(y), p_i \circ I^U_R(y))]\}\hat{\cap}S$$
$$= (Q\hat{\cap}R)\hat{\cap}S = R.H.S$$

$$for\ all\ i = 1, 2, \cdots, m$$

Distributive Properties 4.9 Distributive properties hold for mIVF sets Q, R and S over universal set Y as
(1) $Q\hat{\cup}(R\hat{\cap}S) = (Q\hat{\cup}R)\hat{\cap}(Q\hat{\cup}S)$
(2) $Q\hat{\cap}(R\hat{\cup}S) = (Q\hat{\cap}R)\hat{\cup}(Q\hat{\cap}S)$

$(1) Proof :-$

$$R.H.S = (Q\hat{\cup}R)\hat{\cap}(Q\hat{\cup}S)$$
$$= \{y, [min(p_i \circ I^L_Q(y), p_i \circ I^L_R(y)), max(p_i \circ I^U_Q(y), p_i \circ I^U_R(y))]\}$$
$$\hat{\cap}\{y, [min(p_i \circ I^L_Q(y), p_i \circ I^L_S(y)), max(p_i \circ I^U_Q(y), p_i \circ I^U_S(y))]\}$$
$$= \{y, [max(min(p_i \circ I^L_Q(y), p_i \circ I^L_R(y)), min(p_i \circ I^L_Q(y), p_i \circ I^L_S(y))),$$
$$min(max(p_i \circ I^U_Q(y), p_i \circ I^U_R(y)), max(p_i \circ I^U_Q(y), p_i \circ I^U_S(y)))]\}$$
$$= \{y, [min(p_i \circ I^L_Q(y), max(p_i \circ I^L_R(y), p_i \circ I^L_S(y))),$$
$$max(p_i \circ I^U_Q(y), min(p_i \circ I^U_R(y), p_i \circ I^U_S(y)))]\}$$
$$=$$
$$\{y, [p_i \circ I^L_Q(y), p_i \circ I^U_Q(y)]\}\hat{\cup}\{y, [max(p_i \circ I^L_R(y), p_i \circ I^L_S(y)), min(p_i \circ I^U_R(y), p_i \circ I^U_S(y))]\}$$
$$= Q\hat{\cup}(R\hat{\cap}S) = L.H.S$$
$$for\ all\ i = 1, 2, \cdots, m$$

$(2) Proof:-$

$$R.H.S = (Q\hat{\cap}R)\hat{\cup}(Q\hat{\cap}S)$$
$$= \{y, [max(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), min(p_i \circ I_Q^U(y), p_i \circ I_R^U(y))]\}$$
$$\hat{\cup}\{y, [max(p_i \circ I_Q^L(y), p_i \circ I_S^L(y)), min(p_i \circ I_Q^U(y), p_i \circ I_S^U(y))]\}$$
$$= \{y, [min(max(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), max(p_i \circ I_Q^L(y), p_i \circ I_S^L(y))),$$
$$max(min(p_i \circ I_Q^U(y), p_i \circ I_R^U(y)), min(p_i \circ I_Q^U(y), p_i \circ I_S^U(y)))]\}$$
$$= \{y, [max(p_i \circ I_Q^L(y), min(p_i \circ I_R^L(y), p_i \circ I_S^L(y))),$$
$$min(p_i \circ I_Q^U(y), max(p_i \circ I_R^U(y), p_i \circ I_S^U(y)))]\}$$
$$=$$
$$\{y, [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cap}\{y, [min(p_i \circ I_R^L(y), p_i \circ I_S^L(y)), max(p_i \circ I_R^U(y), p_i \circ I_S^U(y))]\}$$
$$= Q\hat{\cap}(R\hat{\cup}S) = L.H.S$$

for all $i = 1, 2, \cdots, m$

De Morgan's Laws 4.10 De Morgan's laws for mIVF sets are given as follows
(1) $(Q\hat{\cup}R)^c = Q^c\hat{\cap}R^c$
(2) $(Q\hat{\cap}R)^c = Q^c\hat{\cup}R^c$

$(1) Proof:-$

$$R.H.S = Q^c\hat{\cap}R^c$$
$$= \{y, [0,1] - [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cap}\{y, [0,1] - [p_i \circ I_R^L(y), p_i \circ I_R^U(y)]\}$$
$$= \{y, [0, p_i \circ I_Q^L(y)) \cup (p_i \circ I_Q^U(y), 1]\}\hat{\cap}\{y, [0, p_i \circ I_R^L(y)) \cup (p_i \circ I_R^U(y), 1]\}$$
$$= \{y, [max(0,0), min(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)))\}$$
$$\cup \{y, (max(p_i \circ I_Q^U(y), p_i \circ I_R^U(y)), min(1,1)]\}$$
$$= \{y, [0, min(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)) \cup (max(p_i \circ I_Q^U(y), p_i \circ I_R^U(y)), 1]\}$$
$$= \{y, [0,1] - [min(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), max(p_i \circ I_Q^U(y), p_i \circ I_Q^U(y))]\}$$
$$= (Q\hat{\cup}R)^c$$
$$= L.H.S$$

for all $i = 1, 2, \cdots, m$

$(2) Proof:-$

$$R.H.S = Q^c\hat{\cup}R^c$$
$$= \{y, [0,1] - [p_i \circ I_Q^L(y), p_i \circ I_Q^U(y)]\}\hat{\cup}\{y, [0,1] - [p_i \circ I_R^L(y), p_i \circ I_R^U(y)]\}$$
$$= \{y, [0, p_i \circ I_Q^L(y)) \cup (p_i \circ I_Q^U(y), 1]\}\hat{\cup}\{y, [0, p_i \circ I_R^L(y)) \cup (p_i \circ I_R^U(y), 1]\}$$
$$= \{y, [min(0,0), max(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)))\}$$
$$\cup \{y, (min(p_i \circ I_Q^U(y), p_i \circ I_R^U(y)), max(1,1)]\}$$
$$= \{y, [0, max(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)) \cup (min(p_i \circ I_Q^U(y), p_i \circ I_R^U(y)), 1]\}$$
$$= \{y, [0,1] - [max(p_i \circ I_Q^L(y), p_i \circ I_R^L(y)), min(p_i \circ I_Q^U(y), p_i \circ I_Q^U(y))]\}$$
$$= (Q\hat{\cap}R)^c$$
$$= L.H.S$$

for all $i = 1, 2, \cdots, m$

5. Distance and Similarity Measure

Definition 5.1 (Distances of mIVF sets). *Let Q and R be two mIVF sets over $Y = \{y_1, y_2, \cdots, y_n\}$ defined as.*

$$Q = \{y_j, p_i \circ I_Q(y_j)\}$$
$$R = \{y_j, p_i \circ I_R(y_j)\}$$
$$\text{for all } i = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots, n$$

where,

$$p_i \circ I_Q(y_j) = [p_i \circ I_Q^L(y_j), p_i \circ I_Q^U(y_j)],$$
$$p_i \circ I_R(y_j) = [p_i \circ I_R^L(y_j), p_i \circ I_R^U(y_j)]$$
$$\text{for all } i = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots, n$$

Their distances will defined as
Hamming distance

$$dis_H(Q, R) = \frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|p_i \circ \alpha_j - p_i \circ \beta_j|\right\} \tag{5. 1}$$

Normalized Hamming distance

$$dis_{NH}(Q, R) = \frac{1}{mn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|p_i \circ \alpha_j - p_i \circ \beta_j|\right\} \tag{5. 2}$$

Euclidean distance

$$dis_E(Q, R) = \sqrt{\frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(p_i \circ \alpha_j - p_i \circ \beta_j)^2\right\}} \tag{5. 3}$$

Normalized Euclidean distance

$$dis_{NE}(Q, R) = \sqrt{\frac{1}{mn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(p_i \circ \alpha_j - p_i \circ \beta_j)^2\right\}} \tag{5. 4}$$

where,

$$p_i \circ \alpha_j = \frac{p_i \circ I_Q^L(y_j) + p_i \circ I_Q^U(y_j)}{2} \tag{5. 5}$$

and

$$p_i \circ \beta_j = \frac{p_i \circ I_R^L(y_j) + p_i \circ I_R^U(y_j)}{2} \tag{5. 6}$$

Theorem 5.2. *The distance measure between mIVF sets Q and R satisfy the following inequalities.*
$(1) dis_H(Q, R) \leq n$,
$(2) dis_{NH}(Q, R) \leq 1$,

$(3) dis_E(Q, R) \leq \sqrt{n}$,
$(4) dis_{NE}(Q, R) \leq 1$,

Theorem 5.3. *The distance functions dis_H, dis_{NH}, dis_E and dis_{NE} defined from $mIVF^Y \to R^+$, are metric distances.*

Proof. Let Q, R and S be three mIVF sets over universal set Y, then
(1) $dis_H(Q, R) \geq 0$
(2) Suppose $dis_H(Q, R) = 0$

$$\Leftrightarrow \frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|p_i \circ \alpha_j - p_i \circ \beta_j|\right\} = 0$$

for all i, j, k

$$\Leftrightarrow |p_i \circ \alpha_j - p_i \circ \beta_j| = 0$$

$$\Leftrightarrow p_i \circ \alpha_j = p_i \circ \beta_j$$

$$\Leftrightarrow Q = R$$

(3) $dis_H(Q, R) = dis_H(R, Q)$
(4) For any three mIVF sets Q, R and S,

$$|p_i \circ \alpha_j - p_i \circ \beta_j|$$

$$= |p_i \circ \alpha_j - p_i \circ \gamma_j + p_i \circ \gamma_j - p_i \circ \beta_j|$$

$$\leq |p_i \circ \alpha_j - p_i \circ \gamma_j| + |p_i \circ \gamma_j - p_i \circ \beta_j|$$

Thus,

$$dis_H(Q, R) \leq dis_H(Q, S) + dis_H(S, R)$$

Definition 5.4 (Similarity Measure)**.** *[6]*
The Similarity Measure($Sim.M$) of two mIVF set Q and R can be defined as

$$Sim.M(Q, R) = \frac{1}{1 + dis(Q, R)} \quad (5.\ 7)$$

Definition 5.5 (Similarity)**.** *[6] Two $mIVF$ sets Q and R are σ similar if and only if $Sim.M(Q, R) \geq \sigma$, i.e.*

$$Q \approx^{\sigma} R \Leftrightarrow Sim.M(Q, R) \geq \sigma, \sigma \in (0, 1) \quad (5.\ 8)$$

Q and R are significantly similar if $Sim.M(Q, R) \geq \frac{1}{2}$.

Example 5.6. *If Q and R be two mIVF set over $Y = \{y_1, y_2\}$ such that*

$$Q = \{(y_1, [0.22, 0.35], [0.40, 0.53]), (y_2, [0.32, 0.36], [0.44, 0.54])\}$$
$$R = \{(y_1, [0.37, 0.69], [0.23, 0.53]), (y_2, [0.10, 0.80], [0.22, 0.24])\}$$

then Hamming distance is

$$dis_H(Q, R) = 0.405$$

and similarity measure will be

$$Sim.M(Q,R) = \frac{1}{1+0.405} = 0.7117$$

It shows Set Q is significantly similar to set R.

Theorem 5.7. *The $Sim.M$ of two mIVF sets Q and R satisfies the following.*
(1). $0 \leq Sim.M(Q,R) \leq 1$
(2). $Sim.M(Q,R) = Sim.M(R,Q)$
(3). *If* $Sim.M(Q,R) = 1 \Leftrightarrow$ *Q=R*

5.8. **A Numerical Example.** In this section, an algorithm is given to compute similarity measure for the structure of mIVF to the application of pattern/brand recognition. Proposed algorithm can be applicable in any field of the patten recognition where more than one opinions are given to single attributive value in the form of interval-valued fuzzy membership.

Algorithm

Step 1 : Assume that there are n number of watches, which are represented by mIVF set H_k, $k = 1, 2, 3, \cdots n$, in feature space ET.
Step 2 : Consider an mIVF Set ϱ is unknown brand watch, that is needed to be recognized.
Step 3 : Convert the interval-valued membership values into multipolar mF set, by using the formula

$$p_i \circ \alpha_{Hkj} = \frac{p_i \circ I^L_{H_k}(y_j) + p_i \circ I^U_{H_k}(y_j)}{2}$$

and

$$p_i \circ \beta_j = \frac{p_i \circ I^L_{\varrho}(y_j) + p_i \circ I^U_{\varrho}(y_j)}{2}$$

which changes $I_{H_k}(y_j)$ and $I_{\varrho}(y_j)$ the interval-valued data into H_k and ϱ the mF set.
Step 4 : After that calculate dis_H, dis_{NH}, dis_E, dis_{NE} distance between H_k and ϱ
Step 5 : Calculate $Sim.M$ (H_k, ϱ) between H_k and ϱ , using the formula

$$Sim.M(H_k, \varrho) = \frac{1}{1+dis(H_k, \varrho)}$$

Step 6 : Lastly, analyze the findings by selecting the Hk, which has the greatest value of similarity measure with unknown brand watch ϱ.
Assume that there are four brands of watches denoted by H_1, H_2, H_3 and H_4. Let $ET = \{et_1$ = Material, et_2 = Glass kind, et_3 = Water Resistance, et_4 = Beautiful Finish$\}$ be feature space of watches. We construct Four $2-$IVF sets (shown in Table 1) and also construct $2-$IVF set of unknown watch as ϱ,

$$\varrho = \{(et_1, [0.37, 0.4][0.92, 0.5]), (et_2, [0.71, 0.5], [0.41, 0.73]), (et_3, [0.93, 0.95][0.62, 0.22]), (et_4[0.13, 0.51][0.11, 0.73])\}$$

Then we convert $2-$IVF sets H_1, H_2, H_3, H_4 set into $2-$F sets (shown in Table 2) and ϱ into $2-$F sets as follows using equation (5.5) and (5.6).

$$\varrho = \{(et_1, (0.385, 0.915)), (et_2, (0.605, 0.57)), (et_3, (0.94, 0.42)), (et_4(0.32, 0.42))\}$$

brands	et_1	et_2	et_3	et_4
H_1	$[0.20, 0.35]$,$[0.40, 0.50]$	$[0.30, 0.60]$,$[0.40.0.35]$	$[0.10, 0.71]$,$[0.92, 0.50]$	$[0.71, 0.92]$,$[0.24, 0.39]$
H_2	$[0.20, 0.40]$,$[0.40, 0.50]$	$[0.30, 0.50]$,$[0.20.0.50]$	$[0.30, 0.50]$,$[0.10, 0.40]$	$[0.17, 0.95]$,$[0.21, 0.40]$
H_3	$[0.30, 0.50]$,$[0.40, 0.50]$	$[0.30, 0.80]$,$[0.40.0.90]$	$[0.40, 0.70]$,$[0.10, 0.20]$	$[0.30, 0.90]$,$[0.40, 0.20]$
H_4	$[0.71, 0.11]$,$[0.30, 0.40]$	$[0.12, 0.90]$,$[0.30.0.92]$	$[0.15, 0.40]$,$[0.50, 0.70]$	$[0.31, 0.57]$,$[0.41, 0.72]$

TABLE 1. Represents four $2-$IVF sets H_1, H_2, H_3, H_4

brands	et_1	et_2	et_3	et_4
H_1	$(0.275, 0.45)$	$(0.45, 0.375)$	$(0.405, 0.71)$	$(0.815, 0.315)$
H_2	$(0.30, 0.45)$	$(0.40, 0.35)$	$(0.40, 0.25)$	$(0.56, 0.34)$
H_3	$(0.40, 0.45)$	$(0.55, 0.65)$	$(0.55, 0.15)$	$(0.60, 0.30)$
H_4	$(0.41, 0.35)$	$(0.51, 0.61)$	$(0.275, 0.60)$	$(0.44, 0.565)$

TABLE 2. Represents four $2-$polar Fuzzy sets H_1, H_2, H_3, H_4

Euclidean distance formula (5.3) gives the distance measure of H_k and ϱ as

$$d_E(H_1, \varrho) = 0.6968$$
$$d_E(H_2, \varrho) = 0.6185$$
$$d_E(H_3, \varrho) = 0.5352$$
$$d_E(H_4, \varrho) = 0.6632$$

By using Similarity Measure formula (5.7) $(Sim.M)$ is

$$Sim.M(H_1, \varrho) = \frac{1}{1 + d_E(H_1, \varrho)} = \frac{1}{1 + 0.6968} = 0.5893$$

$$Sim.M(H_2, \varrho) = \frac{1}{1 + d_E(H_2, \varrho)} = \frac{1}{1 + 0.6185} = 0.6178$$

$$Sim.M(H_3, \varrho) = \frac{1}{1 + d_E(H_3, \varrho)} = \frac{1}{1 + 0.5352} = 0.6513$$

$$Sim.M(H_4, \varrho) = \frac{1}{1 + d_E(H_4, \varrho)} = \frac{1}{1 + 0.6632} = 0.6012$$

While $Sim.M$ is greater than 0.5 for all brands, it is H_3 that is much greater than the others, which is why the unknown watch varrho is too near to brand H_3.

5.9. **Limitation of the Method.** There are several limitations of the method that must be assured before implementing the similarity measure criteria.

(1) Similarity measure can be found between two sets at a time to find comparison among themselves.
(2) The two sets must be independent of each other and must be from the same structure.

6. TOPSIS

A methodology to extend the TOPSIS to m-polar interval-valued set is determined in this section. This process is very applicable to deal with the group decision-making problem under mIVF system.

Each criterion's concern weight can be determined either directly or indirectly by pairwise comparisons. Suppose that there is a group of m decision-makers who assess the scores of alternatives $\ddot{A}_j(j = 1, 2, cdots, p)$ based on criteria $\ddot{C}_k(k = 1, 2, cdots, q)$ in interval-valued membership, while the evaluation of criterion weights is perceived. A matrix representation of a multi-criteria multi-person decision-making problem is as fol- lows:

$$\ddot{G} = \begin{bmatrix} g^I_{11} & g^I_{12} & \cdots & g^I_{1q} \\ g^I_{21} & g^I_{22} & \cdots & g^I_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ g^I_{p1} & g^I_{p2} & \cdots & g^I_{pq} \end{bmatrix}_{p\times q}$$

where p denotes the number of alternatives, q denotes the number of criteria and g^I_{jk} for all j, k; represents the ratings of $jth-$alternatives concering the $kth-$criteria in the interval-valued. For ranting g^I_{jk}, the data of m decision-makers can be represented as

$$g^I_{jk} = [p_i \circ g^L_{jk}, p_i \circ g^U_{jk}]$$
where $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, p$; and $k = 1, 2, \cdots, q$

Convert the given multipolar interval valued membership evaluation of alternatives into multipolar m-single-valued membership evaluation by,

$$p_i \circ g_{jk} = \frac{p_i \circ g^L_{jk} + p_i \circ g^U_{jk}}{2} \tag{6. 9}$$
for all $i = 1, 2, \cdots, m$

then we get rating of alternatives as,

$$\ddot{g}_{jk} = (p_1 \circ g_{jk}, p_2 \circ g_{jk}, \cdots, p_m \circ g_{jk})$$
where $j = 1, 2, \cdots, p$; and $k = 1, 2, \cdots, q$

and the criteria weights are presented in multipolar information as

$$\ddot{w}_k = (p_1 \circ w_k, p_2 \circ w_k, \cdots, p_m \circ w_k)$$
where $k = 1, 2, \cdots, q$

As fuzzy numbers belong to [0,1], then by normalization we get the normalized mF decision matrix denoted as $\ddot{R}$ and

$$\ddot{R} = [\ddot{r}_{jk}]_{p\times q} \text{ where,}$$

$$\ddot{r}_{jk} = \left(p_i \circ r_{jk} = \frac{p_i \circ g_{jk}}{\sqrt{\sum_{j=1}^{p} (p_i \circ g_{jk})^2}} \right) \tag{6. 10}$$
for all $i = 1, 2, \cdots, m$

The above-mentioned normalisation method preserves the property of the mIVF set that the ranges of membership of elements are [0, 1]. We establish the weighted normalised mIVF decision matrix, denoted as $\ddot{V}$, by taking into account the various importance of each criterion,

$$\ddot{V} = [\ddot{v}_{jk}]_{p\times q} \text{ where,}$$

$$\ddot{v}_{jk} = \ddot{r}_{jk}(\cdot)\ddot{w}_k = (p_i \circ r_{jk}(\cdot)p_i \circ w_k) \tag{6. 11}$$

$$\text{for all } i = 1, 2, \cdots, m$$

In accordance with the weighted normalized mF decision matrix, it can be seen that the elements $\ddot{v}_{jk}$ for all i, j; are normalized. Then, we evaluate the mF positive-ideal solution (mFPIS, A^*) and mF negative-ideal solution (mFNIS, A^-) as

$$A^* = (\ddot{v}_1^*, \ddot{v}_2^*, \cdots, \ddot{v}_q^*),$$
$$A^- = (\ddot{v}_1^-, \ddot{v}_2^-, \cdots, \ddot{v}_q^-),$$

where,

$$\ddot{v}_k^* = (p_i \circ v_k^*) = \begin{cases} (1, 1, \cdots, 1) & k \in B; \\ (0, 0, \cdots, 0) & k \in C. \end{cases}$$

and

$$\ddot{v}_k^- = (p_i \circ v_k^-) = \begin{cases} (0, 0, \cdots, 0) & k \in B; \\ (1, 1, \cdots, 1) & k \in C. \end{cases}$$
$$\text{for all } i = 1, 2, \cdots, m \text{ and } k = 1, 2, \cdots, q$$

where B and C denotes the benefit and cost criteria respectively. The Euclidean distance (Separation) of each alternative from A^* and A^- can be calculated as

$$S_j^* = d(\ddot{v}_{jk}, \ddot{v}_k^*) = \sqrt{\frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{k=1}^{q}(p_i \circ v_{jk} - p_i \circ v_k^*)^2\right\}} \tag{6. 12}$$

$$S_j^- = d(\ddot{v}_{jk}, \ddot{v}_k^-) = \sqrt{\frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{k=1}^{q}(p_i \circ v_{jk} - p_i \circ v_k^-)^2\right\}} \tag{6. 13}$$

Next move forward to the closeness coefficient to rank all alternatives after computation of S_j^* and S_j^- of each alternative $\ddot{A}_j$, $(j = 1, 2, \cdots, p)$.

$$C.Cof_j = \frac{S_j^-}{S_j^- + S_j^*} \tag{6. 14}$$

As the closeness coefficient $C.Cof_j$ approaches one, the alternative $\ddot{A}_j$ is similar to mFPIS (A^*) and far from mFNIS (A^-). As a result, using the $C.Cof_j$, we will rate all alternatives and choose the best one that is closest to 1.

Algorithm

In short, the algorithm of the multi-decision maker multi-criteria decision-making in the approach of mIVF set is given as follows.

Step 1: Make a group of decision-makers, then analyzes the evaluation criteria.

Step 2: Evaluate the ratings of alternatives $\ddot{A}_j$ concerning criteria $\ddot{C}_k$ in interval-valued and weight of criterion.

Step 3: Create an mIV decision matrix.

Step 4: Convert the mIV data into m-single-valued.

Step 5: Normalized mSV to get mF decision matrix.

Step 6: Create the weighted normalized mF decision matrix.

	$\ddot{C}_1$	$\ddot{C}_2$	$\ddot{C}_3$	$\ddot{C}_4$
$\ddot{A}_1$	$([3,5],[2,5],[1,4])$	$([5,6],[5,5],[5,7])$	$([4,7],[4,6],[6,7])$	$([7,9],[8,10],[6,9])$
$\ddot{A}_2$	$([6,8],[5,7],[8,8])$	$([6,6],[5,7],[4,7])$	$([2,2],[2,4],[1,2])$	$([1,3],[2,6],[3,5])$
$\ddot{A}_3$	$([6,7],[8,9],[9,10])$	$([5,7],[3,6],[4,4])$	$([4,6],[5,8],[4,6])$	$([6,7],[3,6],[4,8])$
$\ddot{A}_4$	$([2,2],[3,4],[1,3])$	$([6,8],[6,9],[6,7])$	$([9,9],[8,10],[9,9])$	$([2,6],[3,7],[3,6])$
$\ddot{w}$	$(0.32,0.25,0.28)$	$(0.24,0.26,0.26)$	$(0.21,0.29,0.23)$	$(0.23,0.20,0.23)$

TABLE 3. Represents 3-IV decision matrix and 3-polar weights

	$\ddot{C}_1$	$\ddot{C}_2$	$\ddot{C}_3$	$\ddot{C}_4$
$\ddot{A}_1$	$(4,3.5,2.5)$	$(5.5,5,6)$	$(5.5,5,6.5)$	$(8,9,7.5)$
$\ddot{A}_2$	$(7,6,8)$	$(6,6,5.5)$	$(2,3,1.5)$	$(2,4,4)$
$\ddot{A}_3$	$(6.5,8.5,9.5)$	$(6,4.5,4)$	$(5,6.5,5)$	$(6.5,4.5,6)$
$\ddot{A}_4$	$(2,3.5,2)$	$(7,7.5,6.5)$	$(9,9,9)$	$(4,5,4.5)$

TABLE 4. 3-polar decision matrix

Step 7: Determine the fuzzy positive ideal solution (mFPIS) and negative ideal solution (mFNIS).
Step 8: Compute the Separation measure of each alternative from mFPIS and mFNIS, respectively.
Step 9: Compute the closeness coefficient $C.Cof_j$ of each alternative.
Step 10: According to the closeness coefficient, give the rank to all alternatives and select the best one.

6.1. **A Numerical Example.** Assume an investment firm wants to do some investment in best alternative. The company established a committee of three individuals to choose the best option from a list of four potential alternatives for investing the money, $\ddot{A}_1$ is a telecommunication company $\ddot{A}_2$ is a food company $\ddot{A}_3$ is an electronics company $\ddot{A}_4$ is a medicine company
The committee must take following criteria under consideration for the decision making process

- $\ddot{C}_1$ is the environmental impact analysis
- $\ddot{C}_2$ is the risk analysis
- $\ddot{C}_3$ is the social-political impact analysis
- $\ddot{C}_4$ is the growth analysis

The committee of three decision-makers p_i ($i = 1, 2, 3$) evaluate all four possible alternatives using interval-valued data in the range [0,10] and weight of criterion in fuzzy single-valued as shown in Table 3.
Now, converting interval-valued data in above table to single-valued data as shown in Table 4 by using equation (6.9).

	$\ddot{C}_1$	$\ddot{C}_2$	$\ddot{C}_3$	$\ddot{C}_4$
$\ddot{A}_1$	$(0.38, 0.30, 0.19)$	$(0.44, 0.42, 0.53)$	$(0.46, 0.39, 0.53)$	$(0.71, 0.75, 0.66)$
$\ddot{A}_2$	$(0.66, 0.52, 0.62)$	$(0.48, 0.51, 0.49)$	$(0.16, 0.24, 0.12)$	$(0.17, 0.33, 0.35)$
$\ddot{A}_3$	$(0.61, 0.73, 0.74)$	$(0.48, 0.38, 0.35)$	$(0.42, 0.51, 0.40)$	$(0.57, 0.37, 0.53)$
$\ddot{A}_4$	$(0.18, 0.30, 0.15)$	$(0.57, 0.64, 0.58)$	$(0.76, 0.71, 0.73)$	$(0.35, 0.42, 0.39)$

TABLE 5. Normalized 3-F decision matrix

Then we normalized the multi-polar decision matrix to get the normalized mF decision matrix as shown in Table 5 by using (6.10).
Weighted Normalized mF decision matrix is concluded (shown in Table 6) by using equation (6.11).

	$\ddot{C}_1$	$\ddot{C}_2$	$\ddot{C}_3$	$\ddot{C}_4$
$\ddot{A}_1$	$(0.12, 0.07, 0.05)$	$(0.10, 0.10, 0.13)$	$(0.09, 0.11, 0.12)$	$(0.16, 0.15, 0.15)$
$\ddot{A}_2$	$(0.21, 0.13, 0.17)$	$(0.11, 0.13, 0.12)$	$(0.03, 0.06, 0.02)$	$(0.03, 0.06, 0.08)$
$\ddot{A}_3$	$(0.19, 0.18, 0.20)$	$(0.11, 0.09, 0.09)$	$(0.08, 0.14, 0.09)$	$(0.13, 0.07, 0.12)$
$\ddot{A}_4$	$(0.05, 0.07, 0.04)$	$(0.13, 0.16, 0.15)$	$(0.15, 0.20, 0.16)$	$(0.08, 0.08, 0.08)$

TABLE 6. Weighted normalized 3-F decision matrix

As $\ddot{C}_1$ and $\ddot{C}_2$ are cost criteria, and $\ddot{C}_3$ and $\ddot{C}_4$ are benefit criteria. Then the (mFPIS, A^*) and (mFNIS, A^-) are given as follows,

$$A^* = ((0,0,0),(0,0,0),(1,1,1),(1,1,1))$$
$$A^- = ((1,1,1),(1,1,1),(0,0,0),(0,0,0))$$

Computing separation measure (S_j^* and S_j^-) between each attribute with (mFPIS, A^*) and (mFNIS, A^-) by using (6.12) and (6.13) respectively, we get the results (in Table 7),

	S^*	S^-
$\ddot{A}_1$	1.2387	1.2940
$\ddot{A}_2$	1.3649	1.2123
$\ddot{A}_3$	1.2841	1.2229
$\ddot{A}_4$	1.2491	1.2885

TABLE 7. Separation Measures

Computing closeness coefficient $C.Cof_j$, $(j = 1,2,3,4)$ of each alternative by using (6.14), we get results (shown in Table 8)
As $\ddot{A}_1 = 0.51$ is highest among others in Table 8, which is the best ideal solution obtained via the multi-person TOPSIS process, so by the result obtained by the help of decision-making committee of the company is to invest money in a telecommunication company to get best results in earning money.

	$C.Cof$
$\ddot{A}_1$	0.51
$\ddot{A}_2$	0.47
$\ddot{A}_3$	0.48
$\ddot{A}_4$	0.50

TABLE 8. Closeness Coefficients

7. COMPARATIVE ANALYSIS

mIVF set is a comprehensive idea that may be used to tackle real world problems are in the form of m-attributes expressed in interval valued rather than single-valued form. Existing theories cannot be utilized to address or investigate the issues; nevertheless, they do have limitations (see Table 9). Because of these restrictions, they are unable to deal with the data in form of multi-interval-valued data. In Table 9, our suggested model is compared to current methodologies. When the attributes have been studied as an interval-valued form with the decision taken by more than one person, these previous methods fail to execute. The proposed mIVF addresses this shortcoming. It demonstrates that, in comparison to existing methods, our structure is sound and capable of successfully dealing with such challenges. Now, we talk about our proposed strategy and how precise it is.

References	Disadvantage	Ranking
Fuzzy Set [41]	fail to manage multipolar and interval-valued data	Unable to address
Bipolar Fuzzy Set[26]	fail to manage multipolar and interval-valued data	Unable to address
M-Polar Fuzzy Set [9]	fail to manage interval-valued data	Unable to address
Interval-valued Fuzzy Set [20]	fail to manage multipolar data	Unable to address
Proposed Method in this paper	Long calculations in decision-making	Addresses the multipolar and interval-valued data

TABLE 9. Superiority of mIVF set over existing theories

8. CONCLUSION AND RECOMMENDATION

Ordinarily, the problem of multipolar interval-valued fuzzy information occurs, and cannot be adequately elaborated using the current approaches. To overcome the uncertainty, a multipolar interval-valued fuzzy set is introduced, specifically in the interval-valued structure with multipolar. In this article, distance-based $Sim.M$ with the multipolar interval-valued set is used to improve the solution of many stressful situations. It broadens the range of applications in fields such as electronic optimization, industry, and forensic facial portraiture. All of $Sim.M > 0.5$ and too much closeness of Sim.M ensures the importance of its application because it able more efficient and reliable the investigation agency to capture the person, one who is very near to the actual suspect (using mIVF FORENSIC FACIAL PORTRAIT). In addition, based on similarity measure in MCDM, a new method for the best mIVF alternatives is depicted. Furthermore, a TOPSIS technique is defined on the structure of mIVF for the selection of the best attribute. mIVF has defiantly opened the new ways to be applied in various hybrid structure of fuzzy sets such as mIVF Soft set, mIV Intuitionistic Fuzzy Soft Set, mIV Neutrosophic Soft Set, mIVF Hypersoft set, mIVF Plithogenic Hypersoft Set, mIV Intuitionistic Fuzzy Hypersoft Set, mIV Neutrosophic Hypersoft Set, Pythagorean fuzzy uncertain environment, and their hybrid structures in the future. It can also be used in artificial intelligence, medical imaging, data mining, pattern recognition, social understanding, recommender frameworks, machine learning, social networks, signal processing, the monetary framework, neural networks, image processing, quantum geometry, and game theory, among other things.

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