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$Infra - \beta - Closed(Open)$ Sets : New Characterization and Applications

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Abstract. As applications on $infra - \beta - open$ set for new classes of mappings and new concepts topological spaces are introduced, namely, infra- $\beta{-}continuous$, $infra{-}\beta{-}irresolute$, $infra{-}\beta{-}open(closed)$ mapping, $infra - \beta - \tau_i$ $(i = 0, 1, 2, 3, 4)$, $infra - \beta - compact$ and $infra - \beta - connected$ space. Some of special results and properties which belong to these various applications are established and studied. Moreover, the relations and opposite relations between these new concepts and others are discussed and counter examples are given in order to investigate opposite relations.

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Key Words: $infra - \beta - open(closed)$ set;$infra - \beta - continuous$ mapping; $infra - \beta - open(closed)$ mapping; $infra - \beta - irresolute$; $infra - \beta - compact$; $infra - \beta - separation$ axioms; $infra - \beta - connected$.

1. Introduction

Since 1982, Dunham [8] has introduced new operators Cl^* and Int^*. Othman [[14],[15],[16] [17], [18], [19], [20] and [21]] generalized these operators to fuzzy topology and introduced new classes of sets and mappings, such as $infra - \alpha open(closed)$ and $supra - \beta - open(closed)$. Other mathematicians, like, Selvi, Dharani, Missier, Rodrigo and Robert used these new operators in order to introduce [resp. $pre * open$, $\alpha * open$ and $semi * open$] set [resp. [23], [11] and [22]].

In this paper, in using these operators, we introduce various applications on $infra - \beta - open$ set like $infra - \beta - compact$, $infra - \beta - connected$, $infra - \beta - \tau_i$ $(i = 0, 1, 2, 3, 4)$, $infra - \beta - open(closed)$ mapping, $infra - \beta - continuous$ mapping and $infra - \beta - irresolute$ mapping. Moreover, we discuss and study the relations between these new concepts and illustrate the opposite relations with examples. Finally, interesting and spacial results on these new concepts are investigated.

2. PRELIMINARIES

A set $\varphi \subseteq X$ is called a $\beta - open$ [2] (resp. $\alpha - open$ [13], $pre - open$ [12] and semi open [10] set if $\varphi \subseteq cl\ Int\ cl\ (\varphi)$ (resp. $\varphi \subseteq Int\ cl\ Int\ (\varphi)$, $\varphi \subseteq Int\ cl\ (\varphi)$ and $\varphi \subseteq cl\ Int\ (\varphi)$). The family of all $\beta - open$ (resp. $\alpha - open$, $pre - open$ and semi open) sets of X is denoted as $\beta - O(X)$ (resp. $\alpha O(X)$, $PO(X)$ and $SO(X)$).

Definition 2.1. [8] *Let φ any set. Then, (i) Closure*$^*(\varphi)$ $\big(Cl^*(\varphi)\big)$ *is the intersection of all $generalized - closed$ sets containing φ.*

(ii) Interior$^*(\varphi)$ $\big(Int^*(\varphi)\big)$ *is the union of all $generalized - open$ sets contained in φ.*

Definition 2.2. *A set $\varphi \subseteq X$ is called an $infra - \beta - open$* [20] *(resp. $infra - \beta - closed$* [20]*, $infra - pre - open$* [20]*, $infrasemi\ open$* [22] *and $infra - \alpha - open$* [17] *) set if $\varphi \subseteq Cl^*\ Int\ Cl^*\ (\varphi)$ (resp. $Int^*\ Cl\ Int^*\ (\varphi) \subseteq \varphi$, $\varphi \subseteq Int\ Cl^*\ (\varphi)$, $\varphi \subseteq Cl^*\ Int\ (\varphi)$ and $\varphi \subseteq Int\ Cl^*\ Int\ (\varphi)$). The family of all $infra - \beta - open$ (resp. $infra - \beta - closed$, $infra - pre - open$, $infra$ semi open and $infra - \alpha - open$) sets of X is denoted as $I - \beta - O(X)$ (resp. $I - \beta - C(X)$, $I - PO(X)$, $I - SO(X)$ and $I - \alpha - O(X)$).*

Definition 2.3. [9] *In a topological space (X, τ), a subset μ is $generalized - closed$ if $cl(\mu) \subseteq \eta$ whenever $\mu \subseteq \eta$ and η is open set.*

Lemma 2.4. [8] *Let μ any set. Then,*
- $\varphi \subseteq Cl^*\ (\varphi)\ \subseteq\ Cl(\varphi)$.
- $Int\ (\varphi)\ \subseteq Int^*\ (\varphi)\ \subseteq\ (\varphi)$.

Definition 2.5. [1] *A topological space (Y, τ_Y) is called:*
- $\beta - \tau_0$ *if* $\forall x \neq y$, $\exists \beta - open$ *set* $\varphi \in \tau_Y$: *either* $x \in \varphi$ *and* $y \notin \varphi$, *or* $y \in \varphi$ *and* $x \notin \varphi$.
- $\beta - \tau_1$ *if* $\forall x \neq y$, $\exists \beta - open$ *set* $\varphi_1,\ \varphi_2 \in \tau_Y : x \in \varphi_1$, $y \notin \varphi_1$, *and* $y \in \varphi_2$, $x \notin \varphi_2$.
- $\beta - \tau_2$ *if* $\forall x \neq y$, $\exists \beta - open$ *sets* $\varphi_1,\ \varphi_2 \in \tau_Y : x \in \varphi_1$, $y \in \varphi_2$ *and* $\varphi_1 \cap \varphi_2 = \phi$.
- $\beta - Regular$ *space if* $\forall \eta \in \tau_Y^c$ *and* $x \in Y : x \notin \eta$, $\exists \beta - open$ *sets* $\varphi_1,\ \varphi_2 \in \tau_Y$: $x \in \varphi_1$, $\eta \subseteq \varphi_2$ *and* $\varphi_1 \cap \varphi_2 = \phi$.
- $\beta - \tau_3$ *if* (X, τ) *is* $\beta - T_1$ *and* $\beta - regular$ *space.*
- $\beta - Normal$ *space if* $\forall \eta_1,\ \eta_2 \in \tau_Y^c : \eta_1 \cap \eta_2 = \phi$, $\exists \varphi_1,\ \varphi_2 \in \beta - O(X)$: $\eta_1 \subseteq \varphi_1$, $\eta_2 \subseteq \varphi_2$ *and* $\varphi_1 \cap \varphi_2 = \phi$. .
- $\beta - \tau_4$ *if* (X, τ) *is* $\beta - T_1$ *and* $\beta - normal$ *space.*

Definition 2.6. *A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is said to be:*
- $Infra - \alpha - continuous$ [17] *if* $f^{-1}(\varphi) \in I - \alpha O(X)$, $\forall\ \varphi \in \tau_X$.
- $Infra - pre - continuous$ [22] *if* $f^{-1}(\varphi) \in I - PO(X))$, $\forall\ \varphi \in \tau_X$.
- $Infra - semi - continuous$ *if* $f^{-1}(\varphi) \in I - SO(X))$, $\forall\ \varphi \in \tau_X$.

Here, we can construct new topology namely, Infra-topological space by using the operator Int^*.

Definition 2.7. *Let (X, τ) be a topological space. the new topology generated by Int^*. That is, Infra-$\tau_x = \{\lambda \subseteq X : Int^*(\lambda) = \lambda\}$.*
The member of Infra-τ_x is called Infra-open set, the complement of it called Infra-closed set and the family of all Infra-open (resp. Infra-closed) sets can be denoted by infra-O(X) (resp. infra-C(X)).

Remark 2.8.

- **(i):** *The members of Infra-τ_x are closed under the intersection properties and the complement (Infra-τ_x^c) are closed under the union properties.*
- **(ii):** *The union of member of Infra-τ_x is not be member of Infra-τ_x in general.*
- **(iii):** *The intersection of member of Infra-τ_x^c is not be member of Infra-τ_x^c in general.*
- **(iv):** $\tau_x \subseteq Infra - \tau_x$ *and* $\tau_x^c \subseteq Infra - \tau_x^c$.

We can clarify the Remark 2.8 by the following example.

Example 2.9. *Let* $X = \{1, 2, 3\}$*, then*

- $\tau_x = \{\phi, \{3\}, X\}$*, then Infra-*$\tau_x = \{\phi, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, X\}$ *we can see that* $\tau_x \subseteq Infra - \tau_x$*. If* $\lambda_1 = \{1\}$ *and* $\lambda_2 = \{2\}$ *are Infra-open sets but* $\lambda_1 \cup \lambda_2 = \{1, 2\}$ *is not Infra-open set.*
- $\tau_x^c = \{\phi, \{1, 2\}, X\}$*, then* $Infra - \tau_x^c = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, U\}$ *we can see that* $\tau_x^c \subseteq Infra - \tau_x^c$*. If* $\mu_1 = \{1, 3\}$ *and* $\mu_{2,3} = \{3\}$ *are Infra-closed sets but* $\mu_1 \cap \mu_2 = \{3\}$ *is not Infra-closed set.*

3. INFRA-β- CONTINUOUS AND INFRA-β-OPEN (CLOSED) MAPPING

Definition 3.1. *A mapping* $f : (X, \tau_X) \to (Y, \tau_Y)$ *is said to be:*

- $Infra - \beta - continuous$ *if* $f^{-1}(\lambda) \in I\beta - O(X)\big(I\beta - C(X)\big), \forall\lambda \in \tau_Y\big(\tau_Y^c\big)$.
- $Infra - \beta - irresolute$ *if* $f^{-1}(\lambda) \in I\beta - O(X)\big(I\beta - C(X)\big), \forall\lambda \in I\beta - O(X)\big(I\beta - C(X)\big)$.

Theorem 3.2. *For a mapping* $f : (X, \tau_X) \to (Y, \tau_Y)$*, the statements below are the same:*

- **(i):** f *is* $infra - \beta - continuous$*;*
- **(ii):** $\forall\ \lambda \in O(Y)$ *and* $a \in X : f(a) \in \lambda$, $\exists\mu \in I\beta - O(X)$*:* $a \in\ \mu$ *and* $f(\mu) \subseteq \lambda$*;*
- **(iii):** $\forall\ \mu \in C(Y)$*, hence* $f^{-1}(\mu) \in I\beta - C(X)$*;*
- **(vi):** $Int^*ClInt^*(f^{-1}(\lambda)) \subseteq (f^{-1}(Cl\ (\lambda)))$, $\forall\ \lambda \in Y$*;*
- **(v):** $f(Int^*ClInt^*(\mu)) \subseteq Cl(f(\mu))$, $\forall\ \mu \in X$.

Proof.

$(i) \Rightarrow (ii)$**.:** If $a \in X$ and $\forall\lambda \in O(Y) : f(a) \in\ \lambda$, hence $\exists\mu \in I\beta - O(X) : a \in \mu$ and $a \in \mu \subseteq f^{-1}(\lambda)$. Therefore, $f(\mu) \subseteq \lambda$.

$(ii) \Rightarrow (i)$**.:** Let $\lambda \in Y$ and if take $a \in f^{-1}(\lambda)$ and we have $f(a) \in \lambda$. Since $\lambda \in O(Y)$, then $\exists\mu \in I\beta - O(X) : a \in\ \mu$ and $f(\mu) \subseteq\ \lambda$ and we have $a \in\ \mu \subseteq (f^{-1}(\lambda))$. Hence, $f^{-1}(\lambda) \in I\beta - O(X)$.

$(i) \Rightarrow (iii)$**.:** Let $\lambda \in C(Y)$. This show that $\lambda^c \in O(Y)$. This implies that $f^{-1}(\lambda^c) \in I\beta - O(X)$. Hence, $f^{-1}(\lambda) \in I\beta - C(X)$.

$(iii) \Rightarrow (iv)$**.:** Consider $\lambda \in Y$, then $f^{-1}(Cl\ (\lambda)) \in I\beta{-}C(X)$, then $Int^*ClInt^*(f^{-1}(\lambda)) \subseteq (f^{-1}(Cl(\lambda)))$.

$(iv) \Rightarrow (v)$**.:** Suppose $\mu \in X$ and let take $\lambda = f(\mu)$ in (iv), We have $Int^*ClInt^*(f^{-1}(f(\mu))) \subseteq f^{-1}(Cl(f(\mu)))$. Then, $Int^*ClInt^*(\mu) \subseteq f^{-1}(Cl(f(\mu)))$. This show that $f(Int^*ClInt^*(\mu)) \subseteq Cl(f(\mu))$.

$(v) \Rightarrow (i)$**:** Consider $\lambda \in Y$ and $\mu = f^{-1}(\lambda^c)$ in (v). We have $f(Int^*ClInt^*(f^{-1}(\lambda^c))) \subseteq Cl\ f(f^{-1}(\lambda^c)) = \lambda^c$. Hence, $f^{-1}(\lambda^c) \in S\beta - C(X)$ and f is infra-β- continuous.

Corollary 3.3. *For a mapping* $f : (X, \tau_X) \to (Y, \tau_Y)$*, The statements below are the same:*

(i) f *is an* $infra - \beta - continuous$*;*
(ii) For any closed set in Y*, the inverse image of it is an* $infra - \beta - closed$*;*
(iii) $f(I\beta - -Cl(\mu)) \subseteq Cl(f(\mu)), \forall \mu \in X$*;*
(iv) $I\beta - Cl(f^{-1}(\lambda)) \subseteq f^{-1}(Cl(\lambda)), \forall \lambda \in Y$*;*
(v) $f^{-1}(Int(\lambda)) \subseteq I\beta - Cl(f^{-1}(\lambda)), \forall \lambda \in Y$*.*

The "Implication Diagram 1" to give an illustration of the relations between different sorts of continuous mappings.

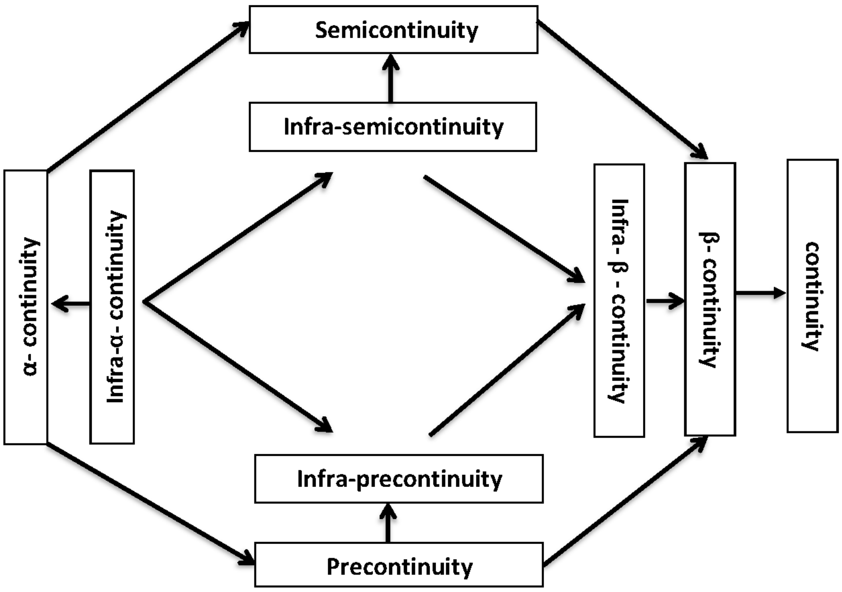

Diagram 1

Remark 3.4. *The opposite relationships must not be necessarily true in the Implication Diagram 1 as appeared by the following examples:*

Example 3.5. *If* $f : (X, \tau_1) \to (Y, \tau_2)$ *be an identity mappings where,* $Y = X = \{a, b, c\}$*,* $\tau_1 = \{\phi, \{a\}, X\}$ *and* $\tau_2 = \{\phi, \{a, c\}, X\}$*. Then,* f *is a* β*-continuous but not an infra-*β*-continuous mapping.*

Example 3.6. *If* $f : (X, \tau_X) \to (Y, \tau_Y)$ *be an identity mappings where,* $Y = X = \{a, b, c\}$*,* $\tau_X = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ *and* $\tau_Y = \{\phi, \{a, c\}, Y\}$*, then:*

- f *is an infra-*$\beta - continuous$ *which is neither an* $infra - precontinuous$ *nor an* $infra - \alpha continuous$*.*
- f *is an* $infra - \beta - continuous$ *which is not continuous mapping.*

Example 3.7. *Consider the space* (X, τ_x) *and* (Y, τ_y) *where,* $Y = X = \{1, 2, 3, 4\}$*.*

Let $\tau_x = \{\phi, \{1\}, \{2\}, \{1,2\}, X\}$, $\tau_y = \{\phi, \{1\}, \{2\}, \{1,2\}, \{2,4\}, X\}$ *and* $f : (X, \tau_x) \to (Y, \tau_y)$ *be an identity mappings. We can see*

- *f is an infra-semicontinuous mapping where as it is not a continuous mapping.*
- *f is an infra-semicontinuous mapping which is not an infra-α-continuous mapping.*

Definition 3.8. *A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is said to be an $infra-\beta-open(closed)$ if $f(\mu) \in I\beta - O(Y)\big(I\beta - C(Y)\big)$, $\forall \mu \in \tau_X\big(\tau_X^c\big)$.*

Theorem 3.9. *For the bijective mapping $f : (X, \tau_X) \to (Y, \tau_Y)$, The statements below are the same:*
(i) f is $an\ infra-\beta-closed$ mapping;
(ii) $f^{-1}(I\beta - Cl(\lambda)) \subseteq Cl(f^{-1}(\lambda))$, $\forall \lambda \in Y$;
(iii) $Int(f^{-1}(\lambda)) \subseteq f^{-1}(I\beta - Int(\lambda))$, $\forall \lambda \in Y$;
(iv) $Int(f^{-1}(\lambda)) \subseteq f^{-1}(Cl^*\ Int\ Cl^*\ (\lambda))$, $\forall \lambda \in Y$.

Proof.

$(i) \Rightarrow (ii)$**.:** If $\lambda \in X$ and f be an infra-$\beta-closed$ mapping, then

$$f^{-1}(\lambda) \subseteq Cl(f^{-1}(\lambda)) \ \Rightarrow\ I\beta - Cl(\lambda)) \subseteq I\beta - Cl(f(Cl(f^{-1}(\lambda))))$$
$$I\beta - Cl(\lambda) \subseteq f(Cl(f^{-1}(\lambda))) \ \Rightarrow\ f^{-1}(I\beta - Cl(\lambda)) \subseteq Cl(f^{-1}(\lambda)).$$

$(ii) \Rightarrow (iii)$**.:** Let $\lambda \in Y$, hence $\lambda^c \in Y$ by (ii) we have

$$f^{-1}(I\beta - Cl(\lambda^c)) \subseteq Cl(f^{-1}(\lambda^c)) \ \Rightarrow\ (f^{-1}(I\beta - Int(\lambda)))^c \subseteq (Int(f^{-1}(\lambda)))^c.$$

Then, $Int\big(f^{-1}(\lambda)\big) \subseteq f^{-1}\big(I\beta - Int(\lambda)\big)$.

$(iii) \Rightarrow (iv)$**.:** By (iii) we can easily get

$$Int(f^{-1}(\lambda)) \subseteq f^{-1}\big(I\beta - Int(\lambda)\big) \subseteq f^{-1}\big(Cl^*\ Int\ Cl^*(I\beta - Int(\lambda))\big)$$
$$Int\big(f^{-1}(\lambda)\big) \subseteq f^{-1}\big(Cl^*\ Int\ Cl^*(\lambda)\big).$$

$(iv) \Rightarrow (i)$**.:** Let $\mu \in C(X)$. Thus, $\big(f(\mu)\big)^c \in Y$

By using (iv), we have

$$Int\big(f^{-1}((f(\mu))^c)\big) \subseteq f^{-1}\big(Cl^*\ Int\ Cl^*((f(\mu))^c)\big)$$
$$\big(Cl(f^{-1}((f(\mu))))\big)^c \subseteq \big(f^{-1}(Int^*\ Cl\ Int^*((f(\mu))))\big)^c.$$
$$Int^*\ Cl\ Int^*\big((f(\mu))\big) \subseteq f(\mu).$$

Then, $f(\mu) \in I\beta - C(Y)$ and f is an infra-$\beta-closed$ mapping.

Corollary 3.10. *For a mapping $f : (X, \tau_X) \to (Y, \tau_Y)$, The statements below are the same:*
(i) f is $an\ infra-\beta-open$ mapping;
(ii) $f(Int\mu) \subseteq I\beta - Int(f(\mu))$, $\forall \mu \in X$;
(iii) $Int\big(f^{-1}(\lambda)\big) \subseteq f^{-1}\big(I\beta - Int(\lambda)\big)$, $\forall \lambda \in Y$;
(iv) $f^{-1}\big(I\beta - Cl(\lambda)\big) \subseteq Cl\big(f^{-1}(\lambda)\big)$, $\forall \lambda \in Y$;
(v) $f(Int\mu) \subseteq Cl^*\ Int\ Cl^*\ (f(\mu))$, $\forall \mu \in X$.

4. INFRA-β- TOPOLOGICAL SPACE

Definition 4.1. *A (X, τ) topological space is called:*

- $Infra - \beta - T_0$ *if* $\forall x \neq y \in X$, $\exists \varphi \subseteq I\beta - O(X)$ *: either* $x \in \varphi$ *and* $y \notin \varphi$, *or* $y \in \varphi$ *and* $x \notin \varphi$.
- $Infra - \beta - T_1$ *if* $\forall x \neq y \in X$, $\exists \varphi_1,\ \varphi_2 \subseteq I\beta - O(X)$ *:* $x \in \varphi_1$, $y \notin \varphi_1$, *and* $y \in \varphi_2$, $x \notin \varphi_2$.
- $Infra - \beta - T_2$ *if* $\forall x \neq y \in X$, $\exists \varphi_1,\ \varphi_2 \subseteq I\beta - O(X)$ *:* $x \in \varphi_1$, $y \in \varphi_2$ *and* $\varphi_1 \cap \varphi_2 = \phi$.
- $Infra - \beta - regular\ space$ *if* $\forall \eta \in \tau^c$ *and* $x \in X$ *:* $x \notin \eta$, $\exists \varphi_1,\ \varphi_2 \subseteq I\beta - O(X)$ *:* $x \in \varphi_1$, $\eta \subseteq \varphi_2$ *and* $\varphi_1 \cap \varphi_2 = \phi$.
- $Infra - \beta - T_3$ *if* (X, τ) *is an* $infra - \beta - T_1$ *and* $infra - \beta - regular\ space$.
- $Infra - \beta - Normal\ space$ *if* $\forall \eta_1,\ \eta_2 \in \tau_Y^c$ *:* $\eta_1 \cap \eta_2 = \phi$, $\exists \varphi_1,\ \varphi_2 \in I\beta - O(X)$ *:* $\eta_1 \subseteq \varphi_1$, $\eta_2 \subseteq \varphi_2$ *and* $\varphi_1 \cap \varphi_2 = \phi$.
- $Infra - \beta - T_4$ *if* (X, τ) *is an* $infra - \beta - T_1$ *and* $infra - \beta - normal\ space$.

Theorem 4.2. *A (X, τ) space is an $infra - \beta - T_0$ iff $I\beta - Cl(\{x\}) \neq I\beta - Cl(\{y\})$, $\forall x \neq y \in X$.*

Proof.

Necessity: Let $I\beta - Cl(\{x\}) \neq I\beta - Cl(\{y\})\ \forall x \neq y \in X$. Hence, $I\beta - Cl(\{x\}) \nsubseteq I\beta - Cl(\{y\})$ or $I\beta - Cl(\{y\}) \nsubseteq I\beta - Cl(\{x\})$.
Suppose that $I\beta - Cl(\{x\}) \nsubseteq I\beta - Cl(\{y\}) \Rightarrow x \notin (I\beta - Cl(\{y\}))$
$\Rightarrow x \in (I\beta - Cl(\{y\}))^c \in I\beta - O(X)$ and $y \notin (I\beta - Cl(\{y\}))^c$. Hence, (X, τ) is an $infra - \beta - T_0$ space.

Sufficiency: Let (X, τ) is an $infra - \beta - T_0$ space, then $\forall x \neq y \in X, \exists \varphi \in I\beta - O(X)$: either $x \in \varphi$ and $y \notin \varphi$, or $y \in \varphi$ and $x \notin \varphi$. If we consider $x \in \varphi$ and $y \notin \varphi$. Therefore, $\varphi^c \in I\beta - C(X)$ and $x \notin \varphi^c$, $y \in \varphi^c$, then $x \notin I\beta - Cl(\{y\})$. Hence, $I\beta - Cl(\{x\}) \neq I\beta - Cl(\{y\})$.

Theorem 4.3. *A (X, τ) space is an $infra - \beta - T_0$, then $I\beta - Int(I\beta - Cl(\{x\})) \cap I\beta - Int(I\beta - Cl(\{y\})) = \phi$, $\forall x \neq y \in X$.*

Proof. Let (X, τ) be an $infra - \beta - T_0$ space, then $\forall x \neq y \in X, \exists \varphi \in I\beta - O(X)$: either $x \in \varphi$ and $y \notin \varphi$, or $y \in \varphi$ and $x \notin \varphi$.
If $x \in \varphi$ and $y \notin \varphi \Rightarrow x \notin \varphi^c$ and $y \in \varphi^c$.
Therefore, $I\beta - Int(I\beta - Cl(\{y\})) \subseteq \varphi^c \Rightarrow I\beta - Int(I\beta - Cl(\{y\})) \cap \varphi = \phi$.
Hence, $x \in \varphi \subseteq (I\beta - Int(I\beta - Cl(\{y\})))^c$
and $I\beta - Int(I\beta - Cl(\{x\})) \subseteq (I\beta - Int(I\beta - Cl(\{y\})))^c$.
Then, $I\beta - Int(I\beta - Cl(\{x\})) \cap I\beta - Int(I\beta - Cl(\{y\})) = \phi$.

Theorem 4.4. *A (X, τ) space is an $infra - \beta - T_1$ iff $\{x\} \in I\beta - C(X)$, $\forall x \in X$.*

Proof.

Necessity: Let $\{x\}, \{y\} \in I\beta - C(X)\ \forall x \neq y \in X$. This show that $x \in \{y\}^c, y \in \{x\}^c$ and $\{x\}^c, \{y\}^c \in I\beta - O(X)$. Then, (X, τ) is an $infra - \beta - T_0$ space.

Sufficiency: Let X is an $infra - \beta - T_1$ space, then $\forall x \neq y \in X, \exists \varphi \in I\beta - O(X)$: $y \in \varphi$ and $x \notin \varphi$. This show that $y \in \varphi \subseteq \{x\}^c$, therefore $\{x\} \in I\beta - C(X)$.

Remark 4.5. *If $g, f : (X, \tau_x) \rightarrow (Y, \tau_y)$ are $irresolute - infra - \beta - continuous$ (resp. $infra - \beta - continuous$) mapping and Y is $infra - \beta - \tau_2$ (resp. τ_2) space, then the set $A = \{x : x \in X, f(x) = g(x)\}$ is not $infra - \beta - closed$ set. We shall illustrate that by the following example:*

Example 4.6. *If $g, f : (X, \tau_x) \to (Y, \tau_y)$ are $irresolute - infra - \beta - continuous$ ($infra - \beta - continuous$) mapping define as following:*

$$f(a) = 2, f(b) = 1 \text{ and } f(c) = 2$$
$$g(a) = 2, g(b) = 1 \text{ and } g(c) = 1$$

and let $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{1\}, \{2\}, Y\}$ where, $X = \{a, b, c\}$ and $Y = \{1, 2\}$, we can see that $A = \{x : x \in X, f(x) = g(x)\} = \{a, b\}$ but is not an $infra - \beta - closed$ set.

The "Implication Diagram 2" to give an illustration of the relations between different sorts of β-topological spaces.

$$\begin{tikzcd}
\boxed{Infra-\beta-\tau_4} \arrow[r] \arrow[d] & \boxed{\beta-\tau_4} \arrow[d] \\
\boxed{Infra-\beta-\tau_3} \arrow[r] \arrow[d] & \boxed{\beta-\tau_3} \arrow[d] \\
\boxed{Infra-\beta-\tau_2} \arrow[r] \arrow[d] & \boxed{\beta-\tau_2} \arrow[d] \\
\boxed{Infra-\beta-\tau_1} \arrow[r] \arrow[d] & \boxed{\beta-\tau_1} \arrow[d] \\
\boxed{Infra-\beta-\tau_0} \arrow[r] & \boxed{\beta-\tau_0}
\end{tikzcd}$$

(Diagram 2)

Remark 4.7. *The opposite relationships must not be necessarily true in the Implication Diagram 2 as appeared by the following examples:*

Example 4.8. *Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, \{2, 3\}, X\}$. We can see*

- *τ is $\beta - \tau_0$, $\beta - \tau_1$, $\beta - \tau_2$, $\beta - \tau_3$ and $\beta - \tau_4$ space.*
- *τ is not $infra - \beta - \tau_0$, $infra - \beta - \tau_1$, $infra - \beta - \tau_2$, $infra - \beta - \tau_3$ and $infra - \beta - \tau_4$ space.*

Example 4.9. *Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, X\}$. We can see τ is an $infra - \beta - \tau_0$ space but not $infra - \beta - \tau_1$ space.*

Example 4.10. *Let $(\mathbb{R}, \tau_{cof})$, where τ_{cof} is Cofinite topological space. We can see τ is an $infra - \beta - \tau_1$ but not $infra - \beta - \tau_2$.*

Example 4.11. *Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, \{1\}, \{1, 2\}, X\}$. We can see τ is an $infra - \beta - normal$ space but not $infra - \beta - regular$ space.*

Theorem 4.12. *A (X, τ_x) space is an $infra - \beta - T_2$ iff $\forall x \neq y \in X$, $\exists \varphi \in I\beta - O(X)$: $x \in \varphi$ and $y \notin I\beta - Cl(\varphi)$.*

Proof.

Necessity:Suppose that $\forall x \neq y \in X \; \exists \varphi \in I\beta - O(X) : x \in \varphi$ and $y \notin I\beta - Cl(\varphi)$. This show that $y \in (I\beta - Cl(\varphi))^c \; \in I\beta - O(X)$ and $\varphi \cap (I\beta - Cl(\varphi))^c = \phi$. Then, X is an $infra - \beta - T_2$ space.

Sufficiency: Let (X, τ_x) is an $infra - \beta - T_2$ space, then $\forall x \neq y \in X \; \exists \varphi$ and λ $\in I\beta - O(X)$ such : $x \in \varphi, y \in \lambda$ and $\varphi \cap \lambda = \phi$. Therefore, $\varphi \subseteq \lambda^c \Rightarrow I\beta - Cl(\varphi) \subseteq \lambda^c$ $\Rightarrow y \notin I\beta - Cl(\varphi)$.

Now, we will discuss some interesting results on an $infra - \beta - compact$ space and an $infra - \beta - connected$ space as follows:

Definition 4.13. *A (X, τ) space is said to be an $infra - \beta - compact$ space if every $infra - \beta - open$ cover has finite subcover.*

Theorem 4.14. *A (X, τ) space is an $infra - \beta - compact$ iff the finite intersection of $infra - \beta - closed$ sets in (X, τ) has non-empty intersection.*

Proof.

Necessity: Let (X, τ) is an $infra - \beta - compact$ space and $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ be a family of $infra - \beta - closed$ sets.
If we assume that $\bigcap_{\alpha \in I} A_\alpha = \phi$, we have $\mathcal{A}^c = \{A_\alpha^c = X - A_\alpha : \alpha \in I\}$ are the family of $infra - \beta - open$ sets in (X, τ).
Then, $\bigcup_{\alpha \in I} A_\alpha^c = \bigcup_{\alpha \in I}(X - A_\alpha) = X - \phi = X$
We get that $\bigcup_{\alpha \in I} A_\alpha^c$ is an $infra - \beta - open$ cover of X but (X, τ) is an $infra - \beta - compact$ space.
Thus, $X \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}^c = \bigcup_{i=1}^{n}(X - A_{\alpha_i}^c) = X - \bigcap_{i=1}^{n} A_{\alpha_i}$ which mean that $\bigcap_{i=1}^{n} A_{\alpha_i}$ should be empty is a contradiction, then $\bigcap_{\alpha \in I} A_\alpha \neq \phi$
Sufficiency: We will prove that (X, τ) is an $infra - \beta - compact$ space so, we assume that $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ is an $infra - \beta - open$ cover for X.
Then, we have $\mathcal{A}^c = \{A_\alpha^c : \alpha \in I\}$ be a family of $infra - \beta - closed$ sets and $\bigcap_{\alpha \in I} A_\alpha^c \neq \phi$ is $infra - \beta - closed$ set see {[20],Theorem 2.4.}.
We put $\bigcap_{\alpha \in I} A_\alpha^c = \mathcal{B} \Rightarrow X - \mathcal{B} = \bigcup_{i=1}^{n} A_{\alpha_i} \Rightarrow (\bigcup_{i=1}^{n} A_{\alpha_i}) \bigcup \mathcal{B}^c = X$.
Then, we get (X, τ) is an $infra - \beta - compact$ space.

Theorem 4.15. *A (X, τ) space is an $infra - \beta - \tau_2$ iff $\forall x \neq y \in X, \exists U \in \tau : x \in I\beta - O(X)$ and $y \in (I\beta - Cl(U))^c$.*

Proof.

Necessity:Let $x \neq y \in X$ and (X, τ) is an $infra - \beta - \tau_2$ hence, $\exists U, V \in I\beta - O(X)$: $x \in U, y \in V$ and $U \cap V = \phi \Rightarrow I\beta - Cl(U) \subseteq V^c \Rightarrow V \subseteq (I\beta - Cl(U))^c$, then $y \in (I\beta - Cl(U))^c$.

Sufficiency: Suppose that $\forall x \neq y \in X, \exists U \in I\beta - O(X) : x \in U$ and $y \in (I\beta - Cl(U))^c \in I\beta - O(X)$. and we have $U \cap (I\beta - Cl(U))^c = \phi$. Then, (X, τ) is an $infra - \beta - \tau_2$ space.

Theorem 4.16. *IF $f : (X, \tau_X) \to (Y, \tau_Y)$ mapping is an $infra - \beta - continuous$ from $infra - \beta - compact$ space to $infra - \beta - \tau_2$ space, then f is an $infra - \beta - closed$ mapping.*

Proof. Let $F \in X$ is an $infra - \beta - closed$ set, then we should prove that $f(F)$ is also $infra - \beta - closed$ set in Y.
Since X is an $infra - \beta - compact$ space then, easy to prove that F is an $infra - \beta -$

$compact$ subset of X.
Now we will prove that $f(F)$ is an $infra-\beta-compact$ subset of Y so we assume that $\mathcal{A}=\{A_\alpha : \alpha \in I\}$ is an $infra-\beta-open$ cover of $f(F)$ i.e $f(F) \subseteq \bigcup_{\alpha\in I} A_\alpha \Rightarrow F \subseteq \bigcup_{\alpha\in I} f^{-1}(A_\alpha)$. Since F is an $infra-\beta-compact$ subset, hence $F \subseteq \bigcup_{i=1}^{n} f^{-1}(A_{ij}) \Rightarrow f(F) \subseteq \bigcup_{i=1}^{n}(A_{ij})$. Then, $f(F)$ is an $infra-\beta-compact$ subset of Y. Since Y is $infra-\beta-\tau_2$ space and $f(F)$ is an $infra-\beta-compact$ subset it and let $x \notin f(F)$ and $x \neq p \in Y, \exists U_x, V_x \in T_y : x \in U_x, p \in V_x$ and $U_x \cap V_x = \phi$ i.e $\{U_x : x \in f(F)\}$ is an $infra-\beta-open$ cover of $f(F)$. But $f(F)$ is an $infra-\beta-compact$ subset of Y, then we have finite sub cover of it i.e $f(F) \subseteq U_{x_1} \cup U_{x_2} \cup .... \cup U_{x_n} = \bigcup_{i=1}^{n} U_{x_i} \subseteq U$. Put $V = V_{p_1} \cap V_{p_2} \cap .... \cap V_{p_n} = \bigcap_{i=1}^{n} V_{p_i}$, then $U \cap V = (\bigcup_{i=1}^{n} U_{x_i}) \bigcap (\bigcap_{i=1}^{n} V_{p_i}) = \phi$. Thus, $f(F) \cap V = \phi \Rightarrow V \subseteq (f(F))^c \Rightarrow p \in V \subseteq (f(F))^c$. So, $(f(F))^c$ is an $infra-\beta-open$ set, hence $f(F)$ is an $infra-\beta-closed$ set in Y. Then f is an $infra-\beta-closed$ mapping.

Definition 4.17. *A set $\mu \in X$ is said to be an $infra-\beta-regular$ set if φ is both $infra-\beta-open$ and $infra-\beta-closed$ set.*

Definition 4.18. *A (X,τ) is said to be an $infra-\beta-connected$ space if X cannot written as the Union of two nonempty disjoint $infra-\beta-open$ sets.*

Theorem 4.19. *If (X,τ) be a topological space, then the following properties are same:*

- **(i):** *(X,τ) is an $infra-\beta-connected$;*
- **(ii):** *(X,τ) cannot expressed as the union of two nonempty disjoint $infra-\beta-closed$ sets in X;*
- **(iii):** *X and ϕ are the only $infra-\beta-regular$ set.*

Proof.

- $(i) \Rightarrow (ii)$**:** Assume (X,τ) is an $infra-\beta-connected$ space. Suppose that φ_1 and φ_2 are nonempty disjoint $infra-\beta-closed$ sets : $X = \varphi_1 \cup \varphi_2$. Therefore, $\varphi_1 = \varphi_2^c$ and $\varphi_2 = \varphi_1^c$ are nonempty disjoint $infra-\beta-open$ sets, which contradicts. Then (ii).
- $(ii) \Rightarrow (iii)$**:** Suppose that φ be a nonempty disjoint $infra-\beta-regular$ set in X. Thus, $X = \varphi \cup \varphi^c$, which contradicts. Then (iii).
- $(iii) \Rightarrow (i)$**:** Let (X,τ) is not an $infra-\beta-connected$ space, then $\exists\varphi_1$ and φ_2 are nonempty disjoint $infra-\beta-open$ sets : $X = \varphi_1 \cup \varphi_2$. Thus, $\varphi_1 = \varphi_2^c$ and $\varphi_2 = \varphi_1^c$. This show that φ_1 is an $infra-\beta-regular$ set, which contradicts. Then, (X,τ) is an $infra-\beta-connected$ space.

Corollary 4.20.

- *Every $\beta-connected$ is $infra-\beta-connected$ space.*
- *Every $infra-\beta-connected$ is connected space.*

Remark 4.21. *The examples below demonstrate that the opposite of Corollary (4.20) is usually not true:*

Example 4.22. *The indiscreet topology (X,τ) is connected and $infra-\beta-connected$ but not $\beta-connected$ space.*

Example 4.23. *Let $X=\{a,b,c\}$ and $\tau=\{\phi,\{a\},\{b\},\{a,b\},X\}$, then it is connected but not $infra-\beta-connected$ space.*

Definition 4.24. *If φ Unable to write two non-empty disjoint $infra-\beta-open$ sets as a Union, a set $\varphi \in X$ is said to be an $infra-\beta-connected$*

Theorem 4.25. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be an $infra - \beta - irresolutesurjective$ mapping. If φ is an $infra - \beta - connected$ subset of X, then $f(\varphi)$ is a connected in Y.*

Proof. Assume $f(\varphi)$ is not an $infra - \beta - connected$ in Y. Hence, $\exists \mu$ and $\lambda \in Y$ are $infra - \beta - open$ sets : $f(\varphi) = \mu \cup \lambda$.
Since f is an $infra - \beta - irresolutesurjective$ mapping, $f^{-1}(\mu)$ and $f^{-1}(\lambda)$ are $infra - \beta - open$ sets in X and $\varphi = f^{-1}(f(\varphi)) = f^{-1}(\mu \cup \lambda) = f^{-1}(\mu) \cup f^{-1}(\lambda)$.
It is clear that $f^{-1}(\mu)$ and $f^{-1}(\lambda)$ are $infra - \beta - open$ in X. Therefore, φ is not an $infra - \beta - connected$ in X, which is a C!.
Then, $f(\varphi)$ is an $infra - \beta - connected.$

5. Conclusion

in this paper, we introduced new mappings and concepts in topological spaces. The relations between these new concepts and mappings are being studied with other topological spaces and mappings. The results of this paper can be extended to fuzzy sets, soft sets, nano sets, leading to the development of the information system and various fields in computational topology see, [[3], [4], [5], [6], [7]].

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