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Mappings related to Hermite-Hadamard type inequalities for harmonically convex functions

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Abstract.: In this study, we define some mappings connected to the Hermite-Hadamard type inequalities constructed for harmonically convex mappings. We investigate some properties of these mappings and provide some refinement inequalities for the Hermite-Hadamard type inequalities that have already been established for harmonic convex functions.

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1. Introduction

For convex functions the following double inequality has great significance in literature and is known as Hermite-Hadamard's inequality:

Let $\psi : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$, be a convex function, then

$$\psi\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \psi(\lambda)d\lambda \leq \frac{\psi(\tau_1)+\psi(\tau_2)}{2}, \tag{1. 1}$$

The inequality (1. 1) holds in reversed direction if ψ is concave.
These inequalities have many extensions and generalizations, see [1]-[25].

Dragomir defined the following mappings $H, F : [0,1] \rightarrow \mathbb{R}$

$$H(\varphi) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \psi\left(\varphi\lambda + (1-\varphi)\left(\frac{\tau_1+\tau_2}{2}\right)\right) d\lambda$$

and

$$F(\varphi) = \frac{1}{(\tau_2-\tau_1)^2}\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \psi(\varphi\lambda + (1-\varphi)\mu)\, d\lambda d\mu,$$

where $\psi : [\tau_1, \tau_2] \to \mathbb{R}$ is a convex function and obtained some refinements between the middle and the left most terms in [2] for (1. 1).

Theorem 1.1. [2] *Let $\psi : [\tau_1, \tau_2] \to \mathbb{R}$ be a convex function on $[\tau_1, \tau_2]$. Then*

(i) *H is convex convex on $[0, 1]$.*

(ii) *The following hold:*

$$\inf_{\varphi\in[0,1]} H(\varphi) = H(0) = \psi\left(\frac{\tau_1+\tau_2}{2}\right)$$

$$\sup_{\varphi\in[0,1]} H(\varphi) = H(1) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \psi(\lambda)\, d\lambda.$$

(iii) *H increases monotonically on $[0, 1]$.*

Theorem 1.2. [2] *Let $\psi : [\tau_1, \tau_2] \to \mathbb{R}$ be a convex function on $[\tau_1, \tau_2]$. Then*

(i) *$F\left(\varphi + \frac{1}{2}\right) = F\left(\frac{1}{2} - \varphi\right)$ for all $\varphi \in \left[0, \frac{1}{2}\right]$*

(ii) *F is convex convex on $[0, 1]$.*

(iii) *The following hold:*

$$\sup_{\varphi\in[0,1]} F(\varphi) = F(1) = F(0) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \psi(\lambda)\, d\lambda$$

$$\inf_{\varphi\in[0,1]} F(\varphi) = F\left(\frac{1}{2}\right) = \frac{1}{(\tau_2-\tau_1)^2}\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \psi\left(\frac{\lambda+\mu}{2}\right) d\lambda d\mu.$$

(iv) *The following inequality is valid:*

$$\psi\left(\frac{\tau_1+\tau_2}{2}\right) \le F\left(\frac{1}{2}\right).$$

(v) *F increases monotonically on $\left[\frac{1}{2}, 1\right]$ and decreases monotonically on $\left[0, \frac{1}{2}\right]$.*

(vi) *We have the inequality $H(\varphi) \le F(\varphi)$ for all $\varphi \in [0, 1]$.*

Yang and Hong [21] provided an improvement between the middle and the right most term by defining the following mapping $P : [0, 1] \to \mathbb{R}$

$$P(\varphi) = \frac{1}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\left[\psi\left(\left(\frac{1+\varphi}{2}\right)\tau_2 + \left(\frac{1-\varphi}{2}\right)\lambda\right) + \psi\left(\left(\frac{1+\varphi}{2}\right)\tau_1 + \left(\frac{1-\varphi}{2}\right)\lambda\right)\right] d\lambda,$$

where $\psi : [\tau_1, \tau_2] \to \mathbb{R}$ is a convex function.

Theorem 1.3. [21] *Let $\psi : [\tau_1, \tau_2] \to \mathbb{R}$ be a convex function on $[\tau_1, \tau_2]$. Then*

(i) *P is convex convex on $[0, 1]$.*

(ii) *P increases monotonically on $[0, 1]$.*

(iii) *The following hold*

$$\inf_{\varphi\in[0,1]} P(\varphi) = P(0) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\psi(\lambda)\,d\lambda$$

and

$$\sup_{\varphi\in[0,1]} P(\varphi) = P(1) = \frac{\psi(\tau_1)+\psi(\tau_2)}{2}.$$

One of the generalizations of the convex functions is harmonic functions:

Definition 1.4. [14] *Define* $I \subseteq \mathbb{R}\setminus\{0\}$ *as an interval of real numbers. The function* $\psi : I \to \mathbb{R}$ *is considered to be harmonically convex, if*

$$\psi\left(\frac{\lambda\mu}{\varphi\lambda+(1-\varphi)\mu}\right) \le \varphi\psi(\mu) + (1-\varphi)\psi(\lambda) \tag{1. 2}$$

for all $\lambda,\mu \in I$ *and* $\varphi \in [0,1]$. *The functions* $\psi : I \to \mathbb{R}$ *is harmonically concave if the inequality in (1. 2) is reversed.*

Using harmonic-convexity, the Hermite-Hadamard type inequalities were proved by İşcan in [14].

Theorem 1.5. [14] *Let* $\psi : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be a harmonically convex function and* $\tau_1,\tau_2 \in I$ *with* $\tau_1 < \tau_2$. *If* $\psi \in L([\tau_1,\tau_2])$ *then the following inequalities hold:*

$$\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right) \le \frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda \le \frac{\psi(\tau_1)+\psi(\tau_2)}{2}. \tag{1. 3}$$

The above results provide us an inspiration to define some mappings similar to the above mappings for harmonically convex functions and to get some refinements between the middle, right most and the left most terms of (1. 3).

2. Main Results

We begin this section by defining the following mappings in connection to the Hermite-Hadamard type inequalities (1. 3):

For a given harmonically convex mapping $\psi : [\tau_1,\tau_2] \to \mathbb{R}$, let $S, U, V : [0,1] \to \mathbb{R}$ be defined by

$$S(\varphi) = \frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\psi\left(\frac{2\tau_1\tau_2\lambda}{2\tau_1\tau_2\varphi+(1-\varphi)\lambda(\tau_1+\tau_2)}\right)d\lambda,$$

$$U(\varphi) = \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right)d\lambda d\mu$$

and

$$V(\varphi) = \frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\left[\psi\left(\frac{2\tau_2\lambda}{(1+\varphi)\lambda+(1-\varphi)\tau_2}\right) + \psi\left(\frac{2\tau_1\lambda}{(1+\varphi)\lambda+(1-\varphi)\tau_1}\right)\right]d\lambda.$$

We sate some important facts which relate harmonically convex and convex functions and use them to prove the main results of this section.

Theorem 2.1. [6, 7] *If* $[\tau_1, \tau_2] \subset I \subset (0, \infty)$ *and if we consider the function* $g : \left[\frac{1}{\tau_2}, \frac{1}{\tau_1}\right] \to \mathbb{R}$ *defined by* $g(\varphi) = \psi\left(\frac{1}{\varphi}\right)$, *then* ψ *is harmonically convex on* $[\tau_1, \tau_2]$ *if and only if* g *is convex in the usual sense on* $\left[\frac{1}{\tau_2}, \frac{1}{\tau_1}\right]$.

Theorem 2.2. [6, 7] *If* $I \subset (0, \infty)$ *and* ψ *is a convex nondecreasing function then* ψ *is HA-convex (harmonically convex) and if* ψ *is a HA-convex (harmonically convex) nonincreasing function then* ψ *is convex.*

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. Let $w : \Omega \to \mathbb{R}$ be a μ-measurable function with $w(x) \geq 0$ for μ -a.e. (almost everywhere) $x \in \Omega$. Cconsider the Lebesgue space

$$L_w(\Omega, \mu) := \left\{\psi : \Omega \to \mathbb{R}, \psi \text{ is } \mu\text{-measurable and } \int_\Omega w(x)\,|\psi(x)|\,d\mu(x) < \infty\right\}.$$

The following result were proved in [6, 7].

Theorem 2.3. [6, 7] *Let* $\psi : I \subset (0, \infty) \to \mathbb{R}$ *be a HA-convex (harmonically convex) function and* $[d, D] \subset I^{\circ}$. *Assume also that* $\lambda : \Omega \to \mathbb{R}$ *satisfying the bounds*

$$0 < d \leq \lambda(\varphi) \leq D < \infty \text{ for } \mu\text{-a.e. } \varphi \in \Omega$$

and $w \geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w\,d\mu = 1$. *If* $\psi \circ \lambda$, $\frac{1}{\lambda} \in L_w(\Omega, \mu)$, *then*

$$\psi\left(\frac{1}{\int_\Omega \frac{w}{\lambda}\,d\mu}\right) \leq \int_\Omega (\psi \circ \lambda)\, w\,d\mu.$$

Related to the above mappings, we have the following results:

Theorem 2.4. *Let* $\psi : [\tau_1, \tau_2] \subseteq (0, \infty) \to \mathbb{R}$ *be a harmonically convex function on* $[\tau_1, \tau_2]$. *Then*

(i) S *is harmonically convex on* $(0, 1]$.

(ii) *The following hold:*

$$\inf_{\varphi\in[0,1]} S(\varphi) = S(0) = \psi\left(\frac{2\tau_1\tau_2}{\tau_1 + \tau_2}\right)$$

$$\sup_{\varphi\in[0,1]} S(\varphi) = S(1) = \frac{\tau_1\tau_2}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \frac{\psi(\lambda)}{\lambda^2}\,d\lambda.$$

(iii) S *increases monotonically on* $[0, 1]$.

Proof. (i) In order to prove that $S : [0, 1] \to \mathbb{R}$ is harmonically convex on $(0, 1]$, where $\psi : [\tau_1, \tau_2] \subseteq (0, \infty) \to \mathbb{R}$ is harmonically convex on $[\tau_1, \tau_2]$, by Theorem 2.1 it suffices to prove that the mapping $G : [0, 1] \to \mathbb{R}$ defined by

$$G(\varphi) = \frac{\tau_1\tau_2}{\tau_2 - \tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g\left(\varphi\lambda + (1 - \varphi)\left(\frac{\tau_1 + \tau_2}{2\tau_1\tau_2}\right)\right) d\lambda$$

is convex for convex function $g:\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]\to\mathbb{R}$ on $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$. Let $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and $\varphi_1,\varphi_2\in[0,1]$. Then

$$\begin{aligned}
&G\left(\alpha\varphi_1+\beta\varphi_2\right)\\
&=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g\left(\left(\alpha\varphi_1+\beta\varphi_2\right)\lambda+\left(1-\left(\alpha\varphi_1+\beta\varphi_2\right)\right)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)d\lambda\\
&=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g\left(\left(\alpha\varphi_1+\beta\varphi_2\right)\lambda+\left(\alpha+\beta-\left(\alpha\varphi_1+\beta\varphi_2\right)\right)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)d\lambda\\
&=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g\left(\alpha\left[\varphi_1\lambda+\left(1-\varphi_1\right)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right]+\beta\left[\varphi_2\lambda+\left(1-\varphi_2\right)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right]\right)d\lambda\\
&\leq\alpha G\left(\varphi_1\right)+\beta G\left(\varphi_2\right).
\end{aligned}$$

This proves that $S:[0,1]\to\mathbb{R}$ is harmonically convex on $(0,1]$.
(ii) By Jensen's inequality for harmonically convex functions (see Theorem 2.3), we obtain

$$S\left(\varphi\right)\geq\psi\left(\frac{1}{\int_{\tau_1}^{\tau_2}\frac{\tau_1\lambda+\tau_2\lambda+2\varphi\tau_1\tau_2-\varphi\tau_1\lambda-\varphi\tau_2\lambda}{2(\tau_2-\tau_1)\lambda^3}d\lambda}\right)=\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right).$$

Using the harmonically convexity of ψ on $[\tau_1,\tau_2]$, we get

$$\begin{aligned}
S\left(\varphi\right)&=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\psi\left(\frac{1}{\frac{1}{\lambda}\varphi+\left(1-\varphi\right)\frac{1}{\frac{2\tau_1\tau_2}{\tau_1+\tau_2}}}\right)d\lambda\\
&\leq\varphi\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi\left(\lambda\right)}{\lambda^2}d\lambda+\left(1-\varphi\right)\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right).
\end{aligned}$$

Since the mapping

$$p\left(\varphi\right):=\varphi\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi\left(\lambda\right)}{\lambda^2}d\lambda+\left(1-\varphi\right)\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)$$

increases monotonically on $[0,1]$, is proved that

$$\begin{aligned}
\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)\leq S\left(\varphi\right)&\leq\varphi\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi\left(\lambda\right)}{\lambda^2}d\lambda\\
&+\left(1-\varphi\right)\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)\leq\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi\left(\lambda\right)}{\lambda^2}d\lambda. \quad (2.\ 4)
\end{aligned}$$

Thus

$$\inf_{\varphi\in[0,1]}S\left(\varphi\right)=S\left(0\right)=\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)$$

and

$$\sup_{\varphi\in[0,1]}S\left(\varphi\right)=S\left(1\right)=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi\left(\lambda\right)}{\lambda^2}d\lambda.$$

(iii) To prove that S increases monotonically on $[0,1]$ is equivalent to prove that G increases monotonically on $[0,1]$. Since G is convex on $[0,1]$, so for $\varphi_1, \varphi_2 \in (0,1)$, we get

$$\frac{G(\varphi_2)-G(\varphi_1)}{\varphi_2-\varphi_1}$$
$$\geq G'_{+}(\varphi_1)=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g'_{+}\left(\varphi_1\lambda+(1-\varphi_1)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)\left(\lambda-\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)d\lambda.$$

The convexity of g on $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ yields

$$g\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)-g\left(\varphi_1\lambda+(1-\varphi_1)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)$$
$$\geq \varphi_1 g'_{+}\left(\varphi_1\lambda+(1-\varphi_1)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}-\lambda\right)$$

for every $\lambda\in\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$. Thus

$$\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g'_{+}\left(\varphi_1\lambda+(1-\varphi_1)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)\left(\lambda-\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)d\lambda$$
$$\geq \frac{1}{\varphi_1}\left[\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}} g\left(\varphi_1\lambda+(1-\varphi_1)\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right)d\lambda-g\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right]$$
$$=\frac{1}{\varphi_1}\left[G(\varphi_1)-g\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right]\geq 0.$$

This proves that $G(\varphi_2)-G(\varphi_1)\geq 0$ for $1\geq\varphi_2\geq\varphi_1$. Hence G is monotonically increasing on $[0,1]$ which implies that S is also monotonically increasing on $[0,1]$. □

Corollary 2.5. *Suppose that the assertions of Theorem 2.4 are satisfied, then*

$$0\leq\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\psi\left(\frac{4\tau_1\tau_2\lambda}{2\tau_1\tau_2+\lambda(\tau_1+\tau_2)}\right)d\lambda-\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)$$
$$\leq\frac{1}{2}\left[\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda-\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)\right]$$
$$\leq\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda-\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right). \quad (2.\ 5)$$

Proof. Taking $\varphi=\frac{1}{2}$ in (2. 4) and rearranging the terms, we get (2. 5). □

Corollary 2.6. *Let $p \le 0$ or $p \ge 1$, $p \ne -1, -3$ and $0 < \tau_1 \le \tau_2$. Then*

$$0 \le \frac{2^{-2p-5}\tau_1\tau_2}{(\tau_2-\tau_1)(p+3)}\left[\left(\frac{\tau_1\tau_2}{\tau_1+3\tau_2}\right)^{-p-3} - \left(\frac{\tau_1\tau_2}{3\tau_1+\tau_2}\right)^{-p-3}\right] - \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)^{-p}$$
$$\le \frac{1}{2}\left[\tau_1\tau_2\left(\frac{\tau_1^{-p-1}-\tau_2^{-p-1}}{(\tau_2-\tau_1)(p+1)}\right) - \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)^{-p}\right]$$
$$\le \tau_1\tau_2\left(\frac{\tau_1^{-p-1}-\tau_2^{-p-1}}{(\tau_2-\tau_1)(p+1)}\right) - \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)^{-p}. \quad (2.\ 6)$$

Proof. Let $\psi : [\tau_1, \tau_2] \subseteq (0, \infty) \to \mathbb{R}$ be defined by $\psi(\lambda) = \lambda^{-p}$, $p \le 0$ or $p \ge 1$ and $p \ne -1, -3$. Then the inequalities (2. 6) follows from (2. 5). □

Corollary 2.7. *Let $0 < \tau_1 \le \tau_2$. Then*

$$0 \le \frac{4\tau_1^2\tau_2^2}{((1-\varphi)\tau_2 - (1+\varphi)\tau_1)((1-\varphi)\tau_1 + (1+\varphi)\tau_2)} - \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)^2$$
$$\le \varphi\left[\tau_1\tau_2 + \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)^2\right] \le \tau_1\tau_2 - \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right)^2. \quad (2.\ 7)$$

Proof. The inequality (2. 7) follows from (2. 5) by considering the harmonically convex function $\psi : [\tau_1, \tau_2] \subseteq (0, \infty) \to \mathbb{R}$ be defined by $\psi(\lambda) = \lambda^2$. □

Theorem 2.8. *Let $\psi : [\tau_1, \tau_2] \subset (0, \infty) \to \mathbb{R}$ be a harmonically convex function on $[\tau_1, \tau_2]$. Then*

(i) *The following identities hold:*
$$U\left(\varphi + \frac{1}{2}\right) = U\left(\frac{1}{2} - \varphi\right) \text{ for all } \varphi \in \left[0, \frac{1}{2}\right].$$
(ii) *U is harmonically convex on $(0, 1]$.*
(iii) *The following identities hold:*
$$\inf_{\varphi\in[0,1]} U(\varphi) = U\left(\frac{1}{2}\right) = \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2 \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\psi\left(\frac{2\lambda\mu}{\lambda+\mu}\right) d\lambda d\mu$$
and
$$\sup_{\varphi\in[0,1]} U(\varphi) = U(0) = U(1) = \frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \frac{\psi(\lambda)}{\lambda^2} d\lambda.$$
(iv) *The following inequality is valid*
$$\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right) \le U\left(\frac{1}{2}\right).$$
(v) *U increases monotonically on $\left[\frac{1}{2}, 1\right]$ and decreases monotonically on $\left[0, \frac{1}{2}\right]$.*
(vi) *$S(\varphi) \le U(\varphi)$ for all $\varphi \in [0, 1]$.*

Proof. (i) Let $\varphi \in \left[0, \frac{1}{2}\right]$, then

$$\begin{aligned} U\left(\varphi+\frac{1}{2}\right) &= \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2 \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\left(\varphi+\frac{1}{2}\right)\mu+\left(\frac{1}{2}-\varphi\right)\lambda}\right) d\lambda d\mu \\ &= \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2 \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\left(\frac{1}{2}-\varphi\right)\mu+\left(\varphi+\frac{1}{2}\right)\lambda}\right) d\lambda d\mu \\ &= U\left(\frac{1}{2}-\varphi\right). \end{aligned}$$

(ii) By similar arguments as that of proving (i) of Theorem 2.4, we can prove that U is harmonically convex on $(0,1]$.
(iii) Since $\psi : [\tau_1,\tau_2] \subset (0,\infty) \to \mathbb{R}$ is a harmonically convex function on $[\tau_1,\tau_2]$, for all λ, $\mu \in [\tau_1,\tau_2]$ and $\varphi \in [0,1]$, we obtain

$$\begin{aligned} &\frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right) \\ &\leq \frac{1}{\lambda^2\mu^2}\left[\varphi\psi(\lambda)+(1-\varphi)\psi(\mu)\right] \\ &= \varphi\frac{1}{\lambda^2\mu^2}\psi(\lambda)+(1-\varphi)\frac{1}{\lambda^2\mu^2}\psi(\mu). \end{aligned}$$

Integrating this inequality over $[\tau_1,\tau_2]\times[\tau_1,\tau_2]$ we get

$$\begin{aligned} &\left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2 \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right) d\lambda d\mu \\ &\leq \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2 \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\left[\varphi\psi(\lambda)+(1-\varphi)\psi(\mu)\right] d\lambda d\mu \\ &= \frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda. \end{aligned}$$

Thus $U(\varphi) \leq U(0) = U(1)$ for all $\varphi \in [0,1]$.
Again the harmonic convexity of $\psi : [\tau_1,\tau_2] \subset (0,\infty) \to \mathbb{R}$ on $[\tau_1,\tau_2]$ gives the following inequality

$$\frac{1}{2}\left[\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right)+\psi\left(\frac{\lambda\mu}{(1-\varphi)\mu+\varphi\lambda}\right)\right] \geq \psi\left(\frac{2\lambda\mu}{\lambda+\mu}\right).$$

Integrating this inequality over $[\tau_1,\tau_2]\times[\tau_1,\tau_2]$ we get

$$\begin{aligned} &\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\psi\left(\frac{2\lambda\mu}{\lambda+\mu}\right) d\lambda d\mu \\ &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\left[\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right)+\psi\left(\frac{\lambda\mu}{(1-\varphi)\mu+\varphi\lambda}\right)\right] d\lambda d\mu \\ &= \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right) d\lambda d\mu \end{aligned}$$

That is $U(\varphi) \geq U\left(\frac{1}{2}\right)$ for all $\varphi \in [0,1]$ and hence the assertion is proven.
(iv) Using Jensen's inequality for double integrals for harmonically convex functions, we have

$$\begin{aligned}&\left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2\mu^2}\psi\left(\frac{2\lambda\mu}{\lambda+\mu}\right)d\lambda d\mu\\ &\geq \psi\left(\frac{1}{\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2}\frac{(\tau_1-\tau_2)^2(\lambda+\mu)}{2\tau_1^2\tau_2^2\lambda^3\mu^3}d\lambda d\mu}\right)=\psi\left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right).\end{aligned}$$

(v) In order to prove that U increases monotonically on $\left[\frac{1}{2},1\right]$ and decreases monotonically on $\left[0,\frac{1}{2}\right]$ it suffices to prove that $K:[0,1]\to\mathbb{R}$ defined by

$$K(\varphi)=\left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}h\left(\varphi\frac{1}{\lambda}+(1-\varphi)\frac{1}{\mu}\right)d\lambda d\mu$$

increases monotonically on $\left[\frac{1}{2},1\right]$ and decreases monotonically on $\left[0,\frac{1}{2}\right]$.
Since K is convex on $(0,1)$ we have for $\varphi_2>\varphi_1$, $\varphi_2,\varphi_1\in\left(\frac{1}{2},1\right)$,

$$\begin{aligned}\frac{K(\varphi_2)-K(\varphi_1)}{\varphi_2-\varphi_1}&\geq K_+^{'}(\varphi_1)\\ &\geq\left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}h^{'}\left(\varphi_1\frac{1}{\lambda}+(1-\varphi_1)\frac{1}{\mu}\right)\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)d\lambda d\mu.\end{aligned}$$

By the convexity of K on $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ we deduce that

$$\begin{aligned}&h\left(\frac{\lambda+\mu}{2\lambda\mu}\right)-h\left(\varphi_1\frac{1}{\lambda}+(1-\varphi_1)\frac{1}{\mu}\right)\\ &\geq h_+^{'}\left(\varphi_1\frac{1}{\lambda}+(1-\varphi_1)\frac{1}{\mu}\right)\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\frac{(1-2\varphi_1)}{2}\end{aligned}$$

for all $\frac{1}{\lambda},\frac{1}{\mu}$ in $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ and $\varphi\in\left(\frac{1}{2},1\right)$, which is equivalent to

$$\begin{aligned}&h_+^{'}\left(\varphi_1\frac{1}{\lambda}+(1-\varphi_1)\frac{1}{\mu}\right)\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\\ &\geq\frac{2}{2\varphi_1-1}\left[h\left(\varphi_1\frac{1}{\lambda}+(1-\varphi_1)\frac{1}{\mu}\right)-h\left(\frac{\lambda+\mu}{2\lambda\mu}\right)\right].\end{aligned}$$

Integrating on $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]\times\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ we obtain

$$U_+^{'}(\varphi_1)\geq\frac{2}{2\varphi_1-1}\left[U(\varphi_1)-U\left(\frac{1}{2}\right)\right]\geq 0$$

which shows that U increases monotonically on $\left[\frac{1}{2},1\right]$. The fact that U decreases monotonically on $\left[0,\frac{1}{2}\right]$ follows from the above conclusion and using the result

$$U\left(\varphi+\frac{1}{2}\right)=U\left(\frac{1}{2}-\varphi\right)\ \text{ for all } \varphi\in\left[0,\frac{1}{2}\right].$$

(vi) We observe that

$$S(\varphi)=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\psi\left(\frac{1}{\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\lambda-\varphi\lambda+\varphi\mu}{\lambda\mu^3}d\mu}\right)d\lambda.$$

Using Jensen's integral inequality (Theorem 2.3) we get that

$$S(\varphi)\le\left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)^2\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2\mu^2}\psi\left(\frac{\lambda\mu}{\varphi\mu+(1-\varphi)\lambda}\right)d\lambda d\mu=U(\varphi)$$

for all $\varphi\in[0,1]$. □

Theorem 2.9. *Let $\psi:[\tau_1,\tau_2]\to\mathbb{R}$ be a harmonically convex function on $[\tau_1,\tau_2]$. Then*

(i) *V is harmonically convex on $(0,1]$.*
(ii) *V increases monotonically on $[0,1]$.*
(iii) *The following hold:*

$$\inf_{\varphi\in[0,1]}V(\varphi)=V(0)=\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda$$

$$\sup_{\varphi\in[0,1]}V(\varphi)=V(1)=\frac{\psi(\tau_1)+\psi(\tau_2)}{2}.$$

Proof. (i) In order to prove that $V:[0,1]\to\mathbb{R}$ is harmonically convex on $(0,1]$, where $\psi:[\tau_1,\tau_2]\to\mathbb{R}$ is harmonically convex on $[\tau_1,\tau_2]$, it suffices to prove that the mapping $Q:[0,1]\to\mathbb{R}$ defined by

$$Q(\varphi)=\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\left[k\left(\left(\frac{1+\varphi}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi}{2}\right)\frac{1}{\lambda}\right)\right.$$
$$\left.+k\left(\left(\frac{1+\varphi}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi}{2}\right)\frac{1}{\lambda}\right)\right]d\lambda$$

is convex for convex function $k:\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]\to\mathbb{R}$ on $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ by Theorem 2.1. Let $\alpha,\beta\ge0$ with $\alpha+\beta=1$ and $\varphi_1,\varphi_2\in[0,1]$. Then

$$Q(\alpha\varphi_1+\beta\varphi_2)$$
$$=\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\left[k\left(\left(\frac{1+\alpha\varphi_1+\beta\varphi_2}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-(\alpha\varphi_1+\beta\varphi_2)}{2}\right)\frac{1}{\lambda}\right)\right.$$
$$\left.+k\left(\left(\frac{1+\alpha\varphi_1+\beta\varphi_2}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-(\alpha\varphi_1+\beta\varphi_2)}{2}\right)\frac{1}{\lambda}\right)\right]d\lambda=\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}$$
$$\times\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\left[k\left(\alpha\left[\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right]+\beta\left[\left(\frac{1+\varphi_2}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_2}{2}\right)\frac{1}{\lambda}\right]\right)\right.$$
$$\left.+k\left(\alpha\left[\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right]+\beta\left[\left(\frac{1+\varphi_2}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_2}{2}\right)\frac{1}{\lambda}\right]\right)\right]d\lambda$$
$$\le\alpha Q(\varphi_1)+\beta Q(\varphi_2).$$

This proves that $Q:[0,1]\to\mathbb{R}$ is harmonically convex on $(0,1]$.
(ii) Let $\varphi\in[0,1]$, then

$$\begin{aligned}
&\frac{Q(\varphi_2)-Q(\varphi_1)}{\varphi_2-\varphi_1}\\
&\geq Q_+^{'}(\varphi_1)\\
&=\frac{\tau_1\tau_2}{4(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\left[k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\tau_2}-\frac{1}{\lambda}\right)\right.\\
&\left.+k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\tau_1}-\frac{1}{\lambda}\right)\right]d\lambda.
\end{aligned}$$

By the convexity of k on $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ we deduce that

$$\begin{aligned}
&k\left(\frac{\tau_2+\lambda}{2\tau_2\lambda}\right)-k\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\\
&\geq\frac{\varphi_1}{2}k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}-\frac{1}{\tau_2}\right)
\end{aligned}$$

for all $\frac{1}{\lambda}$ in $\left[\frac{1}{\tau_2},\frac{1}{\tau_1}\right]$ and $\varphi_1\in(0,1)$, which is equivalent to

$$\begin{aligned}
&k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\tau_2}-\frac{1}{\lambda}\right)\\
&\geq\frac{2}{\varphi_1}\left[k\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)-k\left(\frac{\tau_2+\lambda}{2\tau_2\lambda}\right)\right].
\end{aligned}$$

Similarly, we also get that

$$\begin{aligned}
&k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\tau_1}-\frac{1}{\lambda}\right)\\
&\geq\frac{2}{\varphi_1}\left[k\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)-k\left(\frac{\tau_1+\lambda}{2\tau_1\lambda}\right)\right].
\end{aligned}$$

Thus

$$\frac{\tau_1\tau_2}{4(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\left[k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\tau_2}-\frac{1}{\lambda}\right)\right.$$
$$\left.+k_+^{'}\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\left(\frac{1}{\tau_1}-\frac{1}{\lambda}\right)\right]d\lambda$$
$$\geq\frac{2}{\varphi_1}\frac{\tau_1\tau_2}{4(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\left[k\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_2}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)\right.$$
$$\left.+k\left(\left(\frac{1+\varphi_1}{2}\right)\frac{1}{\tau_1}+\left(\frac{1-\varphi_1}{2}\right)\frac{1}{\lambda}\right)-k\left(\frac{\lambda+\tau_2}{2\lambda\tau_2}\right)-k\left(\frac{\tau_1+\lambda}{2\tau_1\lambda}\right)\right]d\lambda$$
$$=\frac{1}{\varphi_1}\left[Q(\varphi_1)-\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}k\left(\frac{\lambda+\tau_2}{2\lambda\tau_2}\right)d\lambda-\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}k\left(\frac{\tau_1+\lambda}{2\tau_1\lambda}\right)d\lambda\right]$$
$$\geq\frac{1}{\varphi_1}\left[Q(\varphi_1)-k\left(\frac{\tau_1+\tau_2}{2\tau_1\tau_2}\right)\right]\geq 0.\text{ (By Theorem 1.1)}$$

This shows that $Q(\varphi_2)-Q(\varphi_1)\geq 0$ for $1\geq\varphi_2\geq\varphi_1$. Hence Q is monotonically increasing on $[0,1]$ which implies that P is also monotonically increasing on $[0,1]$.
(iii) Since $V(\varphi)$ is monotonically increasing, we have

$$V(\varphi)\geq V(0)=\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\left[\psi\left(\frac{2\tau_2\lambda}{\lambda+\tau_2}\right)+\psi\left(\frac{2\tau_1\lambda}{\lambda+\tau_1}\right)\right]d\lambda$$
$$=\frac{\tau_1\tau_2}{(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{\tau_1+\tau_2}{2\tau_1\tau_2}}\psi(\lambda)\,d\lambda+\frac{\tau_1\tau_2}{(\tau_2-\tau_1)}\int_{\frac{\tau_1+\tau_2}{2\tau_1\tau_2}}^{\frac{1}{\tau_1}}\psi(\lambda)\,d\lambda=\frac{\tau_1\tau_2}{(\tau_2-\tau_1)}\int_{\frac{1}{\tau_2}}^{\frac{1}{\tau_1}}\psi(\lambda)\,d\lambda$$
$$=\frac{\tau_1\tau_2}{(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda.$$

Using the harmonic convexity of ψ on $[\tau_1,\tau_2]$ and Hermite-Hadamard type inequalities for harmonically convex functions, we get

$$V(\varphi)=\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\frac{1}{\lambda^2}\left[\psi\left(\frac{2\tau_2\lambda}{(1+\varphi)\lambda+(1-\varphi)\tau_2}\right)+\psi\left(\frac{2\tau_1\lambda}{(1+\varphi)\lambda+(1-\varphi)\tau_1}\right)\right]d\lambda$$
$$\leq\frac{\tau_1\tau_2}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\frac{1}{2\lambda^2}\left[(1+\varphi)\psi(\tau_2)+(1-\varphi)\psi(\lambda)+(1+\varphi)\psi(\tau_1)+(1-\varphi)\psi(\lambda)\right]d\lambda$$
$$=\left(\frac{1+\varphi}{2}\right)\frac{\psi(\tau_1)+\psi(\tau_2)}{2}+\left(\frac{1-\varphi}{2}\right)\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda\leq\frac{\psi(\tau_1)+\psi(\tau_2)}{2}.$$

Thus

$$\frac{\tau_1\tau_2}{(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda\leq V(\varphi)\leq\left(\frac{1+\varphi}{2}\right)\frac{\psi(\tau_1)+\psi(\tau_2)}{2}$$
$$+\left(\frac{1-\varphi}{2}\right)\frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda\leq\frac{\psi(\tau_1)+\psi(\tau_2)}{2}.$$

It is proved that

$$\inf_{\varphi\in[0,1]} V(\varphi) = V(0) = \frac{\tau_1\tau_2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\psi(\lambda)}{\lambda^2}d\lambda$$

and

$$\sup_{\varphi\in[0,1]} V(\varphi) = V(1) = \frac{\psi(\tau_1)+\psi(\tau_2)}{2}.$$

□

Corollary 2.10. *Let* $0<\tau_1\le\tau_2$. *Then*

$$\tau_1\tau_2 \le \frac{(1+\varphi)\tau_1\tau_2^3+(1+\varphi)\tau_1^3\tau_2+(1-\varphi)\tau_1^4+(1-\varphi)\tau_2^4}{((1+\varphi)\tau_1+(1-\varphi)\tau_2)((1-\varphi)\tau_1+(1+\varphi)\tau_2)}$$
$$\le\left(\frac{1+\varphi}{2}\right)\left(\frac{\tau_1^2+\tau_2^2}{2}\right)+\left(\frac{1-\varphi}{2}\right)\tau_1\tau_2\le\frac{\tau_1^2+\tau_2^2}{2}. \quad (2.\ 8)$$

Proof. The proof follows by choosing $\psi(\varphi)=\lambda^2$, $\lambda\in[\tau_1,\tau_2]\subseteq(0,\infty)$. □

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References

[1] H. Alzer, *A note on Hadamard's inequalities, C. R. Math. Rep. Acad. Sci. Canada*, **11** (1989), 255-258.

[2] S.S. Dragomir, *Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl.*, **167** (1992), 49–56.

[3] S.S. Dragomir, *Further properties of some mappings associated with Hermite-Hadamard inequalities, Tamkang. J. Math.*, **34** (1) (2003), 45–57.

[4] S.S. Dragomir, Y.J. Cho and S.S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications, J. Math. Anal. Appl.*, **245** (2000), 489–501.

[5] S.S. Dragomir, D.S. Milŏsević and J. Sándor, *On some refinements of Hadamard's inequalities and applications, Univ. Belgrad. Publ. Elek. Fak. Sci. Math.*, **4** (1993), 3–10.

[6] S. S. Dragomir, *Inequalities of Jensen type for HA-convex functions, Analele Universităţii Oradea Fasc. Matematica*, Tom **XXVII** (2020), Issue No. 1, 103-124.

[7] S. S. Dragomir, *Inequalities of Hermite-Hadamard Type for HA-Convex Functions, Moroccan J. of Pure and Appl. Anal.*, **3** (1) (2017), 83-101.

[8] S. S. Dragomir, *On Hadamard's inequality for convex functions, Mat. Balkanica*, **6** (1992), 215-222.

[9] S. S. Dragomir, *On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, Math. Ineq. and Appl.*, **3** (2000), 177-187.

[10] S. S. Dragomir, *On Hadamard's inequality on a disk, Journal of Ineq. in Pure and Appl. Math.*, **1**(1) (2000), Article 2.

[11] S. S. Dragomir, *On some integral inequalities for convex functions, Zb.-Rad. (Kragujevac)* (1996), 21-25.

[12] S. S. Dragomir and R. P. Agarwal, *Two new mappings associated with Hadamard's inequalities for convex functions, Appl. Math. Lett.*, **11** (1998), 33-38.

[13] J. Hadamard, *Étude sur les propriétés des fonctions entières en particulier d'une function considérée par Riemann J. Math. Pures and Appl.*, **58** (1893), 171-215.

[14] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics*, **43** (6) (2014), 935-942.

[15] M. A. Latif, Some companions of Fejér type inequalities for harmonically convex functions, Symmetry 2022, 14, 2268. https://doi.org/10.3390/sym14112268

[16] M. A. Latif, *Fejér type inequalities for harmonically convex functions, AIMS Mathematics,* **7,** No. 8 1523415257.

[17] M. A. Latif, S. S. Dragomir, E. Momoniat, *Fejér type inequalities for harmonically-convex functions with applications,* Journal of Applied Analysis & Computation, **7, No. 3** (2017) 795-813. doi: 10.11948/2017050.

[18] T. Sitthiwirattham, M. A. Ali and Hüseyin Budak, S. K. Ntouyas and C. Promsakon, *Fractional Ostrowski type inequalities for differentiable harmonically convex functions, AIMS Mathematics*, **7,** No.3 3939–3958.

[19] X. You, M. A. Ali , H. Budak, J. Reunsumrit and Thanin Sitthiwirattham, *Hermite–Hadamard–Mercer-type inequalities for harmonically convex mappings, Mathematics*, **9**, (2021) 2556.

[20] X.X. You, M. A. Ali, H. Budak, P. Agarwal and Yu-M. Chu, *Extensions of Hermite–Hadamard inequalities for harmonically convex functions via generalized fractional integrals, Journal of Inequalities and Applications*, (2021) 2021:102.

[21] G.S. Yang and M.C. Hong, *A note on Hadamard's inequality, Tamkang. J. Math.*, **28,** No. 1 (1997) 33–37.

[22] G.S. Yang and K.L. Tseng, *On certain integral inequalities related to Hermite-Hadamard inequalities, J. Math. Anal. Appl.*, **239** (1999) 180–187.

[23] G.S. Yang and K.L. Tseng, *Inequalities of Hadamard's type for Lipschitzian mappings, J. Math. Anal. Appl.*, **260** (2001) 230–238.

[24] G.S. Yang and K.L. Tseng, *On certain multiple integral inequalities related to Hermite- Hadamard inequalities, Utilitas Math.*, **62** (2002), 131–142.

[25] D. Zhao, M. A. Ali, A.Kashuri and H. Budak, *Generalized fractional integral inequalities of Hermite–Hadamard type for harmonically convex functions*, *Advances in Difference Equations* (2020) 2020:137.