Punjab University Journal of Mathematics (2021),53(8),527-535
https://doi.org/10.52280/pujm.2021.530801

Several Congruences Related to Harmonic Numbers

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Received: 26 April, 2021 / Accepted: 17 June, 2021 / Published online: 26 August, 2021

Abstract.: Let p be a prime greater than or equal to 5. In this paper, by using the harmonic numbers and Fermat quotient we establish congruences involving the sums

$$\sum_{k=1}^{\frac{p-1}{2}} \binom{k}{r} H_k, \quad \sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} \text{ and } \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{4^k}\binom{2k}{k} H_k^{(3)}.$$

For example,

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} \equiv 4E_{2p-4} - 8E_{p-3} \pmod{p^2},$$

where $H_k^{(m)}$ are the generalized harmonic numbers of order m and E_n are Euler numbers.

AMS (MOS) Subject Classification Codes: 11A07; 11B68
Key Words: Congruences, harmonic numbers, Fermat quotient.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^*$ be the positive natural numbers. The generalized harmonic numbers $H_n^{(m)}$ are the rational numbers defined by

$$H_0^{(m)} := 0, \quad H_n^{(m)} := \sum_{k=1}^{n} \frac{1}{k^m}, \ \ n \in \mathbb{N}^*, \ m \geq 0.$$

As usual,

$$H_0 := H_0^{(1)} = 0, \ H_n := H_n^{(1)} = \sum_{k=1}^{n} \frac{1}{k}, \ \ n \in \mathbb{N}^*.$$

In this paper, we define other generalized harmonic numbers (strict) odd multiple

$$H_r\left(s\right) = \sum_{1 \leq j_1 < j_2 < \ldots < j_s \leq r} \frac{1}{\left(2j_1 - 1\right)\left(2j_2 - 1\right)\cdots\left(2j_s - 1\right)}, \ \ r \geq s \geq 1,$$

It is convenient to set $H_0\left(s\right) = 0$ for $s \geq 0$ and it is seen that $H_r\left(1\right) = \sum_{i=1}^{r} \frac{1}{2i-1}, \ r \geq 1.$

Let p be a prime number and let $\mathbb{Z}_{(p)}$ be the set of rational numbers having denominators co-prime with p. Also, for two reduced rational numbers $\frac{N_1}{D_1}, \ \frac{N_2}{D_2} \in \mathbb{Z}_{(p)}$ such that D_1 and D_2 are co-prime with p, we write $\frac{N_1}{D_1} \equiv \frac{N_2}{D_2} \pmod p$ to mean that the numerator $N_1D_2 - N_2D_1$ is divisible by p.

In 2017, Meštrović and Andjić [7] obtained that for each prime $p > 3$ and $0 \leqslant m \leqslant p-2$ the congruence

$$\sum_{k=m}^{p-1} \binom{k}{m} H_k \equiv \frac{(-1)^m}{m+1} \left(1 - pH_{m+1} + \frac{p^2}{2}\left(H_{m+1}^2 - H_{m+1}^{(2)}\right)\right) \ \left(\operatorname{mod} p^3\right), \quad (1.\ 1)$$

where $\binom{k}{m}$ are the binomial coefficients.

In 2003, Rodriguez-Villegas [8] conjectured the congruence

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(16)^k} \equiv (-1)^{\frac{p-1}{2}} \ \left(\operatorname{mod} p^2\right).$$

This conjecture proved to be true and was improved by Z-W Sun in 2011[11]. The sequence of Bernoulli numbers $(B_n)_{n\geq 0}$ is defined by

$$B_0 = 1, \ \ B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \ n \in \mathbb{N}^*.$$

The sequence of Euler numbers $(E_n)_{n\geq 0}$ is defined by

$$E_0 = 1, \ \ E_n = -\sum_{k=1, \ 2|k}^{n-1} \binom{n}{k} E_{n-k}, \ n \in \mathbb{N}^*.$$

It is known that $B_{2n+1} = E_{2n+1} = 0$ for $n \in \mathbb{N}^*$. Many other properties can be found in the literature, see for instance [3, Chapter15]. For $a \in \mathbb{Z}_{(p)}$, we denote by $q_a = q_p\left(a\right)$ the

Fermat quotient defined for a given prime number p by

$$q_a = \frac{a^{p-1}-1}{p}.$$

In this paper, we exploit some properties of harmonic numbers to establish congruences for sums in terms involving these numbers. Interesting results on this subject can be found in [5, 7]. Our main results are as follows.

Theorem 1.1. *Let $p \neq 3$ be an odd prime number and let $r \in \left\{0, 1, \ldots, \frac{p-1}{2}\right\}$. Then*

$$\sum_{k=r}^{\frac{p-1}{2}} \binom{k}{r} H_k \equiv \frac{(-1)^r \binom{2r}{r}}{(r+1)\, 2^{2r+1}} \left(X_{r,p} - pY_{r,p} + p^2 Z_{r,p}\right) \pmod{p^3}, \tag{1. 2}$$

where

$$X_{r,p} = \frac{2r+1}{r+1} - 2q_2,$$

$$Y_{r,p} = \frac{1}{r+1} + 2q_2 - q_2^2 + \left(\frac{2r+1}{r+1} - 2q_2\right) H_r(1),$$

$$Z_{r,p} = q_2^2 - \frac{2}{3}q_2^3 - \frac{7}{12}B_{p-3} + \left(2q_2 - q_2^2 + \frac{1}{r+1}\right) H_r(1) + \left(\frac{2r+1}{r+1} - 2q_2\right) H_r(2).$$

Theorem 1.2. *Let $p > 3$ be a prime number. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{\frac{p-1}{2}}{k} \binom{\frac{p-1}{2}+k}{k} H_k^{(2)} \equiv 4E_{2p-4} - 8E_{p-3} \pmod{p^2}. \tag{1. 3}$$

Theorem 1.3. *Let $p > 3$ be a prime number. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^{k-1} \binom{\frac{p-1}{2}}{k} H_k^{(3)} \equiv -4q_2^2 \pmod{p}.$$

A simple consequence of Theorem 1.3 is given by the following corollary.

Corollary 1.4. *For any prime $p > 3$ we have*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{4^k} \binom{2k}{k} H_k^{(3)} \equiv 4q_2^2 \pmod{p}.$$

In the next section, we present some lemmas to be used later. In section three, we show the proofs of the main results.

2. PRELIMINARIES

In this section, we first state some basic facts which will be used later.

Lemma 2.1. [1, 6] *and* [9, Thm. 5.2(c)] *Let $p \neq 3$ be an odd prime number. Then*

$$H_{\frac{p-1}{2}} \equiv -2q_2 \pmod{p},$$

$$H_{\frac{p-1}{2}} \equiv -2q_2 + pq_2^2 \pmod{p^2}$$

and

$$H_{\frac{p-1}{2}} \equiv -2q_2 + pq_2^2 - \frac{2}{3}p^2q_2^3 - \frac{7}{12}p^2B_{p-3} \pmod{p^3}. \tag{2. 4}$$

Lemma 2.2. *Let $p \neq 3$ be an odd prime number and let $r \in \left\{0, 1, \ldots, \frac{p-1}{2}\right\}$. Then*

$$\binom{\frac{p-1}{2}}{r} \equiv \frac{(-1)^r}{2^{2r}}\binom{2r}{r}\left(1 - pH_r(1) + p^2H_r(2)\right) \pmod{p^3}, \tag{2. 5}$$

$$\binom{\frac{p-1}{2}+r}{2r} \equiv \frac{\binom{2r}{r}}{(-16)^r} \pmod{p^2}, \tag{2. 6}$$

and for $r \in \{0, 1, \ldots, p-1\}$ we have

$$\binom{\lfloor \frac{p}{6} \rfloor + r}{2r}\binom{2r}{r} \equiv \frac{\binom{6r}{3r}\binom{3r}{r}}{(-432)^r} \pmod{p}. \tag{2. 7}$$

Proof. For $r = 0$, (2. 5), (2. 6) and (2. 7) are true. For $r \in \left\{1, 2, \ldots, \frac{p-1}{2}\right\}$, from the definition of the binomial coefficients, we get

$$\begin{aligned}
\binom{\frac{p-1}{2}}{r} &= \frac{1}{r!}\frac{p-1}{2}\left(\frac{p-1}{2} - 1\right)\cdots\left(\frac{p-1}{2} - r + 1\right) \\
&= \frac{1}{2^r r!}(p-1)(p-3)(p-5)\cdots(p-(2r-1)) \\
&= \frac{1}{2^r r!}(-1)(-3)\cdots(-(2r-1)) \times \left(1 - \frac{p}{1}\right)\left(1 - \frac{p}{3}\right)\cdots\left(1 - \frac{p}{2r-1}\right) \\
&= \frac{(-1)^r}{2^r r!}\prod_{k=1}^{r}(2k-1)\prod_{k=1}^{r}\left(1 - \frac{p}{2k-1}\right) \\
&= \frac{(-1)^r}{2^{2r}}\binom{2r}{r}\prod_{k=1}^{r}\left(1 - \frac{p}{2k-1}\right) \\
&= \frac{(-1)^r}{2^{2r}}\binom{2r}{r}\left(1 - p\sum_{1\le i\le r}\frac{1}{2i-1} + p^2\sum_{1\le i<j\le r}\frac{1}{(2i-1)(2j-1)} + \cdots\right) \\
&\equiv \frac{(-1)^r}{2^{2r}}\binom{2r}{r}\left(1 - pH_r(1) + p^2H_r(2)\right) \pmod{p^3}
\end{aligned}$$

which is the congruence (2. 5). We also have

$$\begin{aligned}
\binom{\frac{p-1}{2}+r}{2r} &= \frac{1}{(2r)!}\left(\frac{p-1}{2} + r\right)\left(\frac{p-1}{2} + r - 1\right)\cdots\left(\frac{p-1}{2} - r + 1\right) \\
&= \frac{1}{2^{2r}(2r)!}(p + 2r - 1)(p + 2r - 3)\cdots(p-(2r-3))(p-(2r-1)) \\
&= \frac{1}{2^{2r}(2r)!}\left(p^2 - 1^2\right)\left(p^2 - 3^2\right)\cdots\left(p^2 - (2r-1)^2\right) \\
&\equiv \frac{\binom{2r}{r}}{(-16)^r} \pmod{p^2}
\end{aligned}$$

which gives the congruence (2. 6).
To prove (2. 7) let $s \in \{1,5\}$ given by $p \equiv s \pmod 6$. Then for $r \in \{1,2,\ldots,p-1\}$

$$\begin{aligned}
\binom{\lfloor \frac{p}{6} \rfloor + r}{2r}\binom{2r}{r} &= \frac{\left(\frac{p-s}{6}+r\right)\left(\frac{p-s}{6}+r-1\right)\cdots\left(\frac{p-s}{6}-r+1\right)}{(2r)!}\frac{(2r)!}{(r!)^2} \\
&= \frac{(p+6r-s)(p+6r-6-s)\cdots(p-6r+6-s)}{6^{2r}\cdot(r!)^2} \\
&\equiv (-1)^r\,\frac{(6r-s)(6r-6-s)\cdots(6-s).s(s+6)\cdots(s+6r-6)}{6^{2r}\cdot(r!)^2} \\
&= \frac{(-1)^r\cdot(6r)!}{(2\cdot 4\cdots 6r)(3\cdot 9\cdot 15\cdots(6r-3))\cdot 6^{2r}\cdot(r!)^2} \\
&= \frac{(-1)^r\cdot(6r)!}{2^{3r}\cdot(3r)!\cdot\left(\frac{3^r\cdot(2r)!}{2^r\cdot r!}\right)\cdot 36^r\cdot(r!)^2} \\
&\equiv \frac{\binom{6r}{3r}\binom{3r}{r}}{(-432)^r} \pmod p.
\end{aligned}$$

□

In Lemma 2.2, more specifically in the proofs of the main results of the last section, we will use two combinatorial identities given in [2]. For reader's convenience, we quote them here.

Lemma 2.3. [2, Id. 1.48 and 1.51] *For all integers a,l,n and r such that $a \le n$, we have*

$$\sum_{k=0}^{n}\binom{k+l}{r} = \binom{n+l+1}{r+1} - \binom{l}{r+1} \tag{2. 8}$$

and

$$\sum_{k=a}^{n}\binom{k}{r} = \binom{n+1}{r+1} - \binom{a}{r+1}. \tag{2. 9}$$

Lemma 2.4. *Let be $p \neq 3$ be an odd prime number. Then*

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{(-1)^{k-1}}{k^2} \equiv (-1)^{\frac{p-1}{2}}\left(2E_{2p-4}-4E_{p-3}\right) \pmod{p^2}. \tag{2. 10}$$

Proof. The result is true for $p=5$. We can assume that $p>5$. The left hand side of (2. 10) can be written as

$$\begin{aligned}\sum_{k=1}^{\frac{p-1}{2}}\frac{(-1)^{k-1}}{k^2} &= \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}-\sum_{k=1}^{\frac{p-1}{2}}\frac{1+(-1)^k}{k^2}\\ &= \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}-\sum_{\substack{k=1\\2|k}}^{\frac{p-1}{2}}\frac{1+(-1)^k}{k^2}\\ &= \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}-\frac{1}{2}\sum_{k=1}^{\lfloor\frac{p}{4}\rfloor}\frac{1}{k^2}\end{aligned}$$

and from Corollary 5.2 (a) given in [9] and Corollary 3.8 given in [12], we finally get

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{(-1)^{k-1}}{k^2}\equiv\frac{7}{3}pB_{p-3}-\frac{1}{2}\left((-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4})+\frac{14}{3}pB_{p-3}\right)\pmod{p^2}$$

which proves (2. 10) for $p>5$. □

Lemma 2.5. [4] *Let* $n\in\mathbb{N}^*$. *Then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}H_k^{(2)}=2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}, \tag{2. 11}$$

$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}H_k^{(3)}=\frac{1}{2n}\left(H_n^2+H_n^{(2)}\right). \tag{2. 12}$$

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. By the identity (2. 9), we obtain

$$\begin{aligned}\sum_{k=1}^{\frac{p-1}{2}}\binom{k}{r}H_k &= \sum_{k=1}^{\frac{p-1}{2}}\binom{k}{r}\sum_{j=1}^{k}\frac{1}{j}\\ &= \sum_{j=1}^{\frac{p-1}{2}}\frac{1}{j}\sum_{k=j}^{\frac{p-1}{2}}\binom{k}{r}\\ &= \sum_{j=1}^{\frac{p-1}{2}}\frac{1}{j}\left(\binom{\frac{p+1}{2}}{r+1}-\binom{j}{r+1}\right)\\ &= \binom{\frac{p+1}{2}}{r+1}H_{\frac{p-1}{2}}-\sum_{j=1}^{\frac{p-1}{2}}\frac{1}{j}\binom{j}{r+1},\end{aligned}$$

by the identity $\binom{n}{r}=\frac{n}{r}\binom{n-1}{r-1}$, we get

$$\sum_{k=1}^{\frac{p-1}{2}}\binom{k}{r}H_k=\binom{\frac{p-1}{2}}{r}\frac{1+p}{2\left(r+1\right)}H_{\frac{p-1}{2}}-\frac{1}{r+1}\sum_{j=1}^{\frac{p-1}{2}}\binom{j-1}{r}$$

and by (2. 8), the same expression can also be expressed as

$$\sum_{k=1}^{\frac{p-1}{2}}\binom{k}{r}H_k=\binom{\frac{p+1}{2}}{r+1}\left(H_{\frac{p-1}{2}}+\frac{2}{1+p}-\frac{1}{r+1}\right).$$

Apply the congruence (2. 5) of Lemma 2.2 to obtain

$$\sum_{k=1}^{\frac{p-1}{2}}\binom{k}{r}H_k\equiv\frac{\left(-1\right)^r}{\left(r+1\right)2^{2r+1}}\binom{2r}{r}\left(1+p\right)\left(1-pH_r\left(1\right)+p^2H_r\left(2\right)\right)$$
$$\times\left(H_{\frac{p-1}{2}}+\frac{2}{1+p}-\frac{1}{r+1}\right)\pmod{p^3}.$$

We note that

$$\left(1+p\right)\left(1-pH_r\left(1\right)+p^2H_r\left(2\right)\right)\equiv1-p\left(H_r\left(1\right)-1\right)+p^2\left(H_r\left(2\right)-H_r\left(1\right)\right)\pmod{p^3},$$

so, by the congruence $\frac{2}{1+p}\equiv2-2p+2p^2\pmod{p^3}$ and the congruence (2. 4) of Lemma2.1, it follows

$$\sum_{k=1}^{\frac{p-1}{2}}\binom{k}{r}H_k\equiv\frac{\left(-1\right)^r}{\left(r+1\right)2^{2r+1}}\binom{2r}{r}\left(1-p\left(H_r\left(1\right)-1\right)+p^2\left(H_r\left(2\right)-H_r\left(1\right)\right)\right)$$
$$\times\left(-2q_2+pq_2^2-\frac{2}{3}p^2q_2^3-\frac{7}{12}p^2B_{p-3}+2-2p+2p^2-\frac{1}{r+1}\right)\pmod{p^3}. \tag{3. 13}$$

After distributing and simplifying the right hand side of the congruence (3. 13) we get congruence (1. 2) of Theorem 1.1, which completes the proof. □

Proof of Theorem 1.2. To obtain (1. 3), it suffices to take $n=\frac{p-1}{2}$ in the identity (2. 11) of Lemma 2.5 and use Lemma 2.4. □

Remark 3.1. *By the identity $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$ and the congruence (2. 6), Theorem 1.2 is reduced*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}H_k^{(2)}\equiv4E_{2p-4}-8E_{p-3}\pmod{p^2}.$$

Taking $n=\left\lfloor\frac{p}{6}\right\rfloor$ in the identity (2. 11) of Lemma 2.5 and by the identity $\binom{\lfloor\frac{p}{6}\rfloor}{k}\binom{\lfloor\frac{p}{6}\rfloor+k}{k}=\binom{2k}{k}\binom{\lfloor\frac{p}{6}\rfloor+k}{2k}$, by the congruence (2. 7) and the congruence (2.8) of Lemma 2.6 given in [10], *we deduce*

$$\sum_{k=1}^{\lfloor\frac{p}{6}\rfloor}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}H_k^{(2)}\equiv-20E_{p-3}\pmod{p}. \tag{3. 14}$$

Proof of Theorem 1.3. To obtain (1. 3), it suffices to take $n = \frac{p-1}{2}$ in the identity (2. 12) of Lemma 2.5 and use the first congruence of Lemma 2.1 noting that $H^{(2)}_{\frac{p-1}{2}} \equiv 0 \pmod p$. □

4. PARTICULAR CASES OF THEOREM 1

In this section, we determine the congruence of $\sum_{k=1}^{\frac{p-1}{2}} k^{\alpha} H_k \pmod{p^3}$, for $\alpha \in \{0,1,2\}$.

(i) Taking $r = 0$ in the congruence (1. 2) and use the fact that $H_0(1) = H_0(2) = 0$ to obtain the congruence

$$\sum_{k=1}^{\frac{p-1}{2}} H_k \equiv \frac{1}{2} - q_2 - p\left(-\frac{1}{2}q_2^2 + q_2 + \frac{1}{2}\right) + p^2\left(-\frac{1}{3}q_2^3 + \frac{1}{2}q_2^2 - \frac{7}{24}B_{p-3}\right) \pmod{p^3}. \tag{4. 15}$$

(ii) Taking $r = 1$ in the congruence (1. 2) to obtain

$$\sum_{k=1}^{\frac{p-1}{2}} kH_k \equiv -\frac{1}{8}\left(X_{1,p} - pY_{1,p} + p^2 Z_{1,p}\right) \pmod{p^3}, \tag{4. 16}$$

where

$$X_{1,p} = \frac{3}{2} - 2q_2, \quad Y_{1,p} = 2 - q_2^2 \text{ and } Z_{1,p} = \frac{1}{2} + 2q_2 - \frac{2}{3}q_2^3 - \frac{7}{12}B_{p-3}.$$

So, the congruence (4. 16) is to be

$$\sum_{k=1}^{\frac{p-1}{2}} kH_k \equiv -\frac{3}{16} + \frac{1}{4}q_2 - p\left(\frac{1}{8}q_2^2 - \frac{1}{4}\right) + p^2\left(\frac{1}{12}q_2^3 - \frac{1}{4}q_2 - \frac{1}{16} + \frac{7}{96}B_{p-3}\right) \pmod{p^3}. \tag{4. 17}$$

(iii) Taking $r = 2$ in the congruence (1. 2) to find

$$\sum_{k=1}^{\frac{p-1}{2}} \binom{k}{2} H_k \equiv \frac{1}{16}\left(X_{2,p} - pY_{2,p} + p^2 Z_{2,p}\right) \pmod{p^3}, \tag{4. 18}$$

where

$$\begin{aligned}
X_{2,p} &= \frac{5}{3} - 2q_2, \\
Y_{2,p} &= \frac{1}{3} + 2q_2 - q_2^2 + \left(\frac{5}{3} - 2q_2\right) H_2(1) = \frac{23}{9} - \frac{2}{3}q_2 - q_2^2, \\
Z_{2,p} &= q_2^2 - \frac{2}{3}q_2^3 - \frac{7}{12}B_{p-3} + \left(\frac{1}{3} + 2q_2 - q_2^2\right) H_2(1) + \left(\frac{5}{3} - 2q_2\right) H_2(2) \\
&= 1 + 2q_2 - \frac{1}{3}q_2^2 - \frac{2}{3}q_2^3 - \frac{7}{12}B_{p-3}.
\end{aligned}$$

By substituting $\binom{k}{2} = \frac{1}{2}k^2 - \frac{1}{2}k$ in the right hand side of the congruence (4. 18), it can be written as

$$\frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}} k^2 H_k - \frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}} kH_k \equiv \frac{1}{16}\left(X_{2,p} - pY_{2,p} + p^2 Z_{2,p}\right) \pmod{p^3},$$

from which we get

$$\sum_{k=1}^{\frac{p-1}{2}} k^2 H_k \equiv \sum_{k=1}^{\frac{p-1}{2}} kH_k + \frac{1}{8}\left(X_{2,p} - pY_{2,p} + p^2 Z_{2,p}\right) \pmod{p^3}. \tag{4. 19}$$

In view of (4. 17) and (4. 19), we have

$$\sum_{k=1}^{\frac{p-1}{2}} k^2 H_k \equiv \frac{1}{48} - p\left(-\frac{1}{12}q_2 + \frac{5}{72}\right) + p^2\left(-\frac{1}{24}q_2^2 + \frac{1}{16}\right) \pmod{p^3}.$$

5. Acknowledgment

I would like to thank the referees for their useful comments and suggestions.

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