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Convergence of Jungck-Kirk Type Iteration Method with Applications

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Abstract.: The aim of this article is to define a new Jungck-Kirk type iteration method and to examine the convergence result under appropriate conditions together with other Jungck-Kirk type iteration methods in the literature. It is also to analyze whether the newly defined iteration method is stable. In addition, it has been shown through numerical examples that the new iteration method has a better convergence rate than the others. Finally, to show the validity of convergence and stability results, some examples are given. The results obtained in this paper may be interpreted as a refinement and improvement of the previously known results.

AMS (MOS) Subject Classification Codes: 47H1; 54H25
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1. Introduction and Preliminaries

Fixed point theory is a significant field of study in the literature. Recently, much attention has been given to develop new iterative methods and study its features like convergence, equivalence in terms of convergence, stability, convergence rate, data dependency etc. (see, e.g., [1, 5, 11, 12, 20, 21, 23] and references therein). Throughout this paper, we assume that B is a Banach space, D an arbitrary nonempty set and H, $G : D \to B$ such that $G(D) \subseteq H(D)$. For $x_0 \in D$, following iteration method:

$$Hx_{n+1}=f\left(G,x_n\right)=Gx_n \tag{1. 1}$$

for all $n\in\mathbb{N}$. This iteration method is called Jungck iteration method [15]. A generalization of the Banach contraction mapping theorem is also obtained by Jungck [15]. The Jungck type iteration method has been introduced by many researchers (see:[3, 6, 10, 17, 18, 22]

and references therein).
In 1971, Kirk [19] introduced the Kirk iteration method as follows:

Let $(B, \|\cdot\|)$ be a normed space and A be a nonempty, convex, closed subset of B and $G : A \to A$ be a selfmap of A and let $x_0 \in B$, the sequence $\{x_n\}_{n=0}^{\infty}$ is as follows:

$$x_{n+1} = \sum_{i=0}^{r} \omega_i G^i x_n, \quad n \geq 0, \quad \sum_{i=0}^{r} \omega_i = 1 \tag{1. 2}$$

Based on the approaches given in (1. 1) and (1. 2), some Jungck-type or Kirk-type iteration methods have been defined. In addition, by combining these two types, a new interesting wide field of study has been revealed. These methods are called Jungck-Kirk type iterations, have led many researchers to form Jungck-Kirk-type iteration methods to obtain various types of fixed point theorems (see:[2, 4, 9]). Also, some of these iteration methods are as follows:
In 2012, the following Jungck-Kirk-Noor iteration method was defined by Chugh and Kumar [9] :

$$\begin{cases} Hx_{n+1} = \omega_{n,0}^{(1)} Hx_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i y_n, & \sum_{i=0}^{r} \omega_{n,i}^{(1)} = 1 \\ Hy_n = \omega_{n,0}^{(2)} Hx_n + \sum_{j=1}^{s} \omega_{n,j}^{(2)} G^j z_n, & \sum_{j=0}^{s} \omega_{n,j}^{(2)} = 1 \\ Hz_n = \omega_{n,0}^{(3)} Hx_n + \sum_{k=1}^{t} \omega_{n,k}^{(3)} G^k x_n, & \sum_{k=0}^{t} \omega_{n,k}^{(3)} = 1 \end{cases} \tag{1. 3}$$

in which $\omega_{n,i}^{(1)}, \omega_{n,j}^{(2)}, \omega_{n,k}^{(3)}$ in $[0, 1]$.
In 2013, the following the Jungck-Kirk-SP iteration method and Jungck-Kirk-CR iteration method were defined by Alotaibi et al. [4] as follow respectively:

$$\begin{cases} Hx_{n+1} = \omega_{n,0}^{(1)} Hy_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i y_n, & \sum_{i=0}^{r} \omega_{n,i}^{(1)} = 1 \\ Hy_n = \omega_{n,0}^{(2)} Hz_n + \sum_{j=1}^{s} \omega_{n,j}^{(2)} G^j z_n, & \sum_{j=0}^{s} \omega_{n,j}^{(2)} = 1 \\ Hz_n = \omega_{n,0}^{(3)} Hx_n + \sum_{k=1}^{t} \omega_{n,k}^{(3)} G^k x_n, & \sum_{k=0}^{t} \omega_{n,k}^{(3)} = 1 \end{cases} \tag{1. 4}$$

and

$$\begin{cases} Hx_{n+1} = \omega_{n,0}^{(1)} Hy_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i y_n, & \sum_{i=0}^{r} \omega_{n,i}^{(1)} = 1 \\ Hy_n = \omega_{n,0}^{(2)} Gx_n + \sum_{j=1}^{s} \omega_{n,j}^{(2)} G^j z_n, & \sum_{j=0}^{s} \omega_{n,j}^{(2)} = 1 \\ Hz_n = \omega_{n,0}^{(3)} Hx_n + \sum_{k=1}^{t} \omega_{n,k}^{(3)} G^k x_n, & \sum_{k=0}^{t} \omega_{n,k}^{(3)} = 1 \end{cases} \tag{1. 5}$$

in which $\omega_{n,i}^{(1)}$, $\omega_{n,j}^{(2)}$, $\omega_{n,k}^{(3)}$ in $[0, 1]$.
In 2013, Akewe et al. [2] introduced the implicit Jungck-Kirk-Mann iteration method, the implicit Jungck-Kirk-Ishikawa iteration method, and implicit Jungck-Kirk-multistep iteration as follow respectively

$$\begin{cases} Hx_{n+1} = \omega_{n,0}^{(1)} Hy_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i y_n, \quad \sum_{i=0}^{r} \omega_{n,i}^{(1)} = 1 \end{cases} \tag{1. 6}$$

and

$$\begin{cases} Hx_{n+1} = \omega_{n,0}^{(1)} Hy_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i y_n, \quad \sum_{i=0}^{r} \omega_{n,i}^{(1)} = 1 \\ Hy_n = \omega_{n,0}^{(2)} Hx_n + \sum_{j=1}^{s} \omega_{n,j}^{(2)} G^j x_n, \quad \sum_{j=0}^{s} \omega_{n,j}^{(2)} = 1 \end{cases} \tag{1. 7}$$

and

$$\begin{cases} Hx_{n+1} = \omega_{n,0}^{(1)} Hy_n^1 + \sum_{i=1}^{m_1} \omega_{n,i}^{(1)} G^i y_n, \quad \sum_{i=0}^{m_1} \omega_{n,i}^{(1)} = 1 \\ Hy_n^l = \omega_{n,0}^{(2)l} Hy_n^{l+1} + \sum_{i=1}^{m_{l+1}} \omega_{n,i}^{(2)l} G^i y_n^{l+1}, \quad \sum_{i=0}^{m_{l+1}} \omega_{n,i}^{(2)l} = 1, \\ \qquad l = 1, 2, ..., s-2 \\ Hy_n^{s-1} = \sum_{i=0}^{m_s} \omega_{n,i}^{(2)s-1} G^i x_n^{s-1}, \quad \sum_{i=0}^{m_s} \omega_{n,i}^{(2)s-1} = 1, \quad s \geq 2, n \geq 0 \end{cases} \tag{1. 8}$$

in which $m_1 \geq m_2 \geq m_3 \geq ... \geq m_s$ for each l, $\omega_{n,i}^{(1)} \geq 0$, $\omega_{n,0}^{(1)} \neq 0$, $\omega_{n,i}^{(2)l} \geq 0$, $\omega_{n,0}^{(2)l} \neq 0$ for each l, $\omega_{n,i}^{(1)}$, $\omega_{n,i}^{(2)l}$ in $[0, 1]$.
We recall some definitions and lemmas in the literature for the main results.

Definition 1.1. [14] *Let (B, d) be a complete metric space and $G : B \to B$ be a mapping such that*

$$d(Gx, Gy) \leq \delta d(x, y) + \psi\left(d(x, Gx)\right), \textit{ for all } x, y \in B, \tag{1. 9}$$

in which $\delta \in [0, 1)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a subadditive, monotone increasing function such that $\psi(0) = 0$, $\psi(tv) = t\psi(v)$, $t \geq 0$, $v \in \mathbb{R}_+$.

Lemma 1.2. [24] *Let $(B, \|\cdot\|)$ be a normed space and $G : B \to B$ be a mapping such that for all $x, y \in B$*

$$\|G^J x - G^j y\| \leq \sum_{l=1}^{j} \binom{j}{l} \delta^{j-l} \psi^l \left(\|Gx - x\|\right) + \delta^j \|x - y\| \tag{1. 10}$$

in which δ in $[0, 1)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a subadditive, monotone increasing function such that $\psi(0) = 0$, $\psi(tv) = t\psi(v)$, $t \geq 0$, $v \in \mathbb{R}_+$.

Definition 1.3. [13] *Let $(B, \|\cdot\|)$ be normed space and the pair H, $G : D \to B$ be mappings such that*

$$\|Gx - Gy\| \leq \delta \|Hx - Hy\| + \psi\left(\|Gx - Hx\|\right), \textit{ for all } x, y \in D, \tag{1. 11}$$

in which δ *in* $[0,1)$ *and* $\psi:\mathbb{R}_+\to\mathbb{R}_+$ *be a continuous function such that* $\psi(0)=0$.

Definition 1.4. [16] *Let* B *be a nonempty set and* $H,G{:}B\to B$ *be mappings. If* $Gx=Hx$, *then* $x\in B$ *is called coincidence point of* G *and* H. *If* $x=Gx=Hx$, *then* $x\in B$ *is called common fixed point of* G *and* H. *If* $p=Gx=Hx$ *for some* $x\in B$, *then* p *is called the point of coincidence of* G *and* H. *If* $GHx=HGx$ *for all* $x\in B$, *then a pair* (H,G) *is called commuting. If* $GHx=HGx$ *whenever* $Gx=Hx$ *for some* $x\in B$, *then a pair* (H,G) *is called weakly compatible.*

Lemma 1.5. [25] *Let* $(B,\|\cdot\|)$ *be a normed space and* $G:D\to B$ *nonself maps of* B *satisfy condition* (1. 11). *Assume that* $G(D)\subseteq H(D)$, $\|H^2x-GHx\|\leq\|Hx-Gx\|$ *and* $\|H^2x-GHy\|\leq\|Hx-Gy\|$ *for all* $x,y\in D$ *and* $Gx_p=Hx_p=p$. *Then, for all* $i\in\mathbb{N}$, *for all* $x,y\in D$

$$\|G^jx-G^jy\|\leq\sum_{l=1}^{j}\binom{j}{l}\delta^{j-l}\psi^l\left(\|Gx-Hx\|\right)+\delta^j\|Hx-Hy\| \tag{1. 12}$$

in which $\delta\in[0,1)$ *and* $\psi:\mathbb{R}_+\to\mathbb{R}_+$ *be a subadditive, monotone increasing function such that* $\psi(0)=0$, $\psi(tv)=t\psi(v)$, $t\geq 0$, $v\in\mathbb{R}_+$.

Definition 1.6. [26] *Let* $H,G:D\to B$, $G(D)\subseteq H(D)$ *and* $p=Gx=Hx$. *For any* $x_0\in D$, *let the sequence* $\{Hx_n\}_{n=0}^{\infty}$ *generated by the iteration method* $Hx_{n+1}=f(G,x_n)$ *converges to* p. *Let* $\{Hy_n\}_{n=0}^{\infty}\subset B$ *be an arbitrary sequence and set*

$$\epsilon_n=d(Hy_{n+1},f(G,y_n)) \tag{1. 13}$$

for all $n\in\mathbb{N}$. *Then the iteration method* $f(G,x_n)$ *is called* (H,G)*-stable if and only if* $\lim\limits_{n\to\infty}\epsilon_n=0$ *implies that* $\lim\limits_{n\to\infty}\|Hy_n-p\|=0$.

Lemma 1.7. [27] *Let* $\{b_n\}_{n=0}^{\infty}$ *and* $\{d_n\}_{n=0}^{\infty}$ *be nonnegative real sequences satisfying the following inequality:*

$$b_{n+1}\leq d_n+(1-r_n)\,b_n$$

where $r_n\in(0,1)$ *for all* $n\in\mathbb{N}$, $\sum\limits_{n=0}^{\infty}r_n=\infty$ *and* $\lim_{n\to\infty}\frac{d_n}{r_n}=0$. *Then* $b_n\to 0$ *as* $n\to\infty$.

Definition 1.8. [7] *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be sequences in* A. *We say that* $\{b_n\}_{n=0}^{\infty}$ *is an approximate sequence of* $\{a_n\}_{n=0}^{\infty}$ *if, for any* $t\in\mathbb{N}$, *there exists* $\varepsilon(t)$ *such that*

$$\|a_n-b_n\|\leq\varepsilon(t)\text{, for all } n\geq t.$$

Definition 1.9. [8] *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be sequences in* A. *We say that these sequences are equivalent if*

$$\lim_{n\to\infty}\|a_n-b_n\|=0.$$

In this study, the convergence of the newly defined iteration sequence given by (2. 14) and the sequences given by (1. 3) and (1. 4) is examined under the same conditions. Also, it has become clear that the newly defined iteration method is stable. Finally, from Example 3.1 and Example 3.2, it is seen that the iteration method given in (2. 14) converges faster than the iteration methods given in (1. 3) and (1. 4).

2. RESULTS

In this section, we define a new Jungck-Kirk type iteration method as follows:

$$\begin{cases} x_0 \in D \\ Hx_{n+1} = \omega_{n,0}^{(1)} G z_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i y_n, \quad \sum_{i=0}^{r} \omega_{n,i}^{(1)} = 1 \\ Hy_n = \omega_{n,0}^{(2)} H z_n + \sum_{j=1}^{s} \omega_{n,j}^{(2)} G^j z_n, \quad \sum_{j=0}^{s} \omega_{n,j}^{(2)} = 1 \\ Hz_n = \omega_{n,0}^{(3)} H x_n + \sum_{k=1}^{t} \omega_{n,k}^{(3)} G^k x_n, \quad \sum_{k=0}^{t} \omega_{n,k}^{(3)} = 1 \end{cases} \tag{2. 14}$$

Theorem 2.1. *Let* $(D, \|.\|)$ *be a norm spaces,* $H, G{:}D{\to}B$ *satisfy condition* (1. 12)*. Assume that* $G\,(D) \subseteq H\,(D)$*,* $H\,(D) \subseteq B$*,* $Gx_p{=}Hx_p{=}p$ *and* $\psi : \mathbb{R}_+ {\to} \mathbb{R}_+$ *be a subadditive, monotone increasing function such that* $\psi(0) = 0$*,* $\psi(tv) = t\psi(v)$*,* $t \geq 0$*,* $\delta \in [0, 1)$*. Let* $\{Hx_n\}_{n=0}^{\infty}$ *be iterative sequence* (2. 14)*. Then,* $\{Hx_n\}_{n=0}^{\infty}$ *converges to* p*. Moreover,* p *is a unique common fixed point of* H *and* G *provided that* $D = B$*, and* H *and* G *are weakly compatible.*

Proof. By using iterative sequence (2. 14) and condition (1. 12), we have

$$\begin{aligned} \|Hx_{n+1} - p\| &\leq \sum_{i=1}^{r} \omega_{n,i}^{(1)} \|G^i y_n - p\| + \omega_{n,0}^{(1)} \parallel G z_n - p\| \\ &\leq \delta \omega_{n,0}^{(1)} \parallel H z_n - p\| + \omega_{n,0}^{(1)} \psi\left(\|Gp - p\|\right) \\ &+ \sum_{i=1}^{r} \omega_{n,i}^{(1)} \left(\sum_{l=1}^{i} \binom{i}{l} \delta^{i-l} \psi^l \left(\|Hp - p\|\right) + \delta^i \|Hy_n - p\| \right) \\ &= \delta \omega_{n,0}^{(1)} \parallel H z_n - p\| + \sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \|Hy_n - p\| \end{aligned} \tag{2. 15}$$

and

$$\begin{aligned} \|Hy_n - p\| &\leq \omega_{n,0}^{(2)} \parallel H z_n - p\| + \sum_{j=1}^{s} \omega_{n,j}^{(2)} \|G^j z_n - p\| \\ &\leq \omega_{n,0}^{(2)} \parallel H z_n - p\| \\ &+ \sum_{j=1}^{s} \omega_{n,j}^{(2)} \left(\sum_{l=1}^{j} \binom{j}{l} \delta^{j-l} \psi^l \left(\|Gp - p\|\right) + \delta^j \|Hz_n - p\| \right) \\ &= \omega_{n,0}^{(2)} \parallel H z_n - p\| + \sum_{j=1}^{s} \omega_{n,j}^{(2)} \delta^j \|Hz_n - p\| \\ &= \sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j \|Hz_n - p\| \end{aligned} \tag{2. 16}$$

on the other hand,

$$\begin{aligned}\|Hz_n - p\| &\le \omega_{n,0}^{(3)} \| Hx_n - p\| + \sum_{k=1}^{t} \omega_{n,k}^{(3)} \|G^k x_n - p\| \\ &\le \omega_{n,0}^{(3)} \| Hx_n - p\| \\ &+ \sum_{k=1}^{t} \omega_{n,k}^{(3)} \left(\sum_{l=1}^{k} \binom{k}{l} \delta^{k-l} \psi^l \left(\|Gp - p\|\right) + \delta^k \|Hx_n - p\| \right) \\ &= \omega_{n,0}^{(3)} \| Hx_n - p\| + \sum_{k=1}^{t} \omega_{n,k}^{(3)} \delta^k \|Hx_n - p\| \\ &= \sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k \|Hx_n - p\| \end{aligned} \tag{2. 17}$$

Substituting (2. 16) and (2. 17) into (2. 15), we obtain

$$\begin{aligned}\|Hx_{n+1} - p\| &\le \delta\omega_{n,0}^{(1)} \| Hz_n - p\| + \sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \|Hy_n - p\| \\ &\le \delta\omega_{n,0}^{(1)} \| Hz_n - p\| + \sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \left(\sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j \|Hz_n - p\| \right) \\ &\le \delta\omega_{n,0}^{(1)} \| Hz_n - p\| \\ &+ \sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \left(\sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j \left(\sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k \|Hx_n - p\| \right) \right) \\ &= \left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \right) \left(\sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j \right) \left(\sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k \right) \right) \\ &\cdot \|Hx_n - p\| \end{aligned} \tag{2. 18}$$

Since δ^i, δ^j, $\delta^k \in [0, 1)$ and $\sum_{i=0}^{r} \omega_{n,i}^{(1)} = \sum_{j=0}^{s} \omega_{n,j}^{(2)} = \sum_{k=0}^{t} \omega_{n,k}^{(3)} = 1$, hence

$$\left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \right) \left(\sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j \right) \left(\sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k \right) \right) < 1 \tag{2. 19}$$

Using (2. 19) and Lemma 1.7, it can be seen that $\lim_{n\to\infty} \|Hx_n - p\| = 0$. We prove that p is a unique common fixed point of H, G. Suppose that there exists another point of coincide q

of the pair (H, G). Then, there exists $x_q \in (H, G)$ such that $Hx_q = Gx_q = q$. We obtain

$$0 \leq \|p-q\| = \left\|G^j x_p - G^j x_q\right\| \leq \sum_{l=1}^{j} \binom{j}{l} \delta^{j-l} \psi^l \left(\|Gx_q - Hx_q\|\right) + \delta^j \|Hx_p - Hx_q\|$$

$$\leq \sum_{l=1}^{j} \delta^j \parallel Hx_p - Hx_q\| = \sum_{l=1}^{j} \delta^j \parallel x_p - x_q\|$$

Therefore,

$$0 \leq \|p-q\| = \|Gx_p - Gx_q\| \leq \sum_{l=1}^{j} \delta^j \parallel p - q\| \tag{2. 20}$$

which implies that $p = q$ and H, G are weakly compatible and $Hx_p = Gx_p = p$, then $Gp = GGx_p = GHx_p = HGx_p$ hence $Gp = Hp$. Hence, Gp is a point of coincidence of the pair (H, G) and because point of coincidence is unique, then $Gp = p$. So, $Hp = Gp = p$ and thus, p is a unique common fixed point of H and G.

□

Theorem 2.2. *Let H, G be the same as in Theorem 2.1 with $Gx_p{=}Hx_p{=}p$. Suppose that $H, G{:}D{\to}B$ satisfy condition* (1. 12). *Then, the iterative sequence*

(1) $\{Hx_n\}_{n=0}^{\infty}$ *defined by* (1. 3) *with real sequences* $\left\{\omega_{n,i}^{(1)}\right\}_{n=0}^{\infty}, \left\{\omega_{n,j}^{(2)}\right\}_{n=0}^{\infty}, \left\{\omega_{n,k}^{(3)}\right\}_{n=0}^{\infty}$ *in* $[0, 1]$ *convergences to* p *with the following estimate:*

$$\|Hx_{n+1} - p\| \leq \left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i\right) \left(\sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j\right) \left(\sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k\right)\right) \cdot \|Hx_n - p\| \tag{2. 21}$$

(2) $\{Hx_n\}_{n=0}^{\infty}$ *defined by* (1. 4) *with respect to real sequences* $\left\{\omega_{n,i}^{(1)}\right\}_{n=0}^{\infty}, \left\{\omega_{n,j}^{(2)}\right\}_{n=0}^{\infty}$, $\left\{\omega_{n,k}^{(3)}\right\}_{n=0}^{\infty}$ *in* $[0, 1]$ *convergences to* p *with the following estimate:*

$$\|Hx_{n+1} - p\| \leq \left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i\right) \left(\sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j\right) \left(\sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k\right)\right) \cdot \|Hx_n - p\| \tag{2. 22}$$

Proof. Similar to that Theorem 2.1. □

Theorem 2.3. *Let H, G be the same as in Theorem 2.1 with $Gx_p{=}Hx_p{=}p$. Suppose that $\{Hx_n\}_{n=0}^{\infty}$ be iterative sequence* (2. 14) *converges to p. Then, iteration method* (2. 14) *is (H, G)-stable.*

Proof. We take follow iteration:

$$\begin{cases} Hu_{n+1} = \omega_{n,0}^{(1)} G\nu_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i v_n \\ Hv_n = \omega_{n,0}^{(2)} H\nu_n + \sum_{j=1}^{s} \omega_{n,j}^{(2)} G^j \nu_n \\ H\nu_n = \omega_{n,0}^{(3)} Hu_n + \sum_{k=1}^{t} \omega_{n,k}^{(3)} G^k u_n \end{cases} \tag{2. 23}$$

First, suppose that $\varepsilon_n = \left\| Hu_{n+1} - \omega_{n,0}^{(1)} G\nu_n - \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i v_n \right\|$ and $\lim_{x\to\infty} \varepsilon_n = 0$. Then, we show that $\lim_{x\to\infty} \|Hu_n - p\| = 0$ as follows:

$$\begin{aligned} \|Hu_{n+1} - p\| &\leq \left\| Hu_{n+1} - \omega_{n,0}^{(1)} G\nu_n - \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i v_n \right\| \\ &+ \left\| \omega_{n,0}^{(1)} G\nu_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i v_n - p \right\| \\ &= \left\| Hu_{n+1} - \omega_{n,0}^{(1)} G\nu_n - \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i v_n \right\| \\ &+ \left\| \omega_{n,0}^{(1)} G\nu_n + \sum_{i=1}^{r} \omega_{n,i}^{(1)} G^i v_n - \sum_{i=0}^{r} G^i p \right\| \\ &\leq \varepsilon_n + \omega_{n,0}^{(1)} \|G\nu_n - Gp\| + \left\| \sum_{i=1}^{r} \omega_{n,i}^{(1)} \left(G^i v_n - G^i p \right) \right\| \\ &\leq \varepsilon_n + \omega_{n,0}^{(1)} \left(\psi \left(\|Gp - p\| \right) + \delta \|H\nu_n - p\| \right) \\ &+ \sum_{i=1}^{r} \omega_{n,i}^{(1)} \left(\sum_{l=1}^{i} \binom{i}{l} \delta^{i-l} \psi^l \left(\|Gp - p\| \right) + \delta^i \|Hv_n - p\| \right) \\ &= \varepsilon_n + \delta \omega_{n,0}^{(1)} \| H\nu_n - p\| + \sum_{i=1}^{r} \omega_{n,i}^{(1)} \delta^i \|Hv_n - p\| \end{aligned} \tag{2. 24}$$

Similar to inequality (2. 16) and we have

$$\|Hv_n - p\| \quad \leq \quad \sum_{j=0}^{s} \omega_{n,j}^{(2)} \delta^j \|H\nu_n - p\| \tag{2. 25}$$

Similar to inequality (2. 17) and we have

$$\|H\nu_n - p\| \quad \leq \quad \sum_{k=0}^{t} \omega_{n,k}^{(3)} \delta^k \|Hu_n - p\| \tag{2. 26}$$

Substituting (2. 25) and (2. 26) into (2. 24), we have

$$\|Hu_{n+1}-p\| \le \left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r}\omega_{n,i}^{(1)}\delta^i\right)\left(\sum_{j=0}^{s}\omega_{n,j}^{(2)}\delta^j\right)\left(\sum_{k=0}^{t}\omega_{n,k}^{(3)}\delta^k\right)\right) \cdot \|Hu_n-p\| + \varepsilon_n \tag{2. 27}$$

Using Lemma 1.7, yields $\lim_{n\to\infty} Hu_n = p$.

Conversely, let $\lim_{n\to\infty} Hu_n = p$. Then, we show that $\lim_{n\to\infty} \varepsilon_n = 0$ as follows:

$$\begin{aligned}\varepsilon_n &= \left\|Hu_{n+1} - \omega_{n,0}^{(1)}G\nu_n - \sum_{i=1}^{r}\omega_{n,i}^{(1)}G^i v_n\right\| \\ &\le \|Hu_{n+1}-p\| + \omega_{n,0}^{(1)}\left(\psi\left(\|Gp-p\|\right)+\delta\,\|Hu_n-p\|\right) \\ &+ \sum_{i=1}^{r}\omega_{n,i}^{(1)}\left(\sum_{l=1}^{i}\binom{i}{l}\delta^{i-l}\psi^l\left(\|Gp-p\|\right)+\delta^i\|Hv_n-p\|\right)\end{aligned} \tag{2. 28}$$

Then using (2. 25) and (2. 26), we have

$$\begin{aligned}\varepsilon_n &\le \|Hu_{n+1}-p\| \\ &+ \left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r}\omega_{n,i}^{(1)}\delta^i\right)\left(\sum_{j=0}^{s}\omega_{n,j}^{(2)}\delta^j\right)\left(\sum_{k=0}^{t}\omega_{n,k}^{(3)}\delta^k\right)\right)\|Hu_n-p\|\end{aligned} \tag{2. 29}$$

Since δ^i, δ^j, $\delta^k \in [0,1)$ and $\sum_{i=0}^{r}\omega_{n,i}^{(1)} = \sum_{j=0}^{s}\omega_{n,j}^{(2)} = \sum_{k=0}^{t}\omega_{n,k}^{(3)} = 1$, hence

$$\left(\delta\omega_{n,0}^{(1)} + \left(\sum_{i=1}^{r}\omega_{n,i}^{(1)}\delta^i\right)\left(\sum_{j=0}^{s}\omega_{n,j}^{(2)}\delta^j\right)\left(\sum_{k=0}^{t}\omega_{n,k}^{(3)}\delta^k\right)\right) < 1 \tag{2. 30}$$

Using Lemma 1.7, it can be seen that $\lim_{n\to\infty} \varepsilon_n = 0$. Thus, the iteration method (2. 14) is (H,G)-stable. □

3. Applications

In this section, by using MATLAB 2015Ra, we examine convergence of the iteration method (2. 14), the Jungck-Kirk-Noor iteration method (1. 3), and the Jungck-Kirk-SP iteration method (1. 4) through examples. Numerical results are listed in the form of Table 1 and Table 2, by taking $A=[0,1]$, $\omega_{n,1}^{(1)} = \omega_{n,2}^{(1)} = \omega_{n,3}^{(1)} = \omega_{n,1}^{(2)} = \omega_{n,2}^{(2)} = \omega_{n,3}^{(2)} = \omega_{n,1}^{(3)} = \omega_{n,2}^{(3)} = \omega_{n,3}^{(3)} = \frac{1}{3(n+1)}$, $\omega_{n,0}^{(1)} = 1-\omega_{n,1}^{(1)}-\omega_{n,2}^{(1)}-\omega_{n,3}^{(1)}$, $\omega_{n,0}^{(2)} = 1-\omega_{n,1}^{(2)}-\omega_{n,2}^{(2)}-\omega_{n,3}^{(2)}$, and $\omega_{n,0}^{(3)} = 1-\omega_{n,1}^{(3)}-\omega_{n,2}^{(3)}-\omega_{n,3}^{(3)}$ for all iteration methods and let B is a Banach space with usual norm.

Example 3.1. *Let $f(x)=2x^3-x^2-1$, and G, $H{:}A\to A$ by $Hx=1-x^3$, $Gx=\frac{1-x^2}{2}$ with coincidence point 1. Let $\delta=\frac{1}{2}$, $\psi(t)=\frac{t}{2}$. It is clear that H and G satisfy the condition (1. 11), $G([0,1])\subseteq H([0,1])$ and $H([0,1])$ is a complete subset of $[0,1]$. By* *taking the initial point* $0.5 \in A$*, the obtained results, which are listed in Table 1, show the convergence of some Jungck-Kirk type iteration methods to* $p = 0 = G1 = H1$.

TABLE 1. Convergence behavior of iteration method (2. 14), iteration method (1. 3), and iteration method (1. 4) for Example 3.1.

Iteration Methods	Iteration Steps	x_{n+1}	Gx_n	Hx_n
The New Jungck-Kirk	0	0.50000000000000	0.37500000000000	0.87500000000000
	1	0.88838440561995	0.10538657392565	0.29886317205457
	2	0.93532651488883	0.06258215527295	0.18174298348254
	3	0.95264524702019	0.04623351666492	0.13544303278754
	⋮	⋮	⋮	⋮
	500	0.99961115621222	0.00038876818804	0.00116607782367
	⋮	⋮	⋮	⋮
Jungck-Kirk-Noor	0	0.50000000000000	0.37500000000000	0.87500000000000
	1	0.72390193934222	0.23798299110829	0.62065075824528
	2	0.77642463540786	0.19858239276589	0.00000099915150
	3	0.80097214773511	0.17922180927631	0.48613120725892
	⋮	⋮	⋮	⋮
	500	0.87428564160416	0.11781230844240	0.33171757774670
	⋮	⋮	⋮	⋮
Jungck-Kirk-SP	0	0.50000000000000	0.37500000000000	0.87500000000000
	1	0.83032340595258	0.15528152076365	0.42754435645137
	2	0.85991131874499	0.13027626194713	0.36414074567925
	3	0.86803803406809	0.12325498570560	0.34594199689388
	⋮	⋮	⋮	⋮
	500	0.87524772631709	0.00000000000000	0.00000000207459
	⋮	⋮	⋮	⋮

Example 3.2. *Consider the solution of* $sinx - \frac{x}{2} = 0$*. Let operators* G*,* $H{:}A \to A$ *by* $Hx = \frac{x}{2}$*,* $Gx = x - sinx$ *with common fixed point* 0*. Let* $\delta = 0.4$*,* $\psi(t) = \frac{t}{2}$*.* H *and* G *satisfy the condition* (1. 11)*. It can be seen that* $G([0,1]) \subseteq H([0,1])$ *and* $H([0,1])$ *is a complete subset of* $[0,1]$*. By taking the initial point* $0.5 \in A$*, the obtained results, which are listed in Table 2, show the convergence of some Jungck-Kirk type iteration methods to* $p = 0 = G0 = H0$.

TABLE 2. Convergence behavior of iteration method (2. 14), iteration method (1. 3), and iteration method (1. 4) for Example 3.2.

Iteration Methods	Iteration Steps	x_{n+1}	Gx_n	Hx_n
The New Jungck-Kirk	0	0.50000000000000	0.02057446139580	0.25000000000000
	1	0.00293531291842	0.00000000421514	0.00146765645921
	2	0.00000000174748	0.00000000000000	0.00000000087374
	3	0.00000000000000	0.00000000000000	0.00000000000000
Jungck-Kirk-Noor	0	0.50000000000000	0.02057446139580	0.25000000000000
	1	0.25087510824023	0.00000147711167	0.01034732200112
	2	0.16742364743513	0.00000000000000	0.00000099915150
	3	0.12562276067641	0.00000000000000	0.00000000000000
	⋮	⋮	⋮	⋮
	500	0.00100567046226	0.00000000016952	0.00050283523113
	⋮	⋮	⋮	⋮
Jungck-Kirk-SP	0	0.50000000000000	0.02057446139580	0.25000000000000
	1	0.06480403102168	0.00004534857248	0.03240201551084
	2	0.01920855124260	0.00000118120308	0.00960427562130
	3	0.00810381562714	0.00000008869844	0.00405190781357
	⋮	⋮	⋮	⋮
	28	0.00002950880610	0.00000000000000	0.00001475440305
	⋮	⋮	⋮	⋮
	500	0.00000000414917	0.00000000000000	0.00000000207459
	⋮	⋮	⋮	⋮

The following example shows the application of the Theorem 2.3.

Example 3.3. *Let H, G be the same as in Example 3.2 with Gx_p=Hx_p=p. G, H:$A \to A$ by $Hx = \frac{x}{2}$, $Gx = x - sinx$ with common fixed point $0 = p = x_p$. Let $\delta = \frac{1}{2}$, $\psi(t) = \frac{t}{2}$. We get*

$$Hx_n = \frac{x_n}{2}, \quad Gx_n = x_n - sinx_n \tag{3. 31}$$

and the iteration methods (2. 14) *associated with the mappings* G *and* H *in* (3. 31) *is as follows*

$$\begin{aligned} z_n &= \frac{6x_n + 3nx_n - 6sin(x_n) - 4sin(x_n - sin(x_n))}{6+6n} \\ &- \frac{2sin(x_n - sin(x_n) - sin(x_n - sin(x_n)))}{6+6n} \\ y_n &= \frac{3z_n + 3nz_n - 3(n+1)sin(z_n) - 2sin(z_n - sin(z_n))}{3+3n} \\ &- \frac{sin(z_n - sin(z_n) - sin(z_n - sin(z_n)))}{3+3n} \\ x_{n+1} &= \frac{6nz_n + 6y_n - 6sin(y_n) - 6nsin(z_n) - 4sin(y_n - sin(y_n))}{3+3n} \\ &- \frac{2sin(y_n - sin(y_n) - sin(y_n - sin(y_n)))}{3+3n} \end{aligned} \quad (3.\ 32)$$

On the other hand, if we take the sequence $\{Hu_n\}_{n=0}^{\infty}$ *as* $Hu_n = \frac{n}{2(n^2+1)}$, *then we get*

$$0 \leq \lim_{n\to\infty} \|Hx_n - Hu_n\| \leq 0,$$

which implies that $\lim_{n\to\infty} \|Hx_n - Hu_n\| = 0$. *It is clear that* $\{Hu_n\}_{n=0}^{\infty}$ *is approximate sequence of* $\{Hx_n\}_{n=0}^{\infty}$ *according to Definition 1.8. By using* (3. 32) *and Definition 1.6, we obtain*

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} ||Hu_{n+1} - f(G, u_n)|| = 0.$$

Consequently, iterative sequence $\{Hx_n\}_{n=0}^{\infty}$ *is* (H, G)*-stable.*

4. Conclusion

In this work, considering from Theorem 2.1 and Theorem 2.2, the convergence of the sequences obtained the newly defined iteration method, iteration method (1. 4), and iteration method (1. 3), respectively, were examined under the same conditions. Along with that, the newly defined Jungck-Kirk type iteration method (2. 14) has been shown with numerical examples to have a better convergence rate than the others. Due to these features, it is seen that the iteration method (2. 14) has a good potential for possible works and Jungck-Kirk type iteration methods that will be defined.

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