Punjab University Journal of Mathematics (2021),53(9),606-620
https://doi.org/10.52280/pujm.2021.530901

On Properties of α-Sumudu Transform and Applications

J.A.Nanware
Department of Mathematics,
Shrikrishna Mahavidyalaya, Gunjoti
Dt. Osmanabad (M.S) - 413 606, India
Email: jag_skmg91@rediffmail.com

N.G.Patil
Department of Applied Sciences,
MBES College of Engineering, Ambajogai
Dt.Beed (M.S)- 413 517, India
Email: ngpatil1608@gmail.com

Received: 26 June, 2020 / Accepted: 09 August, 2021 / Published online: 25 September, 2021

Abstract.: The α-Sumudu transform is defined and its properties are proved. α-Sumudu transform of convolution product and composition of functions is obtained. The α-Sumudu transform of Riemann-Liouville integral and derivatives of fractional order are determined. As an application, the solution of Initial Value Problems with Riemann-Liouville derivative of fractional order is obtained. .

AMS (MOS) Subject Classification Codes: 34A12; 26A33;44A35
Key Words: Riemann-Liouville derivative, Sumudu transform, Initial value problems, Mittag-Leffler functions.

1. Introduction

Grunwald, Letnikov, Riemann-Liouville, Caputo, Miller and Ross, Hadamard [29] and Jumarie etc. types of fractional derivatives were introduced. Many natural phenomena are modeled via fractional differential equations. The concept of fractional calculus was defined in 17^{th} century. Researchers have found and studied several methods for obtaining analytical and approximate solution of fractional differential equations which includes Power series method [2, 13, 27, 26], Iterative method, Monotone iterative method, Homotopy Perturbation method [10], Adomian method [28] and Transforms methods [5, 8] etc. Integral transform methods [31] like New transform [30], Laplace transform [11], Sumudu transform [8], and Natural Transform [6] etc. are applied to study the solutions of differential equations of arbitrary order [16]. The Laplace integral transforms of Mathematical physics with the general scheme for applications was illustrated by Luchko [23]. Analytical solutions of some fractional ordinary differential equations are studied by Bulut et.al.[9]

and applied Sumudu transform technique and found that this technique is direct and valuable to fractional differential equations. Analytical solution of fractional model of HIV infection of $CD4^{+}T$ lymphocite cells was obtained by Bulut et.al.[10] using HPSTM and HATM. It is observed that HATM method is rapid and gives large convergence region by choosing appropriate value of h. Comparative study of fractional models using NDTM and VIM was carried in [7]. Transport models are governed by fractional partial differential equations and are investigated by researchers via fractional Sumudu transform [22], Walsh function [19, 21] and Chebyshev polynomials [20]. Linear and nonlinear partial differential equations using natural transform decomposition method was also studied in [6]. The Conformable fractional Laplace and Sumudu transforms were studied by Hammed et. al. [17, 1]. Hammed et.al. introduced conformable fractional derivative and study its properties. Recently, Al-Zhour et.al.[3] studied fractional differential equations in conformable fractional derivative and obtained series solution for Laguere and Lane-Emden fractional differential equations and nonlinear dispersive PDEs [12, 27]. The conformable fractional natural transform have been studied by Al-Zhour et.al.[4] and applied to obtain solution of fractional differential equations. New technique is introduced by El-Ajou et. al.[14] to obtain solution of non-homogeneous higher order matrix fractional differential equations. Numerical solution of the fractional multi-pantagraph system is studied by El-Ajou et.al. [15] using algorithm of HAM and RPSM.

The α - Laplace transform firstly introduced by Romero et. al.[24] and applied to obtain solution of fractional differential equations. Medina et. al. [25] also applied the idea of α-Laplace transform to find the solution of differential equations of α order [24]. This motivates us to define the α-Sumudu transform and we apply this to find the solution of differential equations of fractional order.

In this paper, the α-Sumudu transform is defined and fundamental properties of α-Sumudu transform are obtained. The α-Sumudu transform of a Riemann-Liouville integral and derivative of fractional order, Mittag-Leffler functions, convolution of two functions are determined. The α-Sumudu transform is applied to obtain solution of initial value problems involving R-L fractional derivative.

The paper is compiled as under: In section 2, basic definitions and results are considered. We define α-Sumudu transform, prove its properties, the inverse α-Sumudu transform, convolution product, α-Sumudu transform composition and α-Sumudu transform of R-L derivative are given in third section. As an application of the α-Sumudu transform, solution of initial value problem is obtained in section 4. Concluding remarks are given at the end.

2. Preliminaries

In this section, we consider basic definitions and results in fractional calculus that are required in further section.

Definition 2.1. [29] *The Gamma function* $\Gamma(z)$ *due to Euler is defined as*

$$\Gamma(z) = \int_0^\infty e^{-p} t^{z-1} dp; \quad R(z) > 0$$

Definition 2.2. [29] *Mittag-Leffler (M-L) function with 1-parameter is defined as*

$$E_\alpha(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\alpha p + 1)}$$

Definition 2.3. [29] *M-L function with 2-parameter is defined as*

$$E_{\alpha,\beta}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\alpha p + \beta)}$$

Mittag-Leffler functions of λt^α are

$$E_\alpha(\lambda t^\alpha) = \sum_{p=0}^{\infty} \frac{(\lambda t^\alpha)^p}{\Gamma(\alpha p + 1)}, \quad E_{\alpha,\beta}(\lambda t^\alpha) = \sum_{p=0}^{\infty} \frac{(\lambda t^\alpha)^p}{\Gamma(\alpha p + \beta)}$$

and m^{th} power of these functions is as follows:

$$E_\alpha^m(\lambda t^\alpha) = \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} \frac{(\lambda t^\alpha)^p}{\Gamma(\alpha p + \alpha m + 1)}$$

$$E_{\alpha,\beta}^m(\lambda t^\alpha) = \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} \frac{(\lambda t^\alpha)^p}{\Gamma(\alpha p + \alpha m + \beta)}$$

Definition 2.4. *If $f(t)$ be defined in a interval $[a, \infty)$ then we say that $f(t)$ is locally integrable in $[a, \infty)$ if for all $a < b, f$ is integrable in $[a, b]$.*

Definition 2.5. *A real valued function $f(t)$ is said to be measurable if, for each $\eta \in R$, the set $[t : f(t) > \eta]$ is measurable.*

Definition 2.6. [29] *If $f(t)$ is locally integrable on $[a, \infty)$, then R-L integral of fractional order α, is*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-p)^{\alpha-1} f(p)dp; \quad t > a; R(\alpha) > 0$$

Similarly, if $f(t)$ is locally integrable on $[-\infty, b)$ then

$$I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (p-t)^{\alpha-1} f(p)dp; \quad R(\alpha) > 0$$

Definition 2.7. [29] *Riemann-Liouville (R-L) derivative of fractional order α, is defined as*

$$D_p^\alpha f(p) = \left(\frac{d}{dp}\right)^m I_p^{m-\alpha} f(p); \quad R(\alpha) > 0, \quad m \in I$$

Definition 2.8. [31] *The Sumudu transform of $f(t), \quad t \geq 0$, denoted by $F(u)$ is defined as*

$$F(u) = S[f(t); u] = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)dt; \quad u \in R$$

Definition 2.9. [31] *Let $A(R_0^+)$ be the function space of Sumudu transformable functions, that is*

$$A(R_0^+) = \left\{ f(t) \mid \exists N, \mu_1, \mu_2 > 0, |f(t)| < Ne^{\frac{|t|}{\mu_j}} \quad \text{if} \quad t \in (-1)^j \times [0, \infty) \right\}$$

for function in $A(R_0^+)$, and $N < \infty, \quad \mu_j, j = 1, 2,$ may be finite or infinite.

Definition 2.10. [31] *If $f(t)$ be defined on R_0^+, then incomplete Sumudu transform $F(u)$ of $f(t)$ is defined as*

$$S[f(t), b](u) = \int_0^b \frac{1}{u} e^{-\frac{t}{u}} f(t) dt; \quad \text{for} \quad b,\, u \in R$$

Theorem 2.11 (Fubini's Theorem)**.** *If $f(x, y)$ is continuous function on a rectangle $R = [a, b] \times [c, d]$, then $\int\int_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$.*

Lemma 2.12. [18] *If $f(t)$ is well-behaved (not violating any assumptions like continuity, differentiability etc.) and $\alpha \in (0, 1)$, then Sumudu transform of R-L fractional integral of $f(u)$ is*

$$S[I^\alpha f](u) = u^\alpha S[f](u); \quad R(\alpha) > 0$$

Lemma 2.13. [8] *If $f(t)$ is well-behaved and $\alpha \in (0, 1)$, then Sumudu transform of R-L fractional derivative of $f(t)$ is*

$$S[D^\alpha f(t)](u) = u^{-\alpha} S[f(t)](u) - \frac{I^{1-\alpha} f(t)}{u}|_{t=0}$$

3. α-SUMUDU TRANSFORM AND PROPERTIES

Here, we define α-Sumudu transform, convolution product and study its properties. So far in the literature α-Sumudu transform is not defined yet.

Definition 3.1. *If $f(t)$ is defined on R_0^+, then the α-Sumudu transform $F_\alpha(u)$ is*

$$F_\alpha(u) = S_\alpha[f(t)](u) = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t) dt; \quad u \in R.$$

The α-Sumudu transform is a generalization of Sumudu transform because as $\alpha \to 1$, we have

$$S_1[f(p)](u) = S[f(p)](u).$$

Thus, we have

Theorem 3.2. *If $f(t) \in A(R_0^+)$, then $F_\alpha(u) = S_\alpha[f(t)](u)$ for $u > a^\alpha$*

Proof. It is obvious from the definition of α-Sumudu transform. □

Definition 3.3. [24] *If $f, g \in L^1(R^+)$(measurable space), then classical convolution product is*

$$(f * g)(t) = \int_0^t f(\mu) g(t - \mu) d\mu, \quad t > 0,$$

where $L^1(R^+) = \{f : R \to C | f$ is measurable and $\int |f(t)| dt < \infty\}$.

Definition 3.4. [25] *If $f, g \in L^1(R^+)$, then convolution product o is defined as*

$$(fog)(t) = \int_t^{\infty} f(\mu - t)g(\mu)d\mu, \quad t > 0$$

Lemma 3.5. *If f is well-behaved and $\alpha \in (0,1)$, then the α- Sumudu transform of $f(u)$ is*

$$S_\alpha[f](u) = S[f](\mu); \mu = u^{\frac{1}{\alpha}}$$

Proof. By α-Sumudu transform

$$\begin{aligned} S_\alpha[f(t)](u) &= \int_0^{\infty} \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t)dt; \quad u \in R \\ &= S[f(t)]u^{\frac{1}{\alpha}} \\ &= S[f(t)]\mu, \quad \text{where} \quad \mu = u^{\frac{1}{\alpha}} \\ &= S[f]\mu; \qquad \mu = u^{\frac{1}{\alpha}} \end{aligned}$$

□

Theorem 3.6. *If $a, c \in R$ and $0 < \alpha \leq 1$, then*

(a) $S_\alpha[c] = c$
(b) $S_\alpha[e^{at}] = \frac{1}{1-au^{\frac{1}{\alpha}}}$
(c) $S_\alpha[sinat] = \frac{au^{\frac{1}{\alpha}}}{1+a^2u^{\frac{2}{\alpha}}}$
(d) $S_\alpha[cosat] = \frac{1}{1+a^2u^{\frac{2}{\alpha}}}$
(e) $S_\alpha[sinhat] = \frac{au^{\frac{1}{\alpha}}}{1-a^2u^{\frac{2}{\alpha}}}$
(f) $S_\alpha[coshat] = \frac{1}{1-a^2u^{\frac{2}{\alpha}}}$
(g) $S_\alpha[t^n] = n!u^{\frac{n}{\alpha}}$

Proof. By Definition 3.1, we have

(a) $S_\alpha[c] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} cdt = \frac{c}{u^{\frac{1}{\alpha}}} \left[\frac{e^{-\frac{t}{u^{\frac{1}{\alpha}}}}}{-\frac{1}{u^{\frac{1}{\alpha}}}} \right]_0^\infty = c$

(b) $S_\alpha[e^{at}] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} e^{at}dt = \frac{1}{u^{\frac{1}{\alpha}}} \left[\frac{e^{-(\frac{1}{u^{\frac{1}{\alpha}}}-a)t}}{-(\frac{1}{u^{\frac{1}{\alpha}}}-a)} \right]_0^\infty = \frac{1}{1-au^{\frac{1}{\alpha}}}$

(c) $S_\alpha[sin(at)] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} sin(at)dt$

$= \frac{1}{u^{\frac{1}{\alpha}}} \left[\frac{e^{-\frac{t}{u^{\frac{1}{\alpha}}}}}{(-\frac{1}{u^{\frac{1}{\alpha}}})^2+a^2} \left[-\frac{1}{u^{\frac{1}{\alpha}}} sin(at) - acos(at) \right] \right]_0^\infty$

$= \frac{1}{u^{\frac{1}{\alpha}}} \left[\frac{1}{(\frac{1}{u^{\frac{1}{\alpha}}})^2+a^2} \right] = \frac{au^{\frac{1}{\alpha}}}{1+a^2u^{\frac{2}{\alpha}}}$

(d) $S_\alpha[cosat] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} cosatdt$

$= \frac{1}{u^{\frac{1}{\alpha}}} \left[\frac{e^{-\frac{t}{u^{\frac{1}{\alpha}}}}}{(-\frac{1}{u^{\frac{1}{\alpha}}})^2+a^2} \left[-\frac{1}{u^{\frac{1}{\alpha}}} cosat + asinat \right] \right]_0^\infty = \frac{1}{u^{\frac{1}{\alpha}}} \left[(-\frac{1}{(\frac{1}{u^{\frac{1}{\alpha}}})^2+a^2} (-\frac{1}{u^{\frac{1}{\alpha}}}) \right] =$

$\frac{1}{1+a^2u^{\frac{2}{\alpha}}}$

Similarly, we can prove (e)and (f)

(g) $S_\alpha[t^n] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} t^n dt$

$Let \quad \frac{t}{u^{\frac{1}{\alpha}}} = x \qquad \therefore t = xu^{\frac{1}{\alpha}}; \qquad \therefore dt = u^{\frac{1}{\alpha}} dx$

$S_\alpha[t^n] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-x} x^n u^{\frac{n}{\alpha}} u^{\frac{1}{\alpha}} dx = u^{\frac{n}{\alpha}} \int_0^\infty e^{-x} x^n dx = u^{\frac{n}{\alpha}} \Gamma(n+1) = n! u^{\frac{n}{\alpha}}$

□

Theorem 3.7. *Let* $f, g : [0,\infty) \to R, \quad \lambda, \mu \in R$ *and* $0 < \alpha \le 1$*. If* $S_\alpha[f(t)] = F_\alpha[u], \quad S_\alpha[g(t)] = G_\alpha[u]$*, then*

(i)

$$S_\alpha[\lambda f(t) + \mu g(t)] = \lambda F_\alpha(u) + \mu G_\alpha(u)$$

(ii)

$$S_\alpha[e^{-at} f(t)] = F_\alpha\left[\frac{1}{u^{\frac{1}{\alpha}}} + a\right]$$

(iii)

$$S_\alpha[f'(t)] = \frac{1}{u^{\frac{1}{\alpha}}} F_\alpha[u] - \frac{1}{u^{\frac{1}{\alpha}}} f(0)$$

(iv)

$$S_\alpha\left[\int_0^t f(t)dt\right] = u^{\frac{1}{\alpha}} F_\alpha[u]$$

Proof. By α-Sumudu transform :

(i)

$$\begin{aligned} S_\alpha[\lambda f(t) + \mu g(t)] &= \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} [\lambda f(t) + \mu g(t)] dt \\ &= \lambda \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t) dt + \mu \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} g(t) dt \\ &= \lambda F_\alpha[u] + \mu G_\alpha[u] \end{aligned}$$

(ii)

$$\begin{aligned} S_\alpha[e^{-at} f(t)] &= \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} e^{-at} f(t) dt \\ &= \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-}\left(\frac{1}{u^{\frac{1}{\alpha}}} + a\right)t \quad f(t) dt \\ &= F_\alpha\left[\frac{1}{u^{\frac{1}{\alpha}}} + a\right] \end{aligned}$$

Similarly, we prove

$$S_\alpha[e^{at} f(t)] = F_\alpha\left[\frac{1}{u^{\frac{1}{\alpha}}} - a\right]$$

(iii)

$$S_\alpha[f'(t)] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f'(t)dt$$
$$= \left[\frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t)\right]_0^\infty - \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}}(-\frac{1}{u^{\frac{1}{\alpha}}})e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t)dt$$
$$= \left[0 - \frac{1}{u^{\frac{1}{\alpha}}} f(0)\right] + \frac{1}{u^{\frac{1}{\alpha}}} \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t)dt$$
$$= \frac{1}{u^{\frac{1}{\alpha}}} F_\alpha[u] - \frac{1}{u^{\frac{1}{\alpha}}} f(0)$$

In general

$$S_\alpha[f^n(t)] = \frac{1}{u^{\frac{n}{\alpha}}} F_\alpha[u] - \frac{1}{u^{\frac{n}{\alpha}}} f(0) - \frac{1}{u^{\frac{n-1}{\alpha}}} f'(0) - \frac{1}{u^{\frac{n-2}{\alpha}}} f''(0)... - f^{n-1}(0).$$

(iv) Let $\phi(t) = \int_0^t f(t)dt, \quad \phi(0) = 0 \qquad \phi'(t) = f(t)$

$$S_\alpha[\phi'(t)] = \frac{1}{u^{\frac{1}{\alpha}}} S_\alpha[\phi(t)] - \frac{1}{u^{\frac{1}{\alpha}}} \phi(0)$$
$$= \frac{1}{u^{\frac{1}{\alpha}}} S_\alpha[\phi(t)](\because \phi(0) = 0)$$
$$S_\alpha[\phi(t)] = u^{\frac{1}{\alpha}} S_\alpha[\phi'(t)]$$

Putting the value of $\phi(t)$ and $\phi'(t)$, to obtain

$$S_\alpha\left[\int_0^t f(t)dt\right] = u^{\frac{1}{\alpha}} F_\alpha[u]$$

□

Theorem 3.8. *For $f^{(k)}(t) \in A(R_0^+), k = 1, 2....n$, we have*

$$S_\alpha\left[\left(\frac{d}{dt}f(t)\right)^n\right](u) = \frac{1}{u^{\frac{n}{\alpha}}} S_\alpha[f(t)](u) - \frac{1}{u^{\frac{n}{\alpha}}} \sum_{k=1}^{n} u^{\frac{n-k}{\alpha}} f^{\frac{n-k}{\alpha}}(0)$$

Proof. Using

$$S\left[\left(\frac{d}{dt}f(t)\right)^n\right](\mu) = \frac{1}{\mu^n} S_\alpha[f(t)](u) - \frac{1}{\mu^n} \sum_{k=1}^{n} \mu^{n-k} f^{n-k}(0)$$

$$S_\alpha[f](u) = S[f](\mu); \quad \mu = u^{\frac{1}{\alpha}}$$

we obtain

$$S_\alpha\left[\left(\frac{d}{dt}f(t)\right)^n\right](u) = \frac{1}{u^{\frac{n}{\alpha}}} F_\alpha(u) - \frac{1}{u^{\frac{n}{\alpha}}} \sum_{k=1}^{n} u^{\frac{n-k}{\alpha}} f^{\frac{n-k}{\alpha}}(0)$$

This proves the Theorem. □

Now, we turn to inversion formula

$$S_\alpha[f](u) = S[f](\mu) = g_1(\mu) \quad \mu = u^{\frac{1}{\alpha}}$$

then

$$f(t) = S_\alpha^{-1}\Big[S_\alpha\big[f\big](u)\Big] = S^{-1}\Big[g_1\big(\mu\big)\Big](t)$$

Applying inverse Sumudu transform, to obtain

$$S^{-1}\Big[g_1\big(\mu\big)\Big](t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\frac{t}{\mu}} g_1(\mu)d\mu$$

$$S^{-1}\Big[g_1\big(\mu\big)\Big](t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\frac{t}{\mu}} S[f](\mu)d\mu$$

Change of variable $\mu = u^{\frac{1}{\alpha}}; d\mu = \frac{1}{\alpha}u^{\frac{1}{\alpha}-1}du$ gives

$$S^{-1}\Big[g_1\big(\mu\big)\Big](t) = \frac{1}{2\pi i}\int_{a^\alpha-i\infty}^{a^\alpha+i\infty} e^{\frac{t}{u^{\frac{1}{\alpha}}}} S_\alpha[f](u)\frac{1}{\alpha}u^{\frac{1}{\alpha}-1}du \tag{3. 1}$$

Thus, we have

Definition 3.9. *If f is well-behaved and $\alpha \in (0,1)$,then inverse α-Sumudu transform is*

$$S^{-1}\Big[F_\alpha\big(u\big)\Big](t) = \frac{1}{2\pi i\alpha}\int_{a^\alpha-i\infty}^{a^\alpha+i\infty} e^{\frac{t}{u^{\frac{1}{\alpha}}}} F_\alpha(u)u^{\frac{1}{\alpha}-1}du$$

It is easily seen that $S_\alpha\big[S_\alpha^{-1}\big] = Id$, by change of variable $\mu = u^{\frac{1}{\alpha}}$.

Theorem 3.10. *If $f(t), g(t) \in A(R_0^+)$ such that $F_\alpha(u) = S_\alpha\big[f(t)\big](u)$ and $G_\alpha(u) = S_\alpha\big[g(t)\big](u)$ then $S_\alpha\big[f(t) * g(t)\big](u) = u^{\frac{1}{\alpha}}F_\alpha(u)G_\alpha(u)$*

Proof. Use α-Sumudu transform and convolution,to get

$$S_\alpha[f(t)](u) = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}}e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t)dt$$

$$S_\alpha[(f*g)(t);u] = \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}}e^{-\frac{t}{u^{\frac{1}{\alpha}}}} (f*g)(t)dt$$

$$= \frac{1}{u^{\frac{1}{\alpha}}}\int_0^\infty e^{-\frac{t}{u^{\frac{1}{\alpha}}}}\int_0^t f(\tau)g(t-\tau)dtd\tau$$

Fubini's theorem gives

$$S_\alpha[(f*g)(t)](u) = \frac{1}{u^{\frac{1}{\alpha}}}\int_0^\infty e^{-\frac{\tau}{u^{\frac{1}{\alpha}}}} f(\mu)d\mu\int_\mu^\infty e^{-\frac{(t-\mu)}{u^{\frac{1}{\alpha}}}} g(t-\mu)dt$$

Let $t-\mu = z$ and extension of upper bound of integrals to $t \to \infty$, gives

$$S_\alpha[(f*g)(t);u] = \frac{1}{u^{\frac{1}{\alpha}}}\int_0^\infty e^{-\frac{\mu}{u^{\frac{1}{\alpha}}}} f(\mu)d\mu\int_0^\infty e^{-\frac{z}{u^{\frac{1}{\alpha}}}} g(z)dz$$

$$S_\alpha[(f*g)(t);u] = u^{\frac{1}{\alpha}}[F_\alpha(u).G_\alpha(u)]$$

□

Theorem 3.11. *If* $f, g \in L^1(R_0^+), 0 < \alpha < 1$ *then* $S_\alpha[fog](u) = S_\alpha[g](u).S[f](-u^{\frac{1}{\alpha}})$

Proof. Using definition of α-Sumudu transform and convolution product o, we obtain

$$\begin{aligned} S_\alpha[f(t)](u) &= \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} f(t)dt; \\ S_\alpha[(fog)(t);u] &= \int_0^\infty \frac{1}{u^{\frac{1}{\alpha}}} e^{-\frac{t}{u^{\frac{1}{\alpha}}}} (fog)(t)dt; \\ &= \frac{1}{u^{\frac{1}{\alpha}}} \int_0^\infty e^{-\frac{t}{u^{\frac{1}{\alpha}}}} \int_t^\infty f(\mu - t)g(\mu)dtd\mu \end{aligned}$$

Fubini's theorem implies

$$S_\alpha[(fog)(t);u] = \frac{1}{u^{\frac{1}{\alpha}}} \int_0^\infty e^{-\frac{\mu}{u^{\frac{1}{\alpha}}}} g(\mu)d\mu \int_0^\mu e^{-\frac{(t-\mu)}{u^{\frac{1}{\alpha}}}} f(\mu - t)dt$$

Let $\mu - t = z$ and extension of upper bound of integrals gives

$$\begin{aligned} S_\alpha[(fog)(t);u] &= \frac{1}{u^{\frac{1}{\alpha}}} \int_0^\infty e^{-\frac{\tau}{u^{\frac{1}{\alpha}}}} g(\tau)d\tau \int_\tau^0 e^{\frac{z}{u^{\frac{1}{\alpha}}}} f(z)(-dz) \\ &= \frac{1}{u^{\frac{1}{\alpha}}} \int_0^\infty e^{-\frac{\tau}{u^{\frac{1}{\alpha}}}} g(\tau)d\tau \int_0^\tau e^{\frac{z}{u^{\frac{1}{\alpha}}}} f(z)(dz) \\ S_\alpha[(fog)(t);u] &= S_\alpha[g](u).S[f](-u^{\frac{1}{\alpha}}) \end{aligned}$$

□

Theorem 3.12. *If* $\lambda \in R^+$ *and* $f, g \in L^1(R^+)$ *and* $e_{\lambda^{\frac{1}{\alpha}}}(t) = e^{\frac{t}{\lambda^{\frac{1}{\alpha}}}}; t \geq 0, t \in R^+$,*then*

(i) $fo\frac{1}{\lambda^{\frac{1}{\alpha}}} e_{\lambda^{\frac{1}{\alpha}}} = S_\alpha[f](\lambda)e_{\lambda^{\frac{1}{\alpha}}}$

(ii) $\frac{1}{\lambda^{\frac{1}{\alpha}}} e_{\lambda^{\frac{1}{\alpha}}} of = S_\alpha[f](\lambda^\alpha)e_{-\lambda^{\frac{1}{\alpha}}} - (e_{-\lambda} * f)$

Proof. (i) By definition 3.4, we have

$$\left(fo\frac{1}{\lambda^{\frac{1}{\alpha}}} e_{\lambda^{\frac{1}{\alpha}}}\right)(t) = \int_t^\infty f(\mu - t)\frac{1}{\lambda^{\frac{1}{\alpha}}} e^{-\frac{\mu}{\lambda^{\frac{1}{\alpha}}}} d\mu$$

If $z = \mu - t$ then $dp = d\mu$

$$\begin{aligned} \left(fo\frac{1}{\lambda^{\frac{1}{\alpha}}} e_{\lambda^{\frac{1}{\alpha}}}\right)(t) &= \int_0^\infty f(z)\frac{1}{\lambda^{\frac{1}{\alpha}}} e^{-\frac{(z+t)}{\lambda^{\frac{1}{\alpha}}}} dz \\ &= \left[\int_0^\infty \frac{1}{\lambda^{\frac{1}{\alpha}}} e^{-\frac{z}{\lambda^{\frac{1}{\alpha}}}} dz\right] e_{\lambda^{\frac{1}{\alpha}}} \\ &= S_\alpha[f](\lambda)e_{\lambda^{\frac{1}{\alpha}}}. \end{aligned}$$

(ii) By definition 3.4, we have

$$\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}e_{\lambda^{\frac{1}{\alpha}}} of\right)(t) = \int_t^\infty \frac{1}{\lambda^{\frac{1}{\alpha}}} e^{-\frac{(\mu-t)}{\lambda^{\frac{1}{\alpha}}}} f(\mu)d\mu$$

as $f, y, e_{-\lambda^{\frac{1}{\alpha}}} \in L^1(R^+)$ then $e_{-\lambda^{\frac{1}{\alpha}}} * f \in L^1(R^+)$, we obtain

$$\begin{aligned}\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}e_{\lambda^{\frac{1}{\alpha}}} of\right)(t) &= \left(\int_0^\infty \frac{1}{\lambda^{\frac{1}{\alpha}}} e^{-\frac{(\tau-t)}{\lambda^{\frac{1}{\alpha}}}} f(\tau)d\tau\right)\big(e_{-\lambda^{\frac{1}{\alpha}}} * f\big)(t)\\ &= \left(\int_0^\infty \frac{1}{\lambda^{\frac{1}{\alpha}}} e^{-\frac{(\tau)}{\lambda^{\frac{1}{\alpha}}}} f(\tau)d\tau\right)e_{-\lambda^{\frac{1}{\alpha}}} - \big(e_{-\lambda^{\frac{1}{\alpha}}} * f\big)(t)\\ &= S_\alpha[f](\lambda)e_{-\lambda^{\frac{1}{\alpha}}} - \big(e_{-\lambda^{\frac{1}{\alpha}}} * f\big)(t)\end{aligned}$$

□

Lemma 3.13. *If f is well-behaved and $\alpha \in (0,1)$, then α-Sumudu transform of R-L fractional integral of f is*

$$S_\alpha\big[I_x^\gamma f\big](u) = u^{\frac{\gamma}{\alpha}} S_\alpha\big[F\big](u) \tag{3. 2}$$

Proof. Recall that $t > 0, \beta \in R$ for

$$S_\alpha\big[t^\gamma\big] = \Gamma(\gamma+1) \quad u^{\frac{\gamma}{\alpha}} \tag{3. 3}$$

Since

$$I_x^\gamma f(x) = J_\gamma(t) * f(t), \text{where } J_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}$$

we have from 3. 3

$$S_\alpha\big[J_\gamma(t)\big](u) = S_\alpha\left[\frac{t^{\gamma-1}}{\Gamma\gamma}\right] = u^{\frac{\gamma-1}{\alpha}} \tag{3. 4}$$

By definition of α-Sumudu transform and using Theorem 3.5, we obtain

$$\begin{aligned}S_\alpha\big[I_x^\gamma f(x)\big] &= S_\alpha\big[J_\gamma(t) * f(t)\big](u)\\ &= u^{\frac{1}{\alpha}} S_\alpha\big[J_\gamma(t)\big](u).S_\alpha\big[f\big](u)\\ &= u^{\frac{1}{\alpha}} u^{\frac{\gamma-1}{\alpha}} S_\alpha\big[f\big](u)\\ &= u^{\frac{\gamma}{\alpha}} S_\alpha\big[f\big](u)\end{aligned}$$

□

Lemma 3.14. *If $\alpha \in (0,1)$, then α-Sumudu transform of fractional R-L derivative of $f(t)$ is*

$$S_\alpha\left[D_x^\gamma f(t)\right](u) = u^{-\frac{\gamma}{\alpha}} S_\alpha[f(t)](u) - \frac{I_x^{1-\gamma}}{u} f(t)|_{t=0} \tag{3. 5}$$

Proof. Consider

$$
\begin{aligned}
S_\alpha\big[D_x^\gamma f(t)\big](u) &= S_\alpha\left[\frac{d}{dx} I_x^{1-\gamma} f(t)\right](u) \\
&= u^{-\frac{1}{\alpha}} S_\alpha\big[I_x^{1-\gamma} f(t)\big](u) - \frac{I_x^{1-\gamma}}{u} f(t)|_{t=0} \\
&= u^{-\frac{1}{\alpha}} u^{\frac{1-\gamma}{\alpha}} S_\alpha[f(t)](u) - \frac{I_x^{1-\gamma}}{u} f(t)|_{t=0} \\
&= u^{-\frac{\gamma}{\alpha}} S_\alpha[f(t)](u) - \frac{I_x^{1-\gamma}}{u} f(t)|_{t=0}
\end{aligned}
$$

□

Theorem 3.15. *If* $\gamma \in C,\ R(\gamma) > 0,\ \lambda \in R$, *then*

$$
S_\alpha\left[t^{\gamma m} E_\alpha^m(\lambda t^\gamma)\right] = \frac{u^{\frac{-\gamma}{\alpha}} m!}{(u^{\frac{-\gamma}{\alpha}} - \lambda)^{m+1}}
$$

Proof. Since

$$
\sum_{p=0}^{\infty} \frac{(p+m)!}{p!}.x^p = \frac{m!}{(1-x)^{m+1}},
$$

we have

$$
\begin{aligned}
S_\alpha\left[t^{\gamma m} E_\alpha^m(\lambda t^\gamma)\right] &= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} \lambda^p . \frac{S_\alpha[t^{\gamma p+\gamma m}]}{\Gamma(\gamma p+\gamma m+1)} \\
&= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} \lambda^p . \frac{\Gamma(\gamma p+\gamma m+1)}{\Gamma(\gamma p+\gamma m+1)} . u^{\frac{\gamma p+\gamma m}{\alpha}} \\
&= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} \lambda^p . u^{\frac{\gamma p+\gamma m}{\alpha}} \\
&= u^{\frac{\gamma m}{\alpha}} . \frac{m!}{(1-\lambda u^{\frac{\gamma}{\alpha}})^{m+1}} \\
&= \frac{u^{\frac{\gamma m}{\alpha}} m!}{(u^{\frac{\gamma}{\alpha}})^{m+1} (u^{-\frac{\gamma}{\alpha}} - \lambda)^{m+1}} \\
&= u^{\frac{\gamma m}{\alpha}} . u^{-\frac{\gamma m}{\alpha}} . u^{-\frac{\gamma}{\alpha}} . \frac{m!}{(u^{-\frac{\gamma}{\alpha}} - \lambda)^{m+1}} \\
&= \frac{u^{-\frac{\gamma}{\alpha}} m!}{(u^{-\frac{\gamma}{\alpha}} - \lambda)^{m+1}}
\end{aligned}
$$

□

Theorem 3.16. *If* $\eta, \beta \in C, \quad R(\eta) > 0, \quad R(\beta) > 0, \quad \lambda \in R$, *then*

$$
S_\alpha\left[t^{\eta m-\beta-1} E_{\eta,\beta}^m(\lambda t^\eta)\right] = \frac{u^{\frac{\beta-\eta-1}{\alpha}} m!}{(u^{\frac{-\eta}{\alpha}} - \lambda)^{m+1}}
$$

Proof. Since

$$\sum_{p=0}^{\infty} \frac{(p+m)!}{p!}.x^p = \frac{m!}{(1-x)^{m+1}}$$

then

$$\begin{aligned}
S_\alpha\left[t^{\eta m+\beta-1}E^m_{\eta,\beta}(\lambda t^\eta)\right] &= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!}\lambda^p.\frac{S_\alpha[t^{\eta p+\eta m+\beta-1}]}{\Gamma(\eta p+\eta m+\beta)} \\
&= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!}\lambda^p.\frac{\Gamma(\eta p+\eta m+\beta)}{\Gamma(\eta p+\eta m+\beta)}.u^{\frac{\eta p+\eta m+\beta-1}{\alpha}} \\
&= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!}\lambda^p.u^{\frac{\eta p+\eta m+\beta-1}{\alpha}} \\
&= u^{\frac{\eta m+\beta-1}{\alpha}} \sum_{p=0}^{\infty} \frac{(p+m)!}{p!}(\lambda u^{\frac{\eta}{\alpha}})^p \\
&= u^{\frac{\eta m+\beta-1}{\alpha}} \frac{m!}{(1-\lambda u^{\frac{\eta}{\alpha}})^{m+1}} \\
&= u^{\frac{\eta m+\beta-1}{\alpha}} \frac{m!}{u^{\frac{\eta(m+1)}{\alpha}}(u^{-\frac{\eta}{\alpha}})^{m+1}} \\
&= u^{\frac{\eta m+\beta-1}{\alpha}}.u^{\frac{-\eta m-\eta}{\alpha}}.\frac{m!}{(u^{-\frac{\eta}{\alpha}}-\lambda)^{m+1}} \\
&= u^{\frac{\eta m+\beta-1-\eta m-\eta}{\alpha}}.\frac{m!}{(u^{-\frac{\eta}{\alpha}}-\lambda)^{m+1}} \\
&= u^{\frac{\beta-\eta-1}{\alpha}}.\frac{m!}{(u^{-\frac{\eta}{\alpha}}-\lambda)^{m+1}}
\end{aligned}$$

□

4. Applications

As an application of α-Sumudu transform, we obtain solution of following differential equations involving R-L fractional derivative:

Example 4.1. *Consider the following fractional differential equation*

$$D^\alpha f(t) + af(t) = 0; \quad I^\alpha|_{t=0} = c. \tag{4. 6}$$

Applying α- Sumudu transform, to obtain

$$\frac{1}{u^{\frac{1}{\alpha}}}S_\alpha[f(t)](u) - \frac{I^\alpha}{u^{\frac{1}{\alpha}}}f(t)|_{t=0} + aS_\alpha[f(t)](u) = S_\alpha[f(t)](u)\left[\frac{1}{u^{\frac{1}{\alpha}}} + a\right] = \frac{c}{u^{\frac{1}{\alpha}}}$$

$$S_\alpha[f(t)](u)\left[\frac{1+au^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}}\right] = \frac{c}{u^{\frac{1}{\alpha}}}$$

$$S_\alpha[f(t)](u) = \frac{c}{1+au^{\frac{1}{\alpha}}}$$

Applying inverse α- Sumudu transform, we obtain the solution of (4. 6)

$$S_\alpha^{-1}\Big[S_\alpha[f(t)]\Big](u) = S_\alpha^{-1}\left[\frac{c}{1+au^{\frac{1}{\alpha}}}\right]$$

$$f(t) = ct^{-\alpha}E_{\alpha,\alpha}(-at^\alpha).$$

Example 4.2. *Consider the following fractional differential equation*

$$D^\alpha f(t) - 1 = t^2; \quad I^\alpha|_{t=0} = 2 \tag{4. 7}$$

Applying α- Sumudu transform, to obtain

$$\frac{1}{u^{\frac{1}{\alpha}}}S_\alpha[f(t)](u) - \frac{I^\alpha}{u^{\frac{1}{\alpha}}}f(t)|_t - S_\alpha[1] = S_\alpha[t^2]$$

$$\frac{1}{u^{\frac{1}{\alpha}}}S_\alpha[f(t)](u) = 2!u^{\frac{2}{\alpha}} + 1 + \frac{2}{u^{\frac{1}{\alpha}}}$$

$$S_\alpha[f(t)](u) = 2!u^{\frac{2}{\alpha}}u^{\frac{1}{\alpha}} + u^{\frac{1}{\alpha}} + 2$$

Applying inverse α- Sumudu transform, we obtain the solution of (4. 7)

$$S_\alpha^{-1}\Big[S_\alpha[f(t)]\Big](u) = S_\alpha^{-1}\Big[2!u^{\frac{3}{\alpha}} + u^{\frac{1}{\alpha}} + 2\Big]$$

$$f(t) = \frac{1}{3}t^3 + t + 2$$

Example 4.3. *Consider the following fractional differential equation*

$$D^\alpha f(t) - 1 = cos2t + t^2; \quad I^\alpha|_{t=0} = 2 \tag{4. 8}$$

Applying α- Sumudu transform, to obtain

$$\frac{1}{u^{\frac{1}{\alpha}}}S_\alpha[f(t)](u) - \frac{I^\alpha}{u^{\frac{1}{\alpha}}}f(t)|_t - S_\alpha[1] = S_\alpha[cos2t] + S_\alpha[t^2]$$

$$\frac{1}{u^{\frac{1}{\alpha}}}S_\alpha[f(t)](u) = \frac{1}{1+4u^{\frac{2}{\alpha}}} + 2!u^{\frac{2}{\alpha}} + 1 + \frac{2}{u^{\frac{1}{\alpha}}}$$

$$S_\alpha[f(t)](u) = \frac{u^{\frac{1}{\alpha}}}{1+4u^{\frac{2}{\alpha}}} + 2!u^{\frac{2}{\alpha}}u^{\frac{1}{\alpha}} + u^{\frac{1}{\alpha}} + 2$$

Applying inverse α- Sumudu transform, we obtain the solution of (4. 8)

$$S_\alpha^{-1}\left[S_\alpha[f(t)]\right](u) = S_\alpha^{-1}\left[\frac{u^{\frac{1}{\alpha}}}{1+4u^{\frac{2}{\alpha}}} + 2u^{\frac{3}{\alpha}} + u^{\frac{1}{\alpha}} + 2\right]$$

$$f(t) = \frac{1}{2}sin2t + \frac{1}{3}t^3 + t + 2$$

5. Conclusion

The α- Sumudu transform for Sumudu transformable functions is defined. The properties of α- Sumudu transform such as convolution product and composition are proved. The α- Sumudu transform of elementary functions are also obtained. The α- Sumudu transform of Riemann- Liouville fractional integral and derivative, Mittag-Leffler function with one parameter and two parameters are established. The inverse α-Sumudu transform is defined. The α-Sumudu transform and inverse α-Sumudu transform are used to obtain solution of fractional differential equations with initial conditions.

6. Acknowledgements

Authors are grateful to Reviewer's for reading whole manuscript carefully and for their valuable comments that helps to improve the writing of this paper.

References

[1] Z. Al-Zhour, F. Alrawajeh, N. Al-Mutairi and R. Alkhasawneh, *New Results on the Conformable Fractional Sumudu Transform, Theories and Applications*, Int. J. Anal. Appl., **17,** No. 6(2019) 1019-1033.

[2] Z. Al-Zhour, N. Al-Mutairi, F. Alrawajeh, R. Alkhasawneh, *Series Solutions for the Laguerre and Lane-Emden Fractional Differential Equations in the Sense of Conformable Fractional Derivative,* Alexandria Eng. J., **8,** No.4 (2019) 1413-1420.

[3] Z. Al-Zhour, *Fundamental fractional exponential matrix: New computational formulae and electrical applications*, Int.J. Electron. Commun.(AEU), **129** (2021) 153557.

[4] Z. Al-Zhour, N. Al-Mutairi, F. Alrawajeh, R. Alkhasawneh, *New Theoretical Results and Applications on Conformable Fractional Natural Transform*, Ain Shams Eng. J., **21,** No.1 (2021) 927-933.

[5] M. Baloch, S. Iqbal ,F. Sarwar and Abdul Rehman,*The Relationship of Fractional Transform with Fractional Fourier,Mellin and Sumudu Transforms*, American Scie. Research J. Engi., Tech. Sci.(ASRJETS), **55,**No. 1(2019) 11-16.

[6] H. M. Baskonus,H.Bulut and Y.Pandir, *The Natural Transform Decomposition Method for Linear and Nonlinear Partial Differential Equations*, Math. Engi. Sci. Aero., **5**, No.1(2014) 111-126.

[7] H.M.Baskonus and H. Bulut, *A comparison between NTDM and VIM for modified Cammassa-Holm and Modified Degasperis-Procesi Equations*, Nonl. Studies,**22,**No. 4(2015) 601-611.

[8] D.S. Bodkhe and S.K. Panchal, *On Sumudu Transform of Fractional Derivatives and its Applications to Fractional Differential Equations,* Asian J. Math. Computer Research, **11,** No.1(2016) 69-77.

[9] H.Bulut,H.M.Baskonus and F.B.M.Belgacem, *The Analytical Solution of Some Fractional Ordinary Differential Eqautions by the SUmudu Transform Method*, Abs. Appl. Anal., Vol.2013, ArticleID 203875, 6 Pages, http://dx.doi.org/10.1155/2013/203875.

[10] H.Bulut, Devendra Kumar,Jagdev Singh, Ram Swroop and H.M.Baskonous, *Analytic Study for Fractional Model of HIV Infection of CD4+T Lymphocyte Cells,* Math. Natural Science, **2**(2018) 33-43, doi:10.22436/mns.02.01.04

[11] L. Debnath, and D. Bhatta, *Integral Transforms and Their Applications*, Chapman and Hall/CRC, Taylor and Francis Group,New York; 2007.

[12] A. El-Ajou, M. Oqielat, Z. Al-Zhour, S. Kumar and S. Momani, *Solitary solutions for time-fractional nonlinear dispersive PDEs in the sense of conformable fractional derivative*, Chaos, **29** (2019) 093102.

[13] A. El-Ajou, Z. Al-Zhour M. Oqielat and S. Momani, T. Hayat, *Series Solutions of Nonlinear Conformable Fractional KdV-Burgers Equation with Some Applications*, The Eur. Phys. J. Plus, **134** (2019), 402.

[14] A. El-Ajou, M. Oqielat, Z. Al-Zhour and S. Momani, *A Class of Linear Non-Homogenous Higher Order Matrix Fractional Differential Equations: Analytical Solutions and New Technique*, Fract. Calc. Appl. Anal., **23,** No.2 (2020) 356-377.

[15] A. El-Ajou, M. Oqielat, Z. Al-Zhour, S. Momani, *Analytical Numerical Solutions of the Fractional Multi-Pantograph System: Two Attractive Methods and Comparisons, Results Phys.*, **14** (2019) 102500.

[16] H. Eltayeb and A. Kilicman, *A Note on The Sumudu Transforms and Di?erential Equations*, Appl. Math.Sciences, **4,**No. 22(2010) 1089-1098.

[17] M. Abu Hammad and R. Khalil, *Conformable Fractional Heat Differential Equation*, Int.J. Pure Appl. Math,**94,** No.2(2014) 215-221.

[18] Q. D.Katatbeha and Fethi Bin Muhammad Belgacem, *Applications of the Sumudu Transform to Fractional Differential Equations*, Nonl. Studies, **18,** No. 1(2011) 99-112.

[19] A.Kadem, *Solving Transport Equation via Walsh Function and Sumudu Transform,* Apppl. Comput.Math.**7,** No. 1(2008) 107-118.

[20] A.Kadem and Analele, *Solving the One-dimensional Neutron Transport Equation Using Chebyshev Polynomials and The Sumudu Transform*, Universitatii Oradea Fasc.Matematica, Tom XII (2005) 153-171.

[21] A.Kadem, *Analytic Investigations of the Walsh Functions combined with the Sumudu Transform and Applications to the Transport Equation,* Asian-European J. Math., **4,** No.2(2011) 199-215.

[22] A.Kadem, A. Kilicman, *Note on Transport Equation and Fractional Sumudu Transform,* Comp. Math. Appl. **62**(2011) 2992-3003. https://doi.org/10.1016/s0304-0208(06)x8001-5

[23] Y. Luchko, *Some Schemata for Applications of the Integral Transforms of Mathematicl physics*, Mathematics, **7**, No. 254 (2019); doi: 10.3390/math7030254.

[24] G. D. Medina and N.R. Ojeda, J.H. Pereira and L. G. Romero, *Fractional Laplace Transform and Fractional Calculus,* Int. Mathematical Forum, **12,** No.20(2017) 991-1000.

[25] P.J. Miana, *Convolution Products in $L^1(R^+)$, Integral Transforms and Fractional Calculus*, Frac. Cal. Appl. Anal., **8,** No.4(2005) 361-370.

[26] J.A. Nanware, Gunwant A. Birajdar, *Methods of Solving Fractional Differential Equations of Order $\alpha(0 < \alpha < 1 <)$*, Bull. Marathwada Mathematical Soci.,**15,** No.2(2014) 40-53.

[27] M. Oqielat, A. El-Ajou, Z. Al-Zhour, R. Alkhasawneh, H. Alrabaiah, *Series solutions for nonlinear time-fractional Schrodinger equations: Comparisions between conformable and Caputo derivatives,* Alexandria Eng. J., **59,** No.4 (2020) 2101-2114.

[28] T. Patel and Ramakanta Meher, *Adomian Decomposition Sumudu Transform Method for Solving a Solid and Porous Fin with Temperature Dependent Internal Heat Generation,* SpringerPlus **5**489 (2016) DOI 10.1186/s40064-016-2106-8.

[29] I. Podlubny, *Fractional Differential Equations*, Academic Press, United States, 1999.

[30] R. Saadeh, Ahmad Qazza and Aliaa Burqan, *A New Integral Transform: ARA Transform and Its Properties and Applications*, Symmetry **12**(2020) 925.

[31] G.K. Watugala, *Sumudu transform- A New Integral Transform to Solve Differential Equations and Control Engineering Problems*, Int.J.Math.Educ.Sci.Technol. **24,** No. 1(1993) 35-43. https://doi.org/10.1080/0020739930240105