Punjab University Journal of Mathematics (2022),54(12),702-711
https://doi.org/10.52280/pujm.2022.541201

Translation Hypersurfaces in Euclidean 4-Spaces

Ipek AKKILIN
Department of Mathematics,
Yildiz Technical University, Trkiye,
Email: ipekakkilinc@gmail.com

Salim YUCE
Department of Mathematics,
Yildiz Technical University, Trkiye,
Email: sayuce@yildiz.edu.tr

Received: 17 November, 2022 / Accepted: 23 December, 2022 / Published online: 28 December, 2022

Abstract.: In this article, the translation hypersurfaces in Euclidean 4-space are defined as the sum of three curves with distinct parameters with unit speed, and non-planar. These curves are called the generator curves of the hypersurface. Utilizing the hypersurface theory in Euclidean 4-space, unit normal vector field, shape (Weingarten) operator matrix, fundamental forms, Gaussian curvature and mean curvature have been expressed for the translation hypersurfaces. Finally, the computational example is stated to efficiency of the theoretical results.

AMS (MOS) Subject Classification Codes: 53A35
Key Words: Translation hypersurface, Euclidean 4-space, shape operator, fundamental form, Gaussian curvature.

1. Introduction

The theory of surfaces is the most attractive branch of differential geometry dealing with certain characteristic properties of surfaces. In the literature, much attention has been given to several types of surfaces, such as canal surface, ruled surface, rotation surface etc. to determine their internal or exterior features. Translation surfaces (Scherk surfaces) has become the focus of this attention in recent years. Translation surfaces in Euclidean 3-space were first introduced by Heinrich Ferdinand Scherk in 1835, [17]. Translation surfaces in $\mathbb{E}^3$ are defined by the following immersion

$$\phi : \mathbb{E}^2 \to \mathbb{E}^3 : (u,v) \to \phi(u,v) = \alpha(u) + \beta(v),$$

where $\alpha(u)$ and $\beta(v)$ are unit speed, non-planar curves. The idea of investigating the translation surfaces by considering them from various perspectives in different dimensional spaces is a remarkable area for the geometers, [21, 7]. First of all, the translation surfaces in 3-dimensional Euclidean space were discussed [10, 5, 1], then these studies were generalized and carried up to 4-dimensional and n-dimensional [7, 19, 20]. In addition, there are many studies in non-Euclidean Geometry (especially Lorentz-Minkowski and Galilean spaces) [25, 24, 3, 2, 4]. We highlight some papers that make an important contribution to the study of translation surfaces, more precisely, on minimal translation surfaces [13, 6, 16, 15], affine translation surfaces [18, 14, 12], Weingarten translation surfaces [8, 23] and constant curvature translation surfaces [2, 11]. Nowadays, translation surfaces are generally used for design purposes in the architectural field. One of the best examples of this is the design of the glass ceiling of the bank named DZ Bank in Berlin, Germany. In addition, cylinder, elliptical paraboloid, egg box surface and helicoid surface are some examples of translation surfaces.

Is it possible to define translation hypersurfaces as the sum of three curves with distinct parameters with unit speed, and non-planar? The answer to this question is given in this study. With this purpose, the paper is organized as 3 main sections and conclusions section. Section 2 is devoted to the general information about hypersurfaces in Euclidean 4-space and its associated objects such as the unit normal vector field, shape operator matrix, fundamental forms, Gaussian curvature and mean curvature. In Section 3, the original concept and its theoretical results are stated. The efficiency of the theoretical results is supported by a computational example.

2. Preliminaries

In this section, we recall some basic notations and the related results in [26].

Let $x = \sum\limits_{i=1}^{4} x_i e_i$, $y = \sum\limits_{i=1}^{4} y_i e_i$, $z = \sum\limits_{i=1}^{4} z_i e_i$ be vectors in $\mathbb{R}^4$, equipped with the standard inner product given by $\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. The norm of a vector $x \in \mathbb{R}^4$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$. The vector product (or the ternary product or cross product) of the vectors $x, y, z \in \mathbb{R}^4$ is defined by

$$x \otimes y \otimes z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}. \tag{2. 1}$$

(see details in [22]).

Let M^3 be a hypersurface in Euclidean 4-space whose parametrization is given by

$$\begin{aligned} &\phi : U \subset \mathbb{R}^3 \to \mathbb{E}^4 \\ &\quad (u, v, w) \to \phi(u, v, w) = (\phi_1(u, v, w), \phi_2(u, v, w), \phi_3(u, v, w), \phi_4(u, v, w)), \end{aligned}$$

where ϕ_i, $1 \leq i \leq 4$ are differentiable real valued functions defined on $U \subset \mathbb{R}^3$. $\phi(U) = M^3 \subset \mathbb{E}^4$ is a hypersurface if and only if system $\{\phi_u, \phi_v, \phi_w\}$ is linearly independent, where the partial derivatives ϕ_u, ϕ_v and ϕ_w can be expressed as $\phi_u = \frac{\partial \phi}{\partial u}$,

$\phi_v = \frac{\partial\phi}{\partial v}$ and $\phi_w = \frac{\partial\phi}{\partial w}$. The unit normal vector field N of the hypersurface M^3 given by the parametric equation is expressed by

$$N(u,v,w) = \frac{\phi_u \otimes \phi_v \otimes \phi_w}{\|\phi_u \otimes \phi_v \otimes \phi_w\|}. \tag{2. 2}$$

The first fundamental form I on the space of vector fields $\chi(M^3)$ is defined by the function

$$\begin{aligned} I : \chi(M^3) \times \chi(M^3) &\to C^\infty(M^3, \mathbb{R}) \\ (X,Y) &\mapsto I(X,Y) = \langle X, Y \rangle \end{aligned}$$

and also referred to as

$$I = \phi_{11}du^2 + \phi_{22}dv^2 + \phi_{33}dw^2 + 2\left(\phi_{12}dudv + \phi_{13}dudw + \phi_{23}dvdw\right), \tag{2. 3}$$

where ϕ_{ij}, $1 \leq i,j \leq 3$ are the coefficients of the first fundamental form. Hence, the matrix **I** corresponding to the first fundamental form coefficients of the hypersurface M^3 is expressed as

$$\mathbf{I} = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{pmatrix}. \tag{2. 4}$$

The shape operator of the hypersurface M^3 is defined by

$$\begin{aligned} S : \chi\left(M^3\right) &\to \chi\left(M^3\right) \\ X &\mapsto S\left(X\right) = D_X N, \end{aligned}$$

where D is the Riemannian connection in $\mathbb{E}^4$ and N is the unit normal of M^3. By using the definition of the shape operator, the below equalities can be written

$$\begin{cases} S\left(\phi_u\right) & = D_{\phi_u}N = N_u, \\ S\left(\phi_v\right) & = D_{\phi_v}N = N_v, \\ S\left(\phi_w\right) & = D_{\phi_w}N = N_w. \end{cases}$$

For any curve α on the hypersurface M^3, we can write $\langle \alpha'', N \rangle = -\left\langle S\left(\alpha'\right), \alpha' \right\rangle$ with the help of these equalities.

The second fundamental form II of the hypersurface M^3 is expressed as follows:

$$\begin{aligned} II : \chi(M^3) \times \chi(M^3) &\to C^\infty(M^3, \mathbb{R}) \\ (X,Y) &\mapsto II(X,Y) = \langle S(X), Y \rangle, \end{aligned}$$

and also referred to as

$$II = \varphi_{11}du^2 + \varphi_{22}dv^2 + \varphi_{33}dw^2 + 2\left(\varphi_{12}dudv + \varphi_{13}dudw + \varphi_{23}dvdw\right), \tag{2. 5}$$

where φ_{ij}, $1 \leq i,j \leq 3$ are the coefficients of the second fundamental form. Similar with the matrix **I**, the matrix **II** corresponding to the second fundamental form coefficients of the hypersurface M^3 is calculated as

$$\mathbf{II} = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{12} & \varphi_{22} & \varphi_{23} \\ \varphi_{13} & \varphi_{23} & \varphi_{33} \end{pmatrix}. \tag{2. 6}$$

Taking into account all of these, the shape operator matrix S of the hypersurface M^3 can be presented as $S = \mathbf{I}^{-1}\mathbf{II}$ using equations (2. 4) and (2. 6).

The Gaussian curvature K of hypersurface M^3 is given by

$$\begin{aligned} K : M^3 &\to \mathbb{R} \\ P &\mapsto K(P) = \det S(P) \end{aligned} \tag{2. 7}$$

and can be calculated as $K = \det S = \dfrac{\det \mathbf{II}}{\det \mathbf{I}}$ using equations (2. 4) and (2. 6).

The mean curvature H of hypersurface M^3 is defined by

$$\begin{aligned} H : M^3 &\to \mathbb{R} \\ P &\mapsto H(P) = \frac{1}{3}\mathrm{Tr}\, S(P). \end{aligned} \tag{2. 8}$$

For a tangent vector $\vec{0} \neq X_P$, if $\langle S(X_P), X_P \rangle = 0$, then the direction X_P is called the asymptotic direction of the hypersurface M^3 at the point P.

A curve, whose tangent vector at each point is asymptotic, is called an asymptotic line on the hypersurface M^3. A curve α on the hypersurface M^3 is an asymptotic line if and only if

$$\langle \alpha'', N \rangle = 0. \tag{2. 9}$$

If the mean curvature H of the hypersurface M^3 is equal to zero, $H = 0$, then the hypersurface M^3 is called minimal hypersurface.

3. Translation Hypersurfaces Generated by Unit Speed, Non-planar Curves in Euclidean 4-Space

In this original section, the definition of the 3-parametric translation surface in Euclidean 4-space is given, and the definitions of unit normal vector field, shape operator matrix, fundamental forms, Gaussian curvature and mean curvature are mentioned.

Definition 3.1. *A hypersurface M^3 produced by non-planar curves with unit speed*

$$\begin{cases} \alpha(u) = & (\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)) \\ \beta(v) = & (\beta_1(v), \beta_2(v), \beta_3(v), \beta_4(v)) \\ \gamma(w) = & (\gamma_1(w), \gamma_2(w), \gamma_3(w), \gamma_4(w)) \end{cases}$$

is called a translation hypersurface in Euclidean 4-space. Thus, a translation hypersurface is given by an immersion

$$\begin{aligned} \phi : \mathbb{E}^3 &\to \mathbb{E}^4 \\ (u,v,w) &\mapsto \phi(u,v,w) = \alpha(u) + \beta(v) + \gamma(w) \\ &= (\alpha_1(u) + \beta_1(v) + \gamma_1(w), \alpha_2(u) + \beta_2(v) + \gamma_2(w), \\ &\quad \alpha_3(u) + \beta_3(v) + \gamma_3(w), \alpha_4(u) + \beta_4(v) + \gamma_4(w)) \\ &= (\phi_1(u,v,w), \phi_2(u,v,w), \phi_3(u,v,w), \phi_4(u,v,w)). \end{aligned}$$

By taking into account the basic recallments given in previous section let us examine the properties for the translation hypersurface given by above immersion.

The tangent space of hypersurface M^3 is spanned by the vector fields

$$\phi_u = \alpha^{'}(u) = T_\alpha, \quad \phi_v = \beta^{'}(v) = T_\beta, \quad \phi_w = \gamma^{'}(w) = T_\gamma. \tag{3. 10}$$

The unit normal vector field of hypersurface M^3 using equalities (3. 10) and (2. 2) is obtained as

$$N = \frac{T_\alpha \otimes T_\beta \otimes T_\gamma}{\|T_\alpha \otimes T_\beta \otimes T_\gamma\|}.$$

By means of the properties of the vector product in $\mathbb{E}^4$, the square of the denominator is calculated as follows:

$$\begin{aligned} \Delta &= \|T_\alpha \otimes T_\beta \otimes T_\gamma\|^2 \\ &= \begin{vmatrix} \langle T_\alpha, T_\alpha\rangle & \langle T_\alpha, T_\beta\rangle & \langle T_\alpha, T_\gamma\rangle \\ \langle T_\beta, T_\alpha\rangle & \langle T_\beta, T_\beta\rangle & \langle T_\beta, T_\gamma\rangle \\ \langle T_\gamma, T_\alpha\rangle & \langle T_\gamma, T_\beta\rangle & \langle T_\gamma, T_\gamma\rangle \end{vmatrix} = \begin{vmatrix} 1 & \cos\sigma_{\alpha\beta} & \cos\sigma_{\alpha\gamma} \\ \cos\sigma_{\alpha\beta} & 1 & \cos\sigma_{\beta\gamma} \\ \cos\sigma_{\alpha\gamma} & \cos\sigma_{\beta\gamma} & 1 \end{vmatrix} \\ &= 1 - \cos^2\sigma_{\alpha\beta} - \cos^2\sigma_{\alpha\gamma} - \cos^2\sigma_{\beta\gamma} + 2\cos\sigma_{\alpha\beta}\cos\sigma_{\alpha\gamma}\cos\sigma_{\beta\gamma}, \end{aligned}$$

where $\sigma_{\alpha\beta}$ is the angle between T_α and T_β, $\sigma_{\alpha\gamma}$ is the angle between T_α and T_γ and $\sigma_{\beta\gamma}$ is the angle between T_β and T_γ. Hence, the unit normal vector field of hypersurface M^3 is

$$N = \frac{1}{\sqrt{\Delta}}\left(T_\alpha \otimes T_\beta \otimes T_\gamma\right).$$

The coefficients of the first fundamental form of the hypersurface M^3 are obtained as

$$\begin{aligned} \phi_{11} &= \langle\phi_u, \phi_u\rangle = \langle T_\alpha, T_\alpha\rangle = 1, \\ \phi_{22} &= \langle\phi_v, \phi_v\rangle = \quad T_\beta, T_\beta \quad = 1, \\ \phi_{33} &= \langle\phi_w, \phi_w\rangle = \langle T_\gamma, T_\gamma\rangle = 1, \\ \phi_{12} &= \langle\phi_u, \phi_v\rangle = \langle\phi_v, \phi_u\rangle = \phi_{21} = \quad T_\alpha, T_\beta \quad = \quad T_\alpha \quad\quad T_\beta \quad \cos\sigma_{\alpha\beta} = \cos\sigma_{\alpha\beta}, \\ \phi_{13} &= \langle\phi_u, \phi_w\rangle = \langle\phi_w, \phi_u\rangle = \phi_{31} = \langle T_\alpha, T_\gamma\rangle = \quad T_\alpha \quad \|T_\gamma\|\cos\sigma_{\alpha\gamma} = \cos\sigma_{\alpha\gamma}, \\ \phi_{23} &= \langle\phi_v, \phi_w\rangle = \langle\phi_w, \phi_v\rangle = \phi_{32} = \quad T_\beta, T_\gamma \quad = \quad T_\beta \quad \|T_\gamma\|\cos\sigma_{\beta\gamma} = \cos\sigma_{\beta\gamma}. \end{aligned} \tag{3. 11}$$

Then if the equalities (3. 11) are substituted in the equation (2. 3),

$$I = du^2 + dv^2 + dw^2 + 2\left(\cos\sigma_{\alpha\beta}dudv + \cos\sigma_{\alpha\gamma}dudw + \cos\sigma_{\beta\gamma}dvdw\right)$$

is found.

The coefficients of the second fundamental form of the hypersurface M^3 are given by

$$\begin{aligned} \varphi_{11} &= -\langle N, \phi_{uu}\rangle = -\langle N, k_{1\alpha}N_\alpha\rangle = -k_{1\alpha}\langle N, N_\alpha\rangle = -k_{1\alpha}\cos\theta_\alpha, \\ \varphi_{22} &= -\langle N, \phi_{vv}\rangle = -\langle N, k_{1\beta}N_\beta\rangle = -k_{1\beta}\langle N, N_\beta\rangle = -k_{1\beta}\cos\theta_\beta, \\ \varphi_{33} &= -\langle N, \phi_{ww}\rangle = -\langle N, k_{1\gamma}N_\gamma\rangle = -k_{1\gamma}\langle N, N_\gamma\rangle = -k_{1\gamma}\cos\theta_\gamma, \\ \varphi_{12} &= -\langle N, \phi_{uv}\rangle = -\langle N, 0\rangle = 0, \\ \varphi_{13} &= -\langle N, \phi_{uw}\rangle = -\langle N, 0\rangle = 0, \\ \varphi_{23} &= -\langle N, \phi_{vw}\rangle = -\langle N, 0\rangle = 0, \end{aligned} \tag{3. 12}$$

where θ_α is the angle between N and N_α, θ_β is the angle between N and N_β and θ_γ is the angle between N and N_γ. Then if the equalities (3. 12) are substituted in the equation (2. 5), we obtain

$$II = -k_{1\alpha}\cos\theta_\alpha du^2 - k_{1\beta}\cos\theta_\beta dv^2 - k_{1\gamma}\cos\theta_\gamma dw^2.$$

The matrix **I** corresponding to the second fundamental form coefficients of the hypersurface M^3 using equation (2. 4) is obtained as

$$\mathbf{I} = \begin{pmatrix} 1 & \cos\sigma_{\alpha\beta} & \cos\sigma_{\alpha\gamma} \\ \cos\sigma_{\alpha\beta} & 1 & \cos\sigma_{\beta\gamma} \\ \cos\sigma_{\alpha\gamma} & \cos\sigma_{\beta\gamma} & 1 \end{pmatrix}.$$

The inverse of the matrix **I** is given by

$$\mathbf{I}^{-1} = \frac{1}{\Delta}$$
$$\begin{pmatrix} \sin^2\sigma_{\beta\gamma} & -\cos\sigma_{\alpha\beta} + \cos\sigma_{\alpha\gamma}\cos\sigma_{\beta\gamma} & -\cos\sigma_{\alpha\gamma} + \cos\sigma_{\alpha\beta}\cos\sigma_{\beta\gamma} \\ -\cos\sigma_{\alpha\beta} + \cos\sigma_{\alpha\gamma}\cos\sigma_{\beta\gamma} & \sin^2\sigma_{\alpha\gamma} & -\cos\sigma_{\beta\gamma} + \cos\sigma_{\alpha\beta}\cos\sigma_{\alpha\gamma} \\ -\cos\sigma_{\alpha\gamma} + \cos\sigma_{\alpha\beta}\cos\sigma_{\beta\gamma} & -\cos\sigma_{\beta\gamma} + \cos\sigma_{\alpha\beta}\cos\sigma_{\alpha\gamma} & \sin^2\sigma_{\alpha\beta} \end{pmatrix}.$$

By utilizing equation (2. 6), the matrix **II**, which corresponds to the second fundamental form coefficients, is computed as

$$\mathbf{II} = \begin{pmatrix} -k_{1\alpha}\cos\theta_\alpha & 0 & 0 \\ 0 & -k_{1\beta}\cos\theta_\beta & 0 \\ 0 & 0 & -k_{1\gamma}\cos\theta_\gamma \end{pmatrix}.$$

As a consequence, using the matrix **II** and the inverse of matrix **I**, the shape operator S of the hypersurface M^3 is written as

$$S = \frac{1}{\Delta}\begin{pmatrix} -k_{1\alpha}\cos\theta_\alpha\sin^2\sigma_{\beta\gamma} & k_{1\beta}\cos\theta_\beta A & -k_{1\gamma}\cos\theta_\gamma B \\ k_{1\alpha}\cos\theta_\alpha A & -k_{1\beta}\cos\theta_\beta\sin^2\sigma_{\alpha\gamma} & k_{1\gamma}\cos\theta_\gamma C \\ -k_{1\alpha}\cos\theta_\alpha B & k_{1\beta}\cos\theta_\beta C & -k_{1\gamma}\cos\theta_\gamma\sin^2\sigma_{\alpha\beta} \end{pmatrix},$$

where

$$\begin{cases} A &= \cos\sigma_{\alpha\beta} - \cos\sigma_{\alpha\gamma}\cos\sigma_{\beta\gamma}, \\ B &= \cos\sigma_{\alpha\beta}\cos\sigma_{\beta\gamma} - \cos\sigma_{\alpha\gamma}, \\ C &= \cos\sigma_{\beta\gamma} - \cos\sigma_{\alpha\gamma}\cos\sigma_{\alpha\beta}. \end{cases}$$

The Gaussian curvature K of the hypersurface M^3 is calculated as

$$\begin{aligned} K &= \frac{1}{\Delta^3}\begin{vmatrix} -k_{1\alpha}\cos\theta_\alpha\sin^2\sigma_{\beta\gamma} & k_{1\beta}\cos\theta_\beta A & -k_{1\gamma}\cos\theta_\gamma B \\ k_{1\alpha}\cos\theta_\alpha A & -k_{1\beta}\cos\theta_\beta\sin^2\sigma_{\alpha\gamma} & k_{1\gamma}\cos\theta_\gamma C \\ -k_{1\alpha}\cos\theta_\alpha B & k_{1\beta}\cos\theta_\beta C & -k_{1\gamma}\cos\theta_\gamma\sin^2\sigma_{\alpha\beta} \end{vmatrix} \\ &= \frac{1}{\Delta^3}\big[k_{1\alpha}\cos\theta_\alpha k_{1\beta}\cos\theta_\beta k_{1\gamma}\cos\theta_\gamma(A^2\sin^2\sigma_{\alpha\beta} + B^2\sin^2\sigma_{\alpha\gamma} \\ &\quad + C^2\sin^2\sigma_{\beta\gamma} - 2ABC - \sin^2\sigma_{\alpha\beta}\sin^2\sigma_{\alpha\gamma}\sin^2\sigma_{\beta\gamma})\big], \end{aligned}$$

by using equation (2. 7). Additionally, by using the equation (2. 8), the mean curvature H is calculated as follows:

$$H = -\frac{1}{3\Delta}\left[k_{1\alpha}\cos\theta_\alpha \sin^2\sigma_{\beta\gamma} + k_{1\beta}\cos\theta_\beta \sin^2\sigma_{\alpha\gamma} + k_{1\gamma}\cos\theta\gamma \sin^2\sigma_{\alpha\beta}\right]. \tag{3. 13}$$

In the light of the above calculations, the following theorem can be given.

Theorem 3.2. *Let M^3 be a translation hypersurface in $\mathbb{E}^4$. Let any two of the generator curves α, β and γ be asymptotic lines , the first curvatures of these curves be non-zero and the tangents of the generator curves do not be in the same direction. Moreover, take the angle between T_α and T_β for $k = 0, 1, 2, ...$ be $\sigma_{\alpha\beta} \neq k\pi$. Under these circumstances, the necessary and sufficient condition for the translation hypersurface to be minimal is that the third generator curve is also an asymptotic line.*

Proof. $(\Rightarrow)$: Let α, β be asymptotic lines and M^3 be minimal translation hypersurface. Let's prove that curve γ is also an asymptotic line:
Since M^3 is a minimal translation hypersurface, one can see that

$$H = -\frac{1}{3\Delta}\left[k_{1\alpha}\cos\theta_\alpha \sin^2\sigma_{\beta\gamma} + k_{1\beta}\cos\theta_\beta \sin^2\sigma_{\alpha\gamma} + k_{1\gamma}\cos\theta\gamma \sin^2\sigma_{\alpha\beta}\right] = 0.$$

Hence,

$$k_{1\alpha}\cos\theta_\alpha \sin^2\sigma_{\beta\gamma} + k_{1\beta}\cos\theta_\beta \sin^2\sigma_{\alpha\gamma} + k_{1\gamma}\cos\theta\gamma \sin^2\sigma_{\alpha\beta} = 0 \tag{3. 14}$$

is concluded. Since the first curvatures of the generator curves are non-zero, and α and β are asymptotic lines, we obtain that

$$k_{1\alpha} \neq 0, \quad k_{1\beta} \neq 0, \quad k_{1\gamma} \neq 0.$$

According to equation (2. 9), we have that $\langle \alpha'', N\rangle = 0$ and $\langle \beta'', N\rangle = 0$. Therewith,

$$\alpha'', N \;= \left\langle T_\alpha^{'}, N\right\rangle = \langle k_{1\alpha}N_\alpha, N\rangle = k_{1\alpha}\langle N_\alpha, N\rangle = k_{1\alpha}\|N_\alpha\|\,\|N\|\cos\theta_\alpha = k_{1\alpha}\cos\theta_\alpha = 0$$

is obtained. As $k_{1\alpha} \neq 0$, it is clear that $\cos\theta_\alpha = 0$. If the same process is applied for the curve β, then $\cos\theta_\beta = 0$ is calculated. If the found values are replaced in equation (3. 14), $k_{1\gamma}\cos\theta\gamma\sin^2\sigma_{\alpha\beta} = 0$ is obtained. Since $k_{1\gamma} \neq 0$ and the angle between T_α and T_β is $\sigma_{\alpha\beta} \neq k\pi$ for $k = 0, 1, 2, ...$, $\cos\theta_\gamma = 0$, that is $\langle \gamma'', N\rangle = 0$ is obtained. Therefore, according to equation (2. 9), the curve γ is asymptotic line on the hypersurface M^3.

$(\Leftarrow)$: Let α, β and γ be asymptotic lines. Let's prove that the hypersurface M^3 is minimal translation hypersurface:
Since α, β and γ are asymptotic lines according to equation (2. 9) the followings can be written

$$\langle \alpha'', N\rangle = \cos\theta_\alpha = 0, \quad \langle \beta'', N\rangle = \cos\theta_\beta = 0, \quad \langle \gamma'', N\rangle = \cos\theta_\gamma = 0.$$

If the calculated values are replaced in equation (3. 13), then

$$H = -\frac{1}{3\Delta}\left[k_{1\alpha}\cos\theta_\alpha \sin^2\sigma_{\beta\gamma} + k_{1\beta}\cos\theta_\beta \sin^2\sigma_{\alpha\gamma} + k_{1\gamma}\cos\theta\gamma \sin^2\sigma_{\alpha\beta}\right] = 0$$

is obtained. As a result, the hypersurface M^3 is minimal translation hypersurface. □

The following computational example demonstrates the above results.

Example 3.3. *Consider the generator curves given by*

$$\begin{cases} \alpha(u) &= \left(\frac{\sqrt{6}u}{3}-1, \frac{\sqrt{2}u}{3}, \cos\frac{u}{3}, \sin\frac{u}{3}\right), \\ \beta(v) &= \left(\cos\frac{v}{2}, \sin\frac{v}{2}, \frac{\sqrt{2}v}{2}+1, \frac{v}{2}\right), \\ \gamma(w) &= \left(\cos\frac{w}{\sqrt{2}}, \sin\frac{w}{\sqrt{2}}+1, \frac{w}{2}, \frac{w-1}{2}\right), \end{cases}$$

and the translation hypersurface

$$\phi(u, v, w) = \alpha(u) + \beta(v) + \gamma(w) \tag{3. 15}$$

in Euclidean 4*-space. The generator curves* α*,* β *and* γ *have unit speed and are non-planar, since the second curvatures (see in* [9] *)* $k_{2\alpha} = \frac{2\sqrt{2}}{9}$*,* $k_{2\beta} = \frac{3}{4\sqrt{3}}$ *and* $k_{2\gamma} = \frac{1}{2}$ *of these curves are non-zero, respectively. Note that the curves are not asymptotic due to* $\langle\alpha'', N\rangle \neq 0$*,* $\langle\beta'', N\rangle \neq 0$ *and* $\langle\gamma'', N\rangle \neq 0$*. The tangent vectors of the hypersurface* M^3 *are obtained as*

$$\begin{cases} \phi_u = \alpha'(u) &= T_\alpha = \left(\frac{\sqrt{6}}{3}, \frac{\sqrt{2}}{3}, -\frac{1}{3}\sin\frac{u}{3}, \frac{1}{3}\cos\frac{u}{3}\right), \\ \phi_v = \beta'(v) &= T_\beta = \left(-\frac{1}{2}\sin\frac{v}{2}, \frac{1}{2}\cos\frac{v}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right), \\ \phi_w = \gamma'(w) &= T_\gamma = \left(-\frac{1}{\sqrt{2}}\sin\frac{w}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{w}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right), \end{cases} \tag{3. 16}$$

and at point $P = \phi\left(\frac{9\pi}{2}, \pi, 0\right)$ *using the equalities* (3. 16) *are found as*

$$\begin{cases} \alpha'\left(\frac{9\pi}{2}\right) &= T_\alpha\big|_P = \left(\frac{\sqrt{6}}{3}, \frac{\sqrt{2}}{3}, \frac{1}{3}, 0\right), \\ \beta'(\pi) &= T_\beta\big|_P = \left(-\frac{1}{2}, 0, \frac{\sqrt{2}}{2}, \frac{1}{2}\right), \\ \gamma'(0) &= T_\gamma\big|_P = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

The unit normal vector field of hypersurface M^3 *using equation* (2. 2) *is*

$$N\big|_P = \frac{1}{\sqrt{61 - 12\sqrt{2} + 4\sqrt{3} - 6\sqrt{6}}}\left(2, -1 - 2\sqrt{3} + \sqrt{6}, \sqrt{2} - 2\sqrt{3}, 2\sqrt{6}\right).$$

The first and the second fundamental forms of hypersurface M^3 *are given as follows:*

$$I\big|_P = du^2 + dv^2 + dw^2 + \frac{\sqrt{2} - \sqrt{6}}{3}dudv + dudw + \frac{1 + \sqrt{2}}{2}dvdw,$$
$$II\big|_P = \frac{2\sqrt{6}}{9\lambda}du^2 + \frac{-1 - 2\sqrt{3} + \sqrt{6}}{4\lambda}dv^2 + \frac{1}{\lambda}dw^2,$$

where $\lambda = \sqrt{61 - 12\sqrt{2} + 4\sqrt{3} - 6\sqrt{6}}$*. The matrix corresponding to the coefficients of the first fundamental form of M is calculated as*

$$\mathbf{I}\big|_P = \begin{pmatrix} 1 & \frac{\sqrt{2}-\sqrt{6}}{6} & \frac{1}{2} \\ \frac{\sqrt{2}-\sqrt{6}}{6} & 1 & \frac{1+\sqrt{2}}{4} \\ \frac{1}{2} & \frac{1+\sqrt{2}}{4} & 1 \end{pmatrix}.$$

Furthermore, the inverse of the matrix $\mathbf{I}$ *is obtained as*

$$\mathbf{I}^{-1}\big|_P = \begin{pmatrix} \frac{9(2\sqrt{2}-13)}{-\lambda^2} & \frac{6(\sqrt{2}-4\sqrt{6}-3)}{-\lambda^2} & \frac{12(5+\sqrt{3}+\sqrt{2-\sqrt{3}})}{-\lambda^2} \\ \frac{6(\sqrt{2}-4\sqrt{6}-3)}{-\lambda^2} & \frac{108}{\lambda^2} & \frac{12(3+2\sqrt{2}+\sqrt{6})}{\lambda^2} \\ \frac{12(5+\sqrt{3}+\sqrt{2-\sqrt{3}})}{-\lambda^2} & \frac{12(3+2\sqrt{2}+\sqrt{6})}{\lambda^2} & \frac{16(7+\sqrt{3})}{\lambda^2} \end{pmatrix}.$$

The shape operator of the hypersurface M^3 *using equation* $S = \mathbf{I}^{-1}\mathbf{II}$ *is found as*

$$S \quad_P = \begin{pmatrix} \frac{26\sqrt{6}-8\sqrt{3}}{\lambda^3} & \frac{3 \quad 21-23\sqrt{2}-8\sqrt{3}+\sqrt{6}}{2\lambda^3} & -\frac{12 \quad 5+\sqrt{3}+\sqrt{2-\sqrt{2}}}{\lambda^3} \\ \frac{32-\frac{8}{\sqrt{3}}+4\sqrt{6}}{\lambda^3} & \frac{27(-1-2\sqrt{3}+\sqrt{6})}{\lambda^3} & -\frac{12 \quad 3+2\sqrt{2}+\sqrt{6}}{\lambda^3} \\ \frac{8\sqrt{2}(5+\sqrt{3}+\sqrt{2-\sqrt{2}})}{\sqrt{3}\lambda^3} & \frac{3 \quad 8\sqrt{2}+2\sqrt{3}+2\sqrt{6}-3}{\lambda^3} & \frac{16 \quad 7+\sqrt{3}}{\lambda^3} \end{pmatrix}.$$

The Gaussian curvature of the hypersurface M^3 *using equation* $(2.\ 7\ )$ *is obtained as*

$$K \quad_P = \frac{8 \quad 47058-35814\sqrt{2}+14292\sqrt{3}-11557\sqrt{6}+12\sqrt{2-\sqrt{2}} \quad -492+396\sqrt{2}-210\sqrt{3}+175\sqrt{6}}{\lambda^9}.$$

The mean curvature of the hypersurface M^3 *using equation* $(2.\ 8\ )$ *is found*

$$H\big|_P = \frac{85-46\sqrt{3}+53\sqrt{6}}{\lambda^3}.$$

4. CONCLUSIONS

The main point of this paper is to develop the theory of the translation hypersurfaces in Euclidean 4-space. Namely, with this approach, the existence of 3-parametric translation surfaces as the sum of three unit speed non-planar space curves with distinct parameters bring to light along with their characteristic features. The unit normal vector field, the fundamental forms, the shape operator matrix, the Gaussian curvature and the mean curvature are discussed for these new translation hypersurfaces. The calculations were supported with the numerical example.

REFERENCES

[1] A. T. Ali, H. S. A. Aziz, and A. H. Sorour, *On curvatures and points of the translation surfaces in Euclidean 3-space*, Journal of the Egyptian Mathematical Society **23**, No. 1 (2015) 67-172.

[2] M. E. Aydn, *A generalization of translation surfaces with constant curvature in the isotropic space*, Journal of Geometry **107**, No. 3 (2016) 603-615.

[3] A. akmak, M. K. Karacan, S. Kiziltu, and D. W. Yoon, *Translation surfaces in the 3-dimensional Galilean space satisfying* $\Delta^{II}x_i = \lambda_i x_i$, Bulletin of the Korean Mathematical Society **54**, No. 4 (2017) 1241-1254.

[4] M. etin, H. Kocayiit, and M. nder, *Translation surfaces according to Frenet frame in Minkowski 3-space*, International Journal of Physical Sciences **7**, No. 47 (2012) 6135-6143.

[5] M. etin, Y. Tuner, and N. Ekmeki, *Translation surfaces in Euclidean 3-space*, Int. J. Phys. Math. Sci. **5**, No. 4 (2011) 49-56.

[6] F. Dillen, I. V. D. Woestyne, L. Verstraelen, and J. Walrave, *The surface of scherk in* $\mathbb{E}^3$*: A special case in the class of minimal surfaces defined as the sum of two curves*, Bulletin of the Institute of Mathematics Academia Sinica **26**, No. 4 (1998) 257-267.

[7] F. Dillen, L. Verstraelen, L. Vrancken, and G. Zafindratafa, *Classification of polynomial translation hypersurfaces of finite type*, Results in Mathematics **27**, No. 3 (1995) 244-249.

[8] F. Dillen, W. Goemans, and I. V. D. Woestyne, *Translation surfaces of Weingarten type in 3-space*, Bull. Transilv. Univ. Brasov. **15**, No. 50 (2008) 1-14.

[9] H. Gluck, *Higher curvatures of curves in Euclidean space*, The American Mathematical Monthly **73**, No. 7 (1966) 699-704.

[10] H. Liu, *Translation surfaces with constant mean curvature in 3-dimensional spaces*, Journal of Geometry **64**, No. 1-2 (1999) 141-149.

[11] H. Liu, and S. D. Jung, *Affine translation surfaces with constant mean curvature in Euclidean 3-space*, Journal of Geometry **108**, No. 2 (2017) 423-428.

[12] H. Liu, and Y. Yu, *Affine translation surfaces in Euclidean 3-space*, Proceedings of the Japan Academy, Ser. A, Mathematical Sciences **89**, No. 9 (2013) 111-113.

[13] R. Lpez, and . Perdomo, *Minimal translation surfaces in Euclidean space*, The Journal of Geometric Analysis **27**, No. 4 (2017) 2926-2937.

[14] M. Magid, and L. Vrancken, *Affine translation surfaces*, Results in Mathematics **35**, No. 1 (1999) 134-144.

[15] M. Moruz, and M. I. Munteanu, *Minimal translation hypersurfaces in* $\mathbb{E}^4$, Journal of Mathematical Analysis and Applications **439**, No. 2 (2016) 798-812.

[16] M. I. Munteanu, O. Palmas, and G. Ruiz-Hernndez, *Minimal translation hypersurfaces in Euclidean space*, Mediterranean Journal of Mathematics **13**, No. 5 (2016) 2659-2676.

[17] H. F. Scherk, *Bemerkungen ber die kleinste Flche innerhalb gegebener Grenzen*, Crelle J. **47**, No. 13 (1835) 185-208.

[18] B. Senoussi, and M. Bekkar, *Affine Translation surfaces in 3-dimensional Euclidean space satisfying* $\Delta r_i = \lambda_i r_i$, Konuralp Journal of Mathematics **5**, No. 2 (2017) 47-53.

[19] K. Seo, *Translation hypersurfaces with constant curvature in space forms*, Osaka Journal of Mathematics **50**, No. 3 (2013) 631-641.

[20] G. A. ekerci, S. Sevin, and A. C. ken, *On the translation hypersurfaces with Gauss map* G *satisfying* $\Delta G = AG$, Miskolc Mathematical Notes **20**, No.2 (2019) 1215-1225.

[21] L. Verstraelen, J. Walrave, and S. Yaprak, *The minimal translation surfaces in Euclidean space*, Soochow J. Math. **20**, No. 1 (1994) 77-82.

[22] M. Z. Williams, and F. M. Stein, *A triple product of vectors in four-space*, Mathematics Magazine **37**, No. 4 (1964) 230-235.

[23] D. Yang, *Some classifications of Weingarten translation hypersurfaces in Euclidean space*, Bulletin Mathmatique de la Socit des Sciences Mathmatiques de Roumanie **61**, No. 1 (2018) 107-115.

[24] D. W. Yoon, *Weighted minimal translation surfaces in Minkowski 3-space with density*, International Journal of Geometric Methods in Modern Physics **14**, No. 12 (2017) 1750178.

[25] Y. Yuan, and H. L. Liu, *Some new translation surfaces in 3-Minkowski space*, Journal of Mathematical Res. Exp. **31**, (2011) 1123-1128.

[26] S. Yce, *Weingarten map of the hypersurface in Euclidean 4-space and its applications*, Hagia Sophia Journal of Geometry **1**, No. 1 (2019) 1-8.