Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 53(2)(2021) pp. 1-17

New Inequalities of Fejér and Hermite-Hadamard type Concerning Convex and Quasi-Convex Functions With Applications

Muhammad Amer Latif
Department of Basic Sciences,
Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia,
Email: m amer latif@hotmail.com

[1,2]Sever Silvestru Dragomir
[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa Email: sever.dragomir@vu.edu.au
Sofian Obeidat
Department of Basic Sciences,
Deanship of Preparatory Year, University of Hail, Hail 2440, Saudi Arabia
Email: obeidatsofian@gmail.com

Received: 23 April, 2019 / Accepted: 27 January, 2021 / Published online: 22 February, 2021

Abstract.: This research contains new integral inequalities of Fejér and Hermite-Hadamard type involving convex and quasi-convex functions. Applications of the newly established results for special means of positive real numbers are given.

AMS (MOS) Subject Classification Codes: Primary 26D15; Secondary 26A51; 26A33
Key Words: convex functions, Hölder inequality, quasi-convex function, Hermite-Hadamard inequality.

1. INTRODUCTION

We should mention that $\mathcal{U}$ is an interval and $\mathcal{U}^\circ$ is the interior of $\mathcal{U}$ wherever they appear in this paper.

The following definitions are well known in the literature.

Definition 1.1. [21] *A function $\gamma_1 : \mathcal{U} \subset \mathcal{R} \to \mathcal{R}$ is called convex function (in the classical sense) if the inequality*

$$\gamma_1 (\alpha\lambda + (1-\alpha)\mu) \leq \alpha\gamma_1 (\lambda) + (1-\alpha)\gamma_1 (\mu)$$

holds for all λ, $\mu \in \mathcal{U}$ *and* $\alpha \in [0,1]$.

Definition 1.2. [21] *A function* $\gamma_1 : \mathcal{U} \subset \mathcal{R} \to \mathcal{R}$ *is called quasi convex function, if the inequality*

$$\gamma_1\left(\alpha\lambda + (1-\alpha)\mu\right) \leq \max\left\{\gamma_1(\lambda), \gamma_1(\mu)\right\}$$

holds for all λ, $\mu \in \mathcal{U}$ *and* $\alpha \in [0,1]$.

It should be noted that a convex function must be a quasi-convex function but not conversely. The past few decades have witnessed remarkable research on inequalities, including a large number of papers and many fertile applications. The subject has evoked considerable interest from many mathematicians, and an extensive number of new results have been studied in the literature. It is recognized that in general some specific inequalities provide a useful and essential contrivance in the growth of various branches of mathematics. A number of interesting results have been proved by using the concept of classical convexity, s-convexity and harmonically s-convex functions, see for instance [1]-[28] and the references therein. Here we recall some of the results for convex and quasi-convex functions which are closely related to the research of our paper.

Dragomir and Agarwal [4] proved the subsequent result for differentiable convex mappings.

Theorem 1.3. [4] *Let* $\gamma_1 : \mathcal{U}^\circ \subseteq \mathcal{R} \to \mathcal{R}$ *be a differentiable mapping on* $\mathcal{U}^\circ$ *and* θ_1, $\theta_2 \in \mathcal{U}^\circ$ *with* $\theta_1 < \theta_2$. *If* $|\gamma_1'|$ *is convex on* $[\theta_1, \theta_2]$, *then*

$$\left|\frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} \gamma_1(\lambda)\, d\lambda\right| \leq \frac{(\theta_2 - \theta_1)}{8}\left[|\gamma_1'(\theta_1)| + |\gamma_1'(\theta_2)|\right] \quad (1.\ 1)$$

Hwang [6] obtained the given results which contains the result of Theorem 1.3 as a special case.

Theorem 1.4. [6] *Let* $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \to \mathcal{R}$ *be a differentiable mapping on* $\mathcal{U}^\circ$ *and* $\gamma_2 : [\theta_1, \theta_2] \to [0, \infty)$ *be a continuous and symmetric mapping with respect to* $\frac{\theta_1+\theta_2}{2}$, *where* $\theta_1, \theta_2 \in \mathcal{U}^\circ$ *with* $\theta_1 < \theta_2$.

(1) *If* $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ *and* $|\gamma_1'|$ *is convex on* $[\theta_1, \theta_2]$, *then*

$$\left|\frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\, d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\,\gamma_1(\lambda)\, d\lambda\right|$$

$$\leq \frac{(\theta_2 - \theta_1)}{2}\left[\frac{|\gamma_1'(\theta_1)| + |\gamma_1'(\theta_2)|}{2}\right]\int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda)\, d\lambda d\alpha. \quad (1.\ 2)$$

(2) *If* $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ *and* $|\gamma_1'|^{\mathbf{q}}$ *is convex on* $[\theta_1, \theta_2]$ *for* $\mathbf{q} \geq 1$, *then*

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\, d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\gamma_1(\lambda)\, d\lambda \right| \leq \frac{(\theta_2 - \theta_1)}{2} \left[\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}} + |\gamma_1'(\theta_2)|^{\mathbf{q}}}{2} \right]^{\frac{1}{\mathbf{q}}} \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda)\, d\lambda d\alpha, \quad (1.\ 3)$$

(3) *If* $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ *and* $|\gamma_1'|$ *is quasi-convex on* $[\theta_1, \theta_2]$, *then*

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\, d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\gamma_1(\lambda)\, d\lambda \right| \leq \frac{(\theta_2 - \theta_1)}{4} \left[\max\left\{ |\gamma_1'(\theta_1)|, \left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right| \right\} + \max\left\{ \left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|, |\gamma_1'(\theta_2)| \right\} \right] \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda)\, d\lambda d\alpha. \quad (1.\ 4)$$

(4) *If* $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ *and* $|\gamma_1'|^{\mathbf{q}}$ *is quasi-convex on* $[\theta_1, \theta_2]$ *for* $\mathbf{q} \geq 1$, *then*

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\, d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda)\gamma_1(\lambda)\, d\lambda \right| \leq \frac{(\theta_2 - \theta_1)}{4} \left[\left(\max\left\{ |\gamma_1'(\theta_1)|^{\mathbf{q}}, \left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|^{\mathbf{q}} \right\} \right)^{\frac{1}{\mathbf{q}}} + \left(\max\left\{ \left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|^{\mathbf{q}}, |\gamma_1'(\theta_2)|^{\mathbf{q}} \right\} \right)^{\frac{1}{\mathbf{q}}} \right] \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda)\, d\lambda d\alpha. \quad (1.\ 5)$$

where

$$T(\alpha) = \frac{1+\alpha}{2}\theta_1 + \frac{1-\alpha}{2}\theta_2 \text{ and } S(\alpha) = \frac{1-\alpha}{2}\theta_1 + \frac{1+\alpha}{2}\theta_2.$$

The subsequent results are due to Hua et al. [7].

Theorem 1.5. [7] *Let* $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \to \mathcal{R}$ *be a differentiable mapping on* $\mathcal{U}^\circ$ *and* $\gamma_2 : [\theta_1, \theta_2] \to [0, \infty)$ *be a continuous and symmetric mapping with respect to* $\frac{\theta_1+\theta_2}{2}$, *where* $\theta_1, \theta_2 \in \mathcal{U}^\circ$ *with* $\theta_1 < \theta_2$.

(1) *If* $\gamma_1' \in \mathcal{L}_1([\theta_1,\theta_2])$ *and* $|\gamma_1'|^{\mathbf{q}}$ *is convex on* $[\theta_1,\theta_2]$ *for* $\mathbf{q} \geq 1$*, then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \leq \frac{(\theta_2-\theta_1)}{2}\left[\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{2}\right]^{\frac{1}{\mathbf{q}}}\int_0^1\int_{M(\alpha)}^{V(\alpha)}\gamma_2(\lambda)\,d\lambda d\alpha, \quad (1.\ 6)$$

(2) *If* $\gamma_1' \in \mathcal{L}_1([\theta_1,\theta_2])$ *and* $|\gamma_1'|^{\mathbf{q}}$ *is convex on* $[\theta_1,\theta_2]$ *for* $\mathbf{q} > 1$*, then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq \frac{(\theta_2-\theta_1)}{4}\left\{\left(\frac{3|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{4}\right)^{\frac{1}{\mathbf{q}}}+\left(\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+3|\gamma_1'(\theta_2)|^{\mathbf{q}}}{4}\right)^{\frac{1}{\mathbf{q}}}\right\}$$
$$\times\min\left\{\left[\int_0^1\left(\int_{T(\alpha)}^{S(\alpha)}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}},\left[\int_0^1\left(\int_{M(\alpha)}^{V(\alpha)}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}}\right\}, \quad (1.\ 7)$$

where

$$M(\alpha)=\alpha\theta_1+(1-\alpha)\frac{\theta_1+\theta_2}{2},\ V(\alpha)=\alpha\theta_2+(1-\alpha)\frac{\theta_1+\theta_2}{2}$$

and $T(\alpha)$ *and* $S(\alpha)$ *are defined as in Theorem 1.4.*

Motivated by the results mentioned above the main objective of this paper is prove new results of Fejér and Hermite-Hadamard type by using the convexity and quasi-convexity based new family identities for a positive integer $\mathbf{m}$. The results proved in this paper may have some relation with those proved earlier for some specific values of $\mathbf{m}$.

2. New Results

An important lemma to prove the results is given as follows.

Lemma 2.1. *Let* $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \to \mathcal{R}$ *be a differentiable mapping on* $\mathcal{U}^\circ$ *and* $\gamma_2 : [\theta_1,\theta_2] \to [0,\infty)$ *be a continuous and symmetric mapping with respect to* $\frac{\theta_1+\theta_2}{2}$*, where* $\theta_1, \theta_2 \in \mathcal{U}^\circ$ *with* $\theta_1 < \theta_2$*. If* $\gamma_1' \in \mathcal{L}_1([\theta_1,\theta_2])$*, then the equality*

$$\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda$$
$$=\left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\int_0^{\frac{\mathbf{m}}{2}}\left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda\right)\left[\gamma_1'(\zeta_2(\alpha,\mathbf{m}))-\gamma_1'(\zeta_1(\alpha,\mathbf{m}))\right]d\alpha\right] \quad (2.\ 8)$$

holds, where $\zeta_1(\alpha,\mathbf{m}) = \frac{\alpha}{\mathbf{m}}\theta_2 + \left(\frac{\mathbf{m}-\alpha}{\mathbf{m}}\right)\theta_1$, $\zeta_2(\alpha,\mathbf{m}) = \frac{\alpha}{\mathbf{m}}\theta_1 + \left(\frac{\mathbf{m}-\alpha}{\mathbf{m}}\right)\theta_2$, $\|\gamma_2\|_\infty = \sup_{\alpha\in[\theta_1,\theta_2]} |\gamma_2(\alpha)|$ *and* $\mathbf{m}$ *is a positive integer.*

Proof. Using integration by parts, we have

$$\begin{aligned}\mathcal{U}_1 &= \int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})} \gamma_2(\lambda)\,d\lambda\right) \gamma_1'(\zeta_1(\alpha,\mathbf{m}))\,d\alpha \\ &= \frac{\mathbf{m}}{\theta_2-\theta_1}\int_0^1 \left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})} \gamma_2(\lambda)\,d\lambda\right) d(\gamma_1(\zeta_1(\alpha,\mathbf{m}))) \\ &= \frac{\mathbf{m}}{\theta_2-\theta_1}\left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})} \gamma_2(\lambda)\,d\lambda\right)\gamma_1(\zeta_1(\alpha,\mathbf{m}))\Bigg|_0^{\frac{\mathbf{m}}{2}} \\ &+ \int_0^{\frac{\mathbf{m}}{2}} \left[\gamma_2(\zeta_2(\alpha,\mathbf{m})) + \gamma_2(\zeta_1(\alpha,\mathbf{m}))\right]\gamma_1(\zeta_1(\alpha,\mathbf{m}))\,d\alpha\end{aligned}$$

Since γ_2 is symmetric with respect to $\frac{\theta_1+\theta_2}{2}$, we have

$$\gamma_2(\zeta_2(\alpha,\mathbf{m})) = \gamma_2(\zeta_1(\alpha,\mathbf{m})),$$

and hence

$$\mathcal{U}_1 = -\frac{\mathbf{m}}{\theta_2-\theta_1}\left(\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda\right)\gamma_1(\theta_1) + 2\int_0^{\frac{\mathbf{m}}{2}}\gamma_2(\zeta_1(\alpha,\mathbf{m}))\,\gamma_1(\zeta_1(\alpha,\mathbf{m}))\,d\alpha.$$

Setting $\zeta_1(\alpha,\mathbf{m}) = \lambda$, we have

$$\mathcal{U}_1 = -\frac{\mathbf{m}}{\theta_2-\theta_1}\left(\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda\right)\gamma_1(\theta_1) + \frac{2\mathbf{m}}{\theta_2-\theta_1}\int_{\theta_1}^{\frac{\theta_1+\theta_2}{2}}\gamma_2(\lambda)\,\gamma_1(\lambda)\,d\lambda. \qquad (2.\ 9)$$

Similarly,

$$\mathcal{U}_2 = \int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})} \gamma_2(\lambda)\,d\lambda\right) \gamma_1'(\zeta_2(\alpha,\mathbf{m}))\,d\alpha \qquad (2.\ 10)$$

$$= \frac{\mathbf{m}}{\theta_2-\theta_1}\left(\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda\right)\gamma_1(\theta_2) - \frac{2\mathbf{m}}{\theta_2-\theta_1}\int_{\frac{\theta_1+\theta_2}{2}}^{\theta_2}\gamma_2(\lambda)\,\gamma_1(\lambda)\,d\lambda. \qquad (2.\ 11)$$

Subtracting (2. 9) from (2. 10) and multiplying the resulting equality by $\frac{\theta_2-\theta_1}{2\mathbf{m}}$, we get (2. 8). □

Remark 2.2. *If $\mathbf{m}=2$ in Lemma 2.1, the following identity*

$$\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda$$
$$=\left(\frac{\theta_2-\theta_1}{4}\right)\left[\int_0^1\left(\int_{\zeta_1(\alpha,2)}^{\zeta_2(\alpha,2)}\gamma_2(\lambda)\,d\lambda\right)\left[\gamma_1'(\zeta_2(\alpha,2))-\gamma_1'(\zeta_1(\alpha,2))\right]d\alpha\right] \quad (2.\ 12)$$

holds, where $\zeta_1(\alpha,2)=\frac{\alpha}{2}\theta_2+\left(\frac{2-\alpha}{2}\right)\theta_1$, $\zeta_2(\alpha,2)=\frac{\alpha}{2}\theta_1+\left(\frac{2-\alpha}{2}\right)\theta_2$, $\|\gamma_2\|_\infty=\sup_{\alpha\in[\theta_1,\theta_2]}|\gamma_2(\alpha)|$.

Remark 2.3. *If $\gamma_2(\lambda)=\frac{1}{\theta_2-\theta_1}$ for all $\lambda\in[\theta_1,\theta_2]$ in Lemma 2.1, we get the equality*

$$\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\gamma_1(\lambda)\,d\lambda$$
$$=\left(\frac{\theta_2-\theta_1}{2\mathbf{m}^2}\right)\int_0^{\frac{\mathbf{m}}{2}}(\mathbf{m}-2\alpha)\left[\gamma_1'(\zeta_2(\alpha,\mathbf{m}))-\gamma_1'(\zeta_1(\alpha,\mathbf{m}))\right]d\alpha, \quad (2.\ 13)$$

where $\zeta_1(\alpha,\mathbf{m})$, $\zeta_2(\alpha,\mathbf{m})$ are defined as in Lemma 2.1 and $\mathbf{m}\geq 1$ is positive integer.

Remark 2.4. *If $\gamma_2(\lambda)=\frac{1}{\theta_2-\theta_1}$ for all $\lambda\in[\theta_1,\theta_2]$ in Lemma 2.1 and $\mathbf{m}=2$, we get the equality*

$$\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\gamma_1(\lambda)\,d\lambda$$
$$=\left(\frac{\theta_2-\theta_1}{4}\right)\int_0^1(1-\alpha)\left[\gamma_1'(\zeta_2(\alpha,2))-\gamma_1'(\zeta_1(\alpha,2))\right]d\alpha, \quad (2.\ 14)$$

where $\zeta_1(\alpha,2)$, $\zeta_2(\alpha,2)$ are defined as in Remark 2.2.

Theorem 2.5. *Let $\gamma_1:\mathcal{U}\subseteq\mathcal{R}\to\mathcal{R}$ be a differentiable mapping on $\mathcal{U}^\circ$ and $\gamma_2:[\theta_1,\theta_2]\to[0,\infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1+\theta_2}{2}$, where $\theta_1,\theta_2\in\mathcal{U}^\circ$ with $\theta_1<\theta_2$. If $\gamma_1'\in\mathcal{L}_1([\theta_1,\theta_2])$ and $|\gamma_1'|^{\mathbf{q}}$ is convex on $[\theta_1,\theta_2]$ for $\mathbf{q}\geq 1$, then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{\mathbf{m}^{1-\frac{1}{\mathbf{q}}}}\right)\left[\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{4}\right]^{\frac{1}{\mathbf{q}}}\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha, \quad (2.\ 15)$$

where $\mathbf{m}$ is a positive integer.

Proof. From (2. 8) and the power-mean integral inequality, we get

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\le\left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\right)^{1-\frac{1}{\mathbf{q}}}$$
$$\times\left\{\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\int_0^{\frac{\mathbf{m}}{2}}\left|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))\right|^{\mathbf{q}}d\alpha\right)^{\frac{1}{\mathbf{q}}}\right.$$
$$\left.+\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\int_0^{\frac{\mathbf{m}}{2}}\left|\gamma_1'(\zeta_1(\alpha,\mathbf{m}))\right|^{\mathbf{q}}d\alpha\right)^{\frac{1}{\mathbf{q}}}\right\}. \quad (2.\ 16)$$

Using the discrete power-mean inequality $\alpha^r+\beta^r\le 2^{1-r}(\alpha+\beta)^r$ for $\alpha>0,\beta>0$, $0<r<1$ and the convexity of $|\gamma_1'|^{\mathbf{q}}$ on $[\theta_1,\theta_2]$ for $\mathbf{q}\ge 1$, we get

$$\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\int_0^{\frac{\mathbf{m}}{2}}\left|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))\right|^{\mathbf{q}}d\alpha\right)^{\frac{1}{\mathbf{q}}}$$
$$+\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\int_0^{\frac{\mathbf{m}}{2}}\left|\gamma_1'(\zeta_1(\alpha,\mathbf{m}))\right|^{\mathbf{q}}d\alpha\right)^{\frac{1}{\mathbf{q}}}$$
$$\le 2^{1-\frac{1}{\mathbf{q}}}\left(\int_0^{\frac{\mathbf{m}}{2}}\left[\left|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))\right|^{\mathbf{q}}+\left|\gamma_1'(\zeta_1(\alpha,\mathbf{m}))\right|^{\mathbf{q}}\right]d\alpha\right)^{\frac{1}{\mathbf{q}}}\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\right)^{\frac{1}{\mathbf{q}}}$$
$$=2^{1-\frac{1}{\mathbf{q}}}\left(\frac{\mathbf{m}}{2}\right)^{\frac{1}{\mathbf{q}}}\left(\left|\gamma_1'(\theta_1)\right|^{\mathbf{q}}+\left|\gamma_1'(\theta_2)\right|^{\mathbf{q}}\right)^{\frac{1}{\mathbf{q}}}\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\right)^{\frac{1}{\mathbf{q}}} \quad (2.\ 17)$$

Combining (2. 17) and (2. 16), we obtain (2. 15). □

Corollary 2.6. *Under the conditions of Theorem 2.5, if* $\mathbf{m}=2$ *then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{2}\right)\left[\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{2}\right]^{\frac{1}{\mathbf{q}}}\int_0^1\int_{\zeta_1(\alpha,2)}^{\zeta_2(\alpha,2)}\gamma_2(\lambda)\,d\lambda d\alpha, \quad (2.\ 18)$$

where $\zeta_1(\alpha,2)=\frac{\alpha}{2}\theta_2+\left(\frac{2-\alpha}{2}\right)\theta_1$, $\zeta_2(\alpha,2)=\frac{\alpha}{2}\theta_1+\left(\frac{2-\alpha}{2}\right)\theta_2$.

Corollary 2.7. *Suppose that the assumptions of Theorem 2.5 are fulfilled. If* $\mathbf{q}=1$, *then the inequality*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{2}\right)\left[\frac{|\gamma_1'(\theta_1)|+|\gamma_1'(\theta_2)|}{2}\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha, \quad (2.\ 19)$$

holds, where $\mathbf{m}$ *is a positive integer and* $\zeta_1(\alpha,\mathbf{m})$ *and* $\zeta_2(\alpha,\mathbf{m})$ *are defined as in Lemma 2.1.*

Corollary 2.8. *If the assumptions of Theorem 2.5 are satisfied and if* $\mathbf{q}=1$, $\mathbf{m}=2$, *the inequality*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{2}\right)\left[\frac{|\gamma_1'(\theta_1)|+|\gamma_1'(\theta_2)|}{2}\right]\int_0^1\int_{\zeta_1(\alpha,2)}^{\zeta_2(\alpha,2)}\gamma_2(\lambda)\,d\lambda d\alpha, \quad (2.\ 20)$$

holds, where $\zeta_1(\alpha,2)=\frac{\alpha}{2}\theta_2+\left(\frac{2-\alpha}{2}\right)\theta_1$, $\zeta_2(\alpha,2)=\frac{\alpha}{2}\theta_1+\left(\frac{2-\alpha}{2}\right)\theta_2$.

Corollary 2.9. *If we combine (1. 3), (1. 6) and (2. 18), we get*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \leq \left(\frac{\theta_2-\theta_1}{2}\right)\left[\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{2}\right]^{\frac{1}{\mathbf{q}}}$$
$$\times\min\left\{\int_0^1\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha,\int_0^1\int_{T(\alpha)}^{S(\alpha)}\gamma_2(\lambda)\,d\lambda d\alpha,\int_0^1\int_{M(\alpha)}^{V(\alpha)}\gamma_2(\lambda)\,d\lambda d\alpha\right\}. \tag{2. 21}$$

Theorem 2.10. *Let $\gamma_1 : \mathcal{U}\subseteq\mathcal{R}\to\mathcal{R}$ be a differentiable mapping on $\mathcal{U}^\circ$ and $\gamma_2 : [\theta_1,\theta_2]\to[0,\infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1+\theta_2}{2}$, where $\theta_1,\theta_2\in\mathcal{U}^\circ$ with $\theta_1<\theta_2$. If $\gamma_1'\in\mathcal{L}_1([\theta_1,\theta_2])$ and $|\gamma_1'|^{\mathbf{q}}$ is convex on $[\theta_1,\theta_2]$ for $\mathbf{q}>1$, then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \leq \left(\frac{\theta_2-\theta_1}{2\mathbf{m}^{1-\frac{1}{\mathbf{q}}}}\right)\left[\int_0^{\frac{\mathbf{m}}{2}}\left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}}$$
$$\times\left\{\left(\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+3|\gamma_1'(\theta_2)|^{\mathbf{q}}}{8}\right)^{\frac{1}{\mathbf{q}}}+\left(\frac{3|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{8}\right)^{\frac{1}{\mathbf{q}}}\right\}, \tag{2. 22}$$

where $\mathbf{m}$ is a positive integer.

Proof. Using (2. 8) and Hölder integral inequality, we get

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \leq \left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\int_0^{\frac{\mathbf{m}}{2}}\left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}}$$
$$\times\left\{\left(\int_0^{\frac{\mathbf{m}}{2}}|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))|^{\mathbf{q}}\,d\alpha\right)^{\frac{1}{\mathbf{q}}}+\left(\int_0^{\frac{\mathbf{m}}{2}}|\gamma_1'(\zeta_1(\alpha,\mathbf{m}))|^{\mathbf{q}}\,d\alpha\right)^{\frac{1}{\mathbf{q}}}\right\}. \tag{2. 23}$$

Using the convexity of $|\gamma_1'|^{\mathbf{q}}$ on $[\theta_1, \theta_2]$ for $\mathbf{q} \geq 1$, we get

$$\left(\int_0^{\frac{\mathbf{m}}{2}} |\gamma_1' (\zeta_2 (\alpha, \mathbf{m}))|^{\mathbf{q}} \, d\alpha \right)^{\frac{1}{\mathbf{q}}} + \left(\int_0^{\frac{\mathbf{m}}{2}} |\gamma_1' (\zeta_1 (\alpha, \mathbf{m}))|^{\mathbf{q}} \, d\alpha \right)^{\frac{1}{\mathbf{q}}}$$
$$\leq \left(\int_0^{\frac{\mathbf{m}}{2}} \left[|\gamma_1' (\theta_1)|^{\mathbf{q}} \frac{\alpha}{\mathbf{m}} + \left(\frac{\mathbf{m} - \alpha}{\mathbf{m}} \right) |\gamma_1' (\theta_2)|^{\mathbf{q}} \right] d\alpha \right)^{\frac{1}{\mathbf{q}}}$$
$$+ \left(\int_0^{\frac{\mathbf{m}}{2}} \left[|\gamma_1' (\theta_2)|^{\mathbf{q}} \frac{\alpha}{\mathbf{m}} + \left(\frac{\mathbf{m} - \alpha}{\mathbf{m}} \right) |\gamma_1' (\theta_1)|^{\mathbf{q}} \right] d\alpha \right)^{\frac{1}{\mathbf{q}}}$$
$$= \left(\frac{\mathbf{m} |\gamma_1' (\theta_1)|^{\mathbf{q}} + 3\mathbf{m} |\gamma_1' (\theta_2)|^{\mathbf{q}}}{8} \right)^{\frac{1}{\mathbf{q}}} + \left(\frac{3\mathbf{m} |\gamma_1' (\theta_1)|^{\mathbf{q}} + \mathbf{m} |\gamma_1' (\theta_2)|^{\mathbf{q}}}{8} \right)^{\frac{1}{\mathbf{q}}}. \quad (2.\ 24)$$

Combining (2. 24) and (2. 23), we obtain (2. 22). □

Corollary 2.11. *Suppose that the assumptions of Theorem 2.10 are satisfied and* $\mathbf{m} = 2$, *then*

$$\left| \frac{\gamma_1 (\theta_1) + \gamma_1 (\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2 (\lambda) \, d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2 (\lambda) \gamma_1 (\lambda) \, d\lambda \right|$$
$$\leq \left(\frac{\theta_2 - \theta_1}{4} \right) \left[\int_0^1 \left(\int_{\zeta_1(\alpha,2)}^{\zeta_2(\alpha,2)} \gamma_2 (\lambda) \, d\lambda \right)^{\frac{\mathbf{q}}{\mathbf{q}-1}} d\alpha \right]^{1-\frac{1}{\mathbf{q}}}$$
$$\times \left\{ \left(\frac{|\gamma_1' (\theta_1)|^{\mathbf{q}} + 3 |\gamma_1' (\theta_2)|^{\mathbf{q}}}{4} \right)^{\frac{1}{\mathbf{q}}} + \left(\frac{3 |\gamma_1' (\theta_1)|^{\mathbf{q}} + |\gamma_1' (\theta_2)|^{\mathbf{q}}}{4} \right)^{\frac{1}{\mathbf{q}}} \right\}, \quad (2.\ 25)$$

where $\zeta_1 (\alpha, 2) = \frac{\alpha}{2}\theta_2 + \left(\frac{2-\alpha}{2}\right) \theta_1$, $\zeta_2 (\alpha, 2) = \frac{\alpha}{2}\theta_1 + \left(\frac{2-\alpha}{2}\right) \theta_2$.

Corollary 2.12. *Combining (1. 7) and (2. 25), we have*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\frac{(\theta_2-\theta_1)}{4}\left\{\left(\frac{3|\gamma_1'(\theta_1)|^{\mathbf{q}}+|\gamma_1'(\theta_2)|^{\mathbf{q}}}{4}\right)^{\frac{1}{\mathbf{q}}}+\left(\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}}+3|\gamma_1'(\theta_2)|^{\mathbf{q}}}{4}\right)^{\frac{1}{\mathbf{q}}}\right\}$$
$$\times\min\left\{\left[\int_0^1\left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}},\left[\int_0^1\left(\int_{T(\alpha)}^{S(\alpha)}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}},\right.$$
$$\left.\left[\int_0^1\left(\int_{M(\alpha)}^{V(\alpha)}\gamma_2(\lambda)\,d\lambda\right)^{\frac{\mathbf{q}}{\mathbf{q}-1}}d\alpha\right]^{1-\frac{1}{\mathbf{q}}}\right\}.\quad (2.\ 26)$$

Theorem 2.13. *Suppose that the assumptions of Theorem 2.5 are satisfied and $|\gamma_1'|$ is quasi-convex on $[\theta_1,\theta_2]$, then the inequality holds:*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\max\left\{|\gamma_1'(\theta_1)|,\left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|\right\}\right.$$
$$\left.+\max\left\{|\gamma_1'(\theta_2)|,\left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|\right\}\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha,\quad (2.\ 27)$$

where $\zeta_1(\alpha,\mathbf{m})$, $\zeta_2(\alpha,\mathbf{m})$ are defined as in Lemma 2.1 and $\mathbf{m}\geq 1$ is an integer.

Proof. From the identity (2. 8), we have

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\int_0^{\frac{\mathbf{m}}{2}}\left(\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda\right)\left[|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))|+|\gamma_1'(\zeta_1(\alpha,\mathbf{m}))|\right]d\alpha\right].\quad (2.\ 28)$$

By the quasi-convexity of $|\gamma_1'|$ on $[\theta_1,\theta_2]$, we have

$$|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))|\leq\max\left\{|\gamma_1'(\theta_2)|,\left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|\right\}\quad (2.\ 29)$$

and

$$\left|\gamma_1'\left(\zeta_2\left(\alpha,\mathbf{m}\right)\right)\right| \le \max\left\{\left|\gamma_1'\left(\theta_1\right)\right|, \left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|\right\} \tag{2. 30}$$

for all $\alpha \in \left[0, \frac{\mathbf{m}}{2}\right]$.
Combining the inequalities in (2. 28), (2. 29) and (2. 30), we obtain the inequality (2. 27). □

Remark 2.14. *If $|\gamma_1'|$ is non-decreasing in Theorem 2.13, then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda - \int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \le \left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\left(\left|\gamma_1'(\theta_2)\right| + \left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|\right)\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha \tag{2. 31}$$

and if $|\gamma_1'|$ is non-increasing in Theorem 2.13, then

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda - \int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \le \left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\left(\left|\gamma_1'(\theta_1)\right| + \left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|\right)\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha. \tag{2. 32}$$

Theorem 2.15. *Suppose that the assumptions of Theorem 2.5 are satisfied and $|\gamma_1'|^{\mathbf{q}}$ is quasi-convex on $[\theta_1,\theta_2]$ for $\mathbf{q}\ge 1$, then the inequality holds:*

$$\begin{aligned}&\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda - \int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right| \\ &\le \left(\frac{\theta_2-\theta_1}{\mathbf{m}^{1-\frac{1}{\mathbf{q}}}}\right)\left(\frac{1}{2}\right)^{\frac{1}{\mathbf{q}}+1}\left[\left(\max\left\{\left|\gamma_1'(\theta_1)\right|^{\mathbf{q}}, \left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|^{\mathbf{q}}\right\}\right)^{\frac{1}{\mathbf{q}}}\right. \\ &\left.+\left(\max\left\{\left|\gamma_1'(\theta_2)\right|^{\mathbf{q}}, \left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|^{\mathbf{q}}\right\}\right)^{\frac{1}{\mathbf{q}}}\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha,\end{aligned} \tag{2. 33}$$

where $\zeta_1(\alpha,\mathbf{m})$, $\zeta_2(\alpha,\mathbf{m})$ are defined as in Lemma 2.1 and $\mathbf{m}\ge 1$ is an integer.

Proof. Using the identity (2. 8) and the power-mean integral inequality, we get

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{2\mathbf{m}}\right)\left[\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\right]^{1-\frac{1}{\mathbf{q}}}$$
$$\times\left\{\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\int_0^{\frac{\mathbf{m}}{2}}\left|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))\right|^{\mathbf{q}}d\alpha\right)^{\frac{1}{\mathbf{q}}}\right.$$
$$\left.+\left(\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha\int_0^{\frac{\mathbf{m}}{2}}\left|\gamma_1'(\zeta_1(\alpha,\mathbf{m}))\right|^{\mathbf{q}}d\alpha\right)^{\frac{1}{\mathbf{q}}}\right\}. \quad (2.\ 34)$$

By the quasi-convexity of $|\gamma_1'|^{\mathbf{q}}$ on $[\theta_1,\theta_2]$ for $\mathbf{q}\geq 1$, we have

$$\left|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))\right|^{\mathbf{q}}\leq\max\left\{\left|\gamma_1'(\theta_1)\right|^{\mathbf{q}},\left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|^{\mathbf{q}}\right\} \quad (2.\ 35)$$

and

$$\left|\gamma_1'(\zeta_2(\alpha,\mathbf{m}))\right|^{\mathbf{q}}\leq\max\left\{\left|\gamma_1'(\theta_2)\right|^{\mathbf{q}},\left|\gamma_1'\left(\frac{\theta_1+\theta_2}{2}\right)\right|^{\mathbf{q}}\right\} \quad (2.\ 36)$$

for all $\alpha\in\left[0,\frac{\mathbf{m}}{2}\right]$.
Combining the inequalities (2. 34), (2. 35) and (2. 36), we obtain the inequality (2. 33). □

Remark 2.16. *If $|\gamma_1'|$ is non-decreasing in Theorem 2.15, then*

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{\mathbf{m}^{1-\frac{1}{\mathbf{q}}}}\right)\left(\frac{1}{2}\right)^{\frac{1}{\mathbf{q}}+1}\left[\left(\left|\gamma_1'(\theta_2)\right|+\left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|\right)\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha \quad (2.\ 37)$$

and if $|\gamma_1'|$ is non-increasing in Theorem 2.15, then

$$\left|\frac{\gamma_1(\theta_1)+\gamma_1(\theta_2)}{2}\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\,d\lambda-\int_{\theta_1}^{\theta_2}\gamma_2(\lambda)\gamma_1(\lambda)\,d\lambda\right|$$
$$\leq\left(\frac{\theta_2-\theta_1}{\mathbf{m}^{1-\frac{1}{\mathbf{q}}}}\right)\left(\frac{1}{2}\right)^{\frac{1}{\mathbf{q}}+1}\left[\left(|\gamma_1'(\theta_1)|+\left|\gamma_1'\left(\frac{\theta_2+\theta_1}{2}\right)\right|\right)\right]\int_0^{\frac{\mathbf{m}}{2}}\int_{\zeta_1(\alpha,\mathbf{m})}^{\zeta_2(\alpha,\mathbf{m})}\gamma_2(\lambda)\,d\lambda d\alpha. \tag{2. 38}$$

Remark 2.17. *A number of interesting results can be obtained from our results for $\mathbf{m}>2$.*

3. Applications to Special Means

For positive numbers $\theta_1>0$ and $\theta_2>0$, define

$$A(\theta_1,\theta_2)=\frac{\theta_1+\theta_2}{2},$$

$$G(\theta_1,\theta_2)=\sqrt{\theta_1\theta_2}$$

and

$$L_r(\theta_1,\theta_2)=\begin{cases}\left[\frac{\theta_2^{r+1}-\theta_1^{r+1}}{(r+1)(\theta_2-\theta_1)}\right]^{\frac{1}{r}}, & r\neq-1,0,\\ \frac{\theta_2-\theta_1}{\ln\theta_2-\ln\theta_1}, & r=-1,\\ \frac{1}{e}\left(\frac{\theta_2^{\theta_2}}{\theta_1^{\theta_1}}\right)^{\frac{1}{\theta_2-\theta_1}}, & r=0.\end{cases}$$

$A(\theta_1,\theta_2)$, $G(\theta_1,\theta_2)$ and $L_r(\theta_1,\theta_2)$ are called the arithmetic, geometric mean and generalized logarithmic means respectively of θ_1 and θ_2.

Let

$$\gamma_1(\lambda)=\frac{\mathbf{q}\lambda^{1+\frac{1}{\mathbf{q}}}}{\mathbf{q}+1}\text{ for }\lambda>0,\ \mathbf{q}\geq1. \tag{3. 39}$$

Then, obviously

$$|\gamma_1'(\lambda)|^{\mathbf{q}}=\lambda$$

is convex on $[\theta_1,\theta_2]$.

Moreover, the function

$$\gamma_2(\lambda)=\left(\lambda-\frac{\theta_1+\theta_2}{2}\right)^2, \tag{3. 40}$$

where $\theta_1,\theta_2>0$ and $\lambda\in[\theta_1,\theta_2]$, is a symmetric mapping with respect to $\frac{\theta_1+\theta_2}{2}$ on $[\theta_1,\theta_2]$.

Considering the functions (3. 39) and (3. 40), we have the following inequalities of special means $A(\theta_1,\theta_2)$, $G(\theta_1,\theta_2)$ and $L_r(\theta_1,\theta_2)$ using Theorem 2.5 and Theorem 2.10.

Theorem 3.1. *If $\theta_2 > \theta_1 > 0$ and $\mathbf{q} \geq 1$, then*

$$\left| \frac{\mathbf{q}\left(\theta_2-\theta_1\right)^2 A\left(\theta_1^{1+\frac{1}{\mathbf{q}}}, \theta_2^{1+\frac{1}{\mathbf{q}}}\right)}{6(\mathbf{q}+1)} - \frac{\mathbf{q}\left(4\mathbf{q}^2+3\mathbf{q}+1\right) L_{3+\frac{1}{\mathbf{q}}}^{3+\frac{1}{\mathbf{q}}}\left(\theta_1, \theta_2\right)}{2(\mathbf{q}+1)(2\mathbf{q}+1)(3\mathbf{q}+1)} + \frac{\mathbf{q} G^2\left(\theta_1, \theta_2\right) L_{1+\frac{1}{\mathbf{q}}}^{1+\frac{1}{\mathbf{q}}}\left(\theta_1, \theta_2\right)}{(3\mathbf{q}+1)} - \frac{\mathbf{q} G^4\left(\theta_1, \theta_2\right) L_{\frac{1}{\mathbf{q}}-1}^{\frac{1}{\mathbf{q}}-1}\left(\theta_1, \theta_2\right)}{2(\mathbf{q}+1)(2\mathbf{q}+1)} \right| \leq \left[\frac{\left(\theta_2-\theta_1\right)^3}{48}\right]^{1-\frac{1}{\mathbf{q}}} A^{\frac{1}{\mathbf{q}}}\left(\theta_1, \theta_2\right) \quad (3.\ 41)$$

and if $\mathbf{q} > 1$, then

$$\left| \frac{\mathbf{q}\left(\theta_2-\theta_1\right)^2 A\left(\theta_1^{1+\frac{1}{\mathbf{q}}}, \theta_2^{1+\frac{1}{\mathbf{q}}}\right)}{6(\mathbf{q}+1)} - \frac{\mathbf{q}\left(4\mathbf{q}^2+3\mathbf{q}+1\right) L_{3+\frac{1}{\mathbf{q}}}^{3+\frac{1}{\mathbf{q}}}\left(\theta_1, \theta_2\right)}{2(\mathbf{q}+1)(2\mathbf{q}+1)(3\mathbf{q}+1)} + \frac{\mathbf{q} G^2\left(\theta_1, \theta_2\right) L_{1+\frac{1}{\mathbf{q}}}^{1+\frac{1}{\mathbf{q}}}\left(\theta_1, \theta_2\right)}{(3\mathbf{q}+1)} - \frac{\mathbf{q} G^4\left(\theta_1, \theta_2\right) L_{\frac{1}{\mathbf{q}}-1}^{\frac{1}{\mathbf{q}}-1}\left(\theta_1, \theta_2\right)}{2(\mathbf{q}+1)(2\mathbf{q}+1)} \right| \leq \frac{(b-a)^3}{24}\left(\frac{\mathbf{q}-1}{4\mathbf{q}-1}\right)^{1-\frac{1}{\mathbf{q}}} \left\{ \left(\frac{\theta_1+3\theta_2}{8}\right)^{\frac{1}{\mathbf{q}}} + \left(\frac{3\theta_1+\theta_2}{8}\right)^{\frac{1}{\mathbf{q}}} \right\}. \quad (3.\ 42)$$

Corollary 3.2. *If the hypotheses of Theorem 3.1 are satisfied and if $\mathbf{q} = 1$, then*

$$\left| \frac{\left(\theta_2-\theta_1\right)^2 A\left(\theta_1^2, \theta_2^2\right)}{12} - \frac{L_4^4\left(\theta_1, \theta_2\right)}{6} + \frac{G^2\left(\theta_1, \theta_2\right) L_2^2\left(\theta_1, \theta_2\right)}{4} - \frac{G^4\left(\theta_1, \theta_2\right)}{12} \right| \leq A\left(\theta_1, \theta_2\right) \quad (3.\ 43)$$

4. CONCLUSIONS

In this paper, the results are based on a new family of identities for a positive integer $\mathbf{m}$. We have established new family of Fejér and Hermite-Hadamard type inequalities for the functions whose derivatives satisfy assumptions of convexity and quasi-convexity based on the new family of identities for a positive integer $\mathbf{m}$. The ideas and methods to acquire the studies are expected to inspire the interested readers. We suspect that our findings can be extended to obtain various results in convex analysis, special functions, theories related to optimization, mathematical inequalities and can invigorate further research work in various fields of pure and applied sciences.

5. Acknowledgments

The authors are thankful to the anonymous referees for their careful reading of the manuscript and providing their useful comments which have been encompassed in the final version of the manuscript.

References

[1] A. Akkurt and H. Yıldırım, *On Hermite-Hadamard-Fejer inequality type for convex functions via fractional integrals*, Mathematica Moravica, **21**, No.1 (2017) 105-123.

[2] R. Ashraf and A. Suhail, *Certain Analogous for Generalized Geometrically Convex Functions with Applications*, Punjab Univ. j. math., **52**, No. 9 (2020) 31-46.

[3] F. Chen and S. Wu, *Hermite-Hadamard type inequalities for harmonically s-convex functions*, Sci. World (2014), 7, Article ID 279158.

[4] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11**, No.5 (1998) 91-95.

[5] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequat. Math., **48**, No.1 (1994) 100-111.

[6] D. Y. Hwang, *Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables*, Appl. Math. Comput., **217,** No. 23 (2011) 9598-9605.

[7] J. Hua, B.-Y. Xi and F. Qi, *Inequalities of Hermite-Hadamard type involving an s-convex function with applications*, Applied Mathematics and Computation, **246** (2014) 752-760.

[8] İ. İşcan and M. Kunt, *Fejer and Hermite-Hadamard-Fejér type inequalities for harmonically s-convex functions via fractional integrals*, The Australian Journal of Mathematical Analysis and Applications, **12,** No.1 (2015) Article 10, 1-6.

[9] U. S. Kirmaci, M.K. Bakula, M. E. Özdemir and J. Pecaric, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput., **193**, No.1 (2007) 26–35.

[10] M. A. Latif, S. S. Dragomir and E. Momoniat, *Fejér type inequalities for harmonically-convex functions with applications*, Journal of Applied Analysis and Computation, **7**, No. 3 (2017) 795-813.

[11] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejer type inequalities for harmonically-convex functions with applications to special means*, International Journal of Analysis and Applications **13**, No. 1 (2017) 1-14.

[12] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some ϕ-analogues of Hermite-Hadamard inequality for s-convex functions in the second sense and related estimates*, Punjab Univ. j. math. **48,** No.2 (2016) 147-166.

[13] M. A. Latif and W. Irshad, *Some Fejér and Hermite-Hadamard type inequalities considering ϵ-convex and (σ, ϵ)-convex functions*, Punjab Univ. j. math. **50,** No.3 (2018) 13-24.

[14] M. A. Latif, *Estimates of Hermite-Hadamard inequality for twice differentiable harmonically-convex functions with applications*, Punjab Univ. j. math. **50,** No. 1 (2018) 1-13.

[15] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions*, Punjab Univ. j. math. **47,** No. 1 (2015) 57-72.

[16] M. A. Latif, S. Hussain and M. Baloch, *Weighted Simpson's Type Inequalities for HA-convex Functions,* Punjab Univ. j. Math. **52,** No.7 (2020) 11-24.

[17] S. Obeidat and M. A. Latif, *On Fejér and Hermite-Hadamard type inequalities involving h-convex functions and applications,* Punjab Univ. j. math. **52,** No.6 (2020) 1-18.

[18] M. Muddassar and A. Ali, *New integral inequalities through generalized convex functions*, Punjab Univ. j. math. **46**, No. 2 (2014) 47-51.

[19] M. Muddassar and M. I. Bhatti, *Some generalizations of Hermite-Hadamard type integral inequalities and their applications*, Punjab Univ. j. math. **46**, No. 1 (2014) 9-18.

[20] M. A. Noor, K. I. Noor and S. Iftikhar, *Nonconvex functions and integral inequalities*, Punjab Univ. j. math. **47**, No. 2 (2015) 19-27.

[21] J. E. Pecaric and F. Proschan, Y. L. Tong, *Convex Function, Partial Ordering and Statistical Applications*, Academic Press, New York, 1991.

[22] S. Rashid, M. A. Noor and K. I. Noor, *Integral inequalities for exponentially geometrically convex functions via fractional operators*, Punjab Univ. j. math. **52,** No.6 (2020) 65-82.

[23] S. Rashid, M. A. Noor, K. I. Noor, *Some generalize Riemann-Liouville fractional estimates involving functions having exponentially convexity property*, Punjab. Univ. J. Math, **51,** No.11 (2019) 1-15.

[24] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, *Integral inequalities for generalized preinvex functions*, Punjab. Univ. j. math. **51,** No.10 (2019) 77-91.

[25] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, *New Hermite-Hadamard type inequalities for exponentially GA and GG convex functions*, Punjab, Univ. j. math. **52,** No.2 (2020) 15-28.

[26] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., **60,** No.8 (2010) 2191-2199.

[27] Y. Shuang, H. -P. Yin and F. Qi, *Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions*, Analysis (Munich), **33,** No. 2 (2013) 197-208.

[28] M. E. Yıldırım, A. Akkurt and H. Yıldırım, *On some integral inequalities for twice diferentiable φ-convex and quasi-convex functions via k-fractional integrals*, British Journal of Mathematics & Computer Science, **16,** No. 6 (2016) X-XX.