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Curves Lying on Non-lightlike Surface: Differential Equation for Position Vector

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Abstract.: The main purpose of this study is to examine curves lying on a given non-lightlike surface with the help of its position vectors. For this purpose, the darboux frame is used and the position vector of the curve is expressed as a linear combination of the darboux frame with differentiable functions. Then, nonhomogeneous systems of differential equations revealed by the position vector of the curve are obtained for timelike and spacelike surfaces, respectively. For both timelike and spacelike surfaces, the solutions of nonhomogeneous systems of differential equations are obtained depending on the character of the curves and the values k_g, k_n and t_r. The general solutions of the systems of differential equations are obtained separately for each case. Moreover, by considering only the particular solution of the systems of differential equations, new results regarding the differential geometric structure of the curves on the surface are presented with the help of the position vector.

AMS (MOS) Subject Classification Codes: 34A26; 34A34; 53A04; 53A05
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1. Introduction

In the simplest terms, curves can only be thought of as a kind of deformation of straight lines. For this reason, we can treat curves as one-dimensional objects. We are familiar with the concept of curves from other areas of basic mathematics, because the graphs of functions are treated and studied as curves. However, we can express the coordinates of each point of the curve as functions of a parameter. From this point of view, it is the most preferred way of examining the local differential geometric structure of the curve. In many studies dealing with differential geometric properties of curves, some methods and tools of differential calculus are used. This review makes use of the well-known Frenet-Serret frame. Analyzing the geometric structures of curves with the help of vector analysis is very important in this context. Considering the position vectors of the curves from a completely different point of view may produce different results. Even in this context, there are many studies that express the position vector of curves as a linear combination of Frenet-Serret vector fields. If the position vector of a curve is expressed as a linear combination of Frenet-Serret frame fields of the curve, then the characterization of curve depends on solution of the first order system of differential equations [16, 15, 21, 6, 5].

In the literature, darboux frame has been studied in many work in Euclidean and non-Euclidean spaces. In study [17], the relations between the darboux and the Frenet-Serret frame on the ruled surface are examined and some theorems about the ruled surface with darboux frame are given. Also, in study [18], they have studied the Siacci's theorem for the curves on regular surfaces by darboux frame. The spacelike parallel ruled surfaces with darboux frame are introduced in Minkowski 3-space in study [**?**]. The ruled surfaces with a constant slope ruling according to darboux frame in Minkowski space are defined in study [22].

The position vector field also plays important roles in other fields besides geometry, such as physics and mechanics. In any equation of motion, the position vector field defines the motion of a particle. The first and the second derivatives of the position vector field with respect to time give the velocity and acceleration of the particle, respectively. In all aforementioned studies, the position vector of given curve is obtained by means of Frenet-Serret frame. As it is well known, if we want to treat the curve α as an independent object, using the Frenet-Serret frame is one of the most useful ways, but if the curve is lying on a surface, using the Frenet-Serret frame will not be sufficient for research since the differential geometric structure of the surface is also important. Therefore, to study a curve on a surface, it is necessary to introduce a new orthonormal frame containing vectors that reveal the structure of both the curve and the surface. We can take the first vector of the new frame that we will define as the unit tangent vector t of the curve. In order to take the structure of the surface and include it in the new frame, we can take the unit normal vector n of the surface as the second vector. The last one y, can be taken in such a way that $\{t, y, n\}$ forms an orthogonal frame, that is $[t, y, n] = 1$. This correspondence to $y = n \times t$. The new frame $\{t, y, n\}$ is called the darboux frame or the Ribaucour-Darboux frame of the surface M along the curve α. Similarly Frenet-Serret formulas, the derivative of darboux vector fields can be expressed in terms of themselves. At the same time, we can define the normal curvature, geodesic curvature and geodesic torsion of darboux frame. There are relations between geodesic curvature, normal curvature, geodesic torsion and κ and τ. If both surface and curve are timelike or spacelike, then we have

$$k_g(s) = \kappa(s) \cos\theta(s),$$
$$k_n(s) = \kappa(s) \sin\theta(s),$$

where $\theta(s)$ is the angle between the unit normal vector fields of curve and surface. If surface is timelike and curve is spacelike, then we have

$$k_g(s) = \kappa(s) \cosh\theta(s),$$
$$k_n(s) = \kappa(s) \sinh\theta(s),$$

where $\theta(s)$ is the hyperbolic angle between the unit normal vector fields of curve and surface. Furthermore, there is also a relation between geodesic torsion and the torsion functions of the curve as follows:

$$t_r(s) = \tau(s) + \theta'(s).$$

The main purpose of this study will be to examine the position vectors of the curves on the surface using the darboux frame. In this paper, nonlightlike curves $\alpha : I \to M$ lying on a non-lightlike surface M are investigated by its position vectors. For this investigation, it would be appropriate to consider the position vector of the curve as a linear combination of the darboux frame with differentiable functions. As a result, nonhomogeneous systems of differential equations revealed by the position vector of the curve are obtained for timelike and spacelike surfaces, respectively. For both timelike and spacelike surfaces, the solutions of nonhomogeneous systems of differential equations are obtained depending on the character of the curves and the values k_g, k_n and t_r. The general solutions of the systems of differential equations are obtained separately for each cases. Moreover, by considering only the particular solution of the systems of differential equations, new results regarding the differential geometric structure of the curves on the surface are presented with the help of the position vector.

2. Characterizations of Timelike Curves Lying on Timelike Surface

In this section, position vectors of timelike curves on timelike surfaces will be discussed. Let unit speed timelike curve $\alpha : I \to M$ on the timelike surface M be given. In this case, position vector of unit speed timelike curve $\alpha(s)$ is given by

$$\alpha\left(s\right) = r_0\left(s\right) t\left(s\right) + r_1\left(s\right) y\left(s\right) + r_2\left(s\right) n\left(s\right) \tag{1}$$

where r_0, r_1 and r_2 are some differentiable functions of $s \in I \subset \mathbb{R}$. Using the derivative formulas of the darboux vector fields, the following nonhomogeneous system of differential equations is obtained:

$$\begin{aligned} r_0'\left(s\right) &= -k_g(s) r_1\left(s\right) - k_n(s) r_2(s) + 1, \\ r_1'\left(s\right) &= -k_g(s) r_0\left(s\right) - t_r(s) r_2(s), \\ r_2'\left(s\right) &= k_n(s) r_0(s) - t_r(s) r_1(s) \end{aligned} \tag{2}$$

by the equality $\alpha'(s) = t(s)$. If we consider the case where this system of equations has constant coefficients, it is necessary to assume that the curvature functions k_g, k_n and t_r are constant. That is why it is supposed that the curvature functions k_g, k_n and t_r are nonzero constants unless otherwise stated.

There are two cases $k_g^2 + k_n^2 - t_r^2 > 0$ and $k_g^2 + k_n^2 - t_r^2 < 0$ depending on the eigenvalues of the coefficients matrix of the nonhomogeneous linear system of differential Equation 2.

Theorem 1. Assume that $\alpha : I \subset \mathbb{R} \to M$ is a given unit speed timelike curve on the timelike surface M. Then the position vector of the curve α is given by the differentiable

functions as follows:

$$
\begin{aligned}
r_0(s) &= \text{-}c_0t_r\text{+}c_1(k_gt_r\cosh(bs)\text{-}bk_n\sinh(bs)) + c_2(\text{-}bk_n\cosh(bs)\text{+}k_gt_r\sinh(bs))\text{-}\frac{t_r^2 s}{b^2},\\
r_1(s) &= \text{-}c_0k_n\text{+}c_1(k_gk_n\cosh(bs)\text{-}bt_r\sinh(bs))\text{+}c_2(\text{-}bt_r\cosh(bs)\text{+}k_gk_n\sinh(bs))\\
&\qquad \text{+}\frac{k_g\text{-}k_nt_rs}{b^2}, \qquad (3)\\
r_2(s) &= c_0k_g\text{+}c_1(k_n^2\text{-}t_r^2)\cosh(bs)\text{+}c_2(k_n^2-t_r^2)\sinh(bs)\text{+}\frac{k_gt_rs\text{+}k_n}{b^2},
\end{aligned}
$$

where $k_g^2 + k_n^2 - t_r^2 = b^2 > 0$, b is a nonzero real constant, c_0, c_1 and c_2 are arbitrary constants.

Proof. First of all, if Equation 2 is written in matrix form, then

$$
\begin{bmatrix} r_0'(s) \\ r_1'(s) \\ r_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & -k_g & -k_n \\ -k_g & 0 & -t_r \\ k_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} r_0(s) \\ r_1(s) \\ r_2(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \qquad (4)
$$

For the solution of above system of equations, we obtain the eigenvalues $\lambda_1 = 0$,$\lambda_2 = b$, $\lambda_3 = -b$ and the corresponding eigenvectors of the coefficients matrix as follows:

$$
V_1 = \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix}, V_2 = \begin{bmatrix} k_gt_r - k_nb \\ k_gk_n - t_rb \\ k_n^2 - t_r^2 \end{bmatrix}, V_3 = \begin{bmatrix} k_gt_r + k_nb \\ k_gk_n + t_rb \\ k_n^2 - t_r^2 \end{bmatrix},
$$

where $k_g^2 + k_n^2 - t_r^2 = b^2$, respectively. Therefore, the homogenous solution of Equation 4 can be stated as follows:

$$
\begin{aligned}
X_h(s) &= c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + d_1e^{bs} \begin{bmatrix} k_gt_r - k_nb \\ k_gk_n - t_rb \\ k_n^2 - t_r^2 \end{bmatrix} + d_2e^{-bs} \begin{bmatrix} k_gt_r + k_nb \\ k_gk_n + t_rb \\ k_n^2 - t_r^2 \end{bmatrix}\\
&= c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + d_1(\cosh(bs)\text{+}\sinh(bs)) \begin{bmatrix} k_gt_r - k_nb \\ k_gk_n - t_rb \\ k_n^2 - t_r^2 \end{bmatrix}\\
&\quad + d_2(\cosh(bs)\text{-}\sinh(bs)) \begin{bmatrix} k_gt_r + k_nb \\ k_gk_n + t_rb \\ k_n^2 - t_r^2 \end{bmatrix}
\end{aligned}
$$

where c_0, d_1, d_2 are arbitrary constants. If we rearrange the homogeneous solution and perform the necessary operations, then we get the homogenous solution as follows

$$
\begin{aligned}
X_h(s) = c_0 &\begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + c_1 \begin{bmatrix} k_gt_r\cosh(bs) - bk_n\sinh(bs) \\ k_gk_n\cosh(bs) - bt_r\sinh(bs) \\ (k_n^2 - t_r^2)\cosh x \end{bmatrix}\\
&+ c_2 \begin{bmatrix} -bk_n\cosh(bs) + k_gt_r\sinh(bs) \\ -bt_r\cosh(bs) + k_gk_n\sinh(bs) \\ (k_n^2 - t_r^2)\sinh(bs) \end{bmatrix}
\end{aligned}
$$

where $d_1 + d_2 = c_1$, $d_1 - d_2 = c_2$. This gives the fundamental matrix of Equation 4 as follows:

$$\varphi(s) = \begin{bmatrix} -t_r & k_g t_r \cosh(bs) - bk_n \sinh(bs) & -bk_n \cosh(bs) + k_g t_r \sinh(bs) \\ -k_n & k_g k_n \cosh(bs) - bt_r \sinh(bs) & -bt_r \cosh(bs) + k_g k_n \sinh(bs) \\ k_g & (k_n^2 - t_r^2)\cosh(bs) & (k_n^2 - t_r^2)\sinh(bs) \end{bmatrix}.$$

With the use of the equality $X_p(s) = \varphi(s)\, u(s)$, the vector valued function $u(s)$ is found by following equation

$$\varphi(s)\, u'(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Actually, solving the 3×3 linear equation by Crammer's method, the particular solution of the Equation 4 is found as follows:

$$X_p(s) = \frac{1}{b^2} \begin{bmatrix} -t_r^2 s \\ -k_n t_r s + k_g \\ k_g t_r s + k_n \end{bmatrix}.$$

By the homogeneous and particular solution of the equation, the general solution of the system of linear differential equation is expressed as follows:

$$\begin{aligned} r_0(s) &= -c_0 t_r + c_1(k_g t_r \cosh(bs) - bk_n \sinh(bs)) \\ &\quad + c_2(-bk_n \cosh(bs) + k_g t_r \sinh(bs)) - \frac{t_r^2 s}{b^2} \\ r_1(s) &= \text{-}c_0 k_n + c_1(k_g k_n \cosh(bs) - bt_r \sinh(bs)) \\ &\quad + c_2(-bt_r \cosh(bs) + k_g k_n \sinh(bs)) + \frac{k_g - k_n t_r s}{b^2}, \\ r_2(s) &= c_0 k_g + c_1(k_n^2 - t_r^2)\cosh(bs) \\ &\quad + c_2(k_n^2 - t_r^2)\sinh(bs) + \frac{k_g t_r s + k_n}{b^2}. \end{aligned}$$

It is suggested to the readers to see [3] for details of the methods of solving first order nonhomogeneous linear differential systems of equations. □

Theorem 2. Let $\alpha : I \subset \mathbb{R} \to M$ be a unit speed timelike curve on a given timelike surface M. Then the position vector of the curve α is given by the differentiable functions as follows:

$$
\begin{aligned}
r_0(s) &= -c_0 t_r + c_1(bk_n \sin(bs) + k_g t_r \cos(bs)) \\
&\quad + c_2(k_g t_r \sin(bs) - bk_n \cos(bs)) + \frac{t_r^2 s}{b^2}, \\
r_1(s) &= \text{-}c_0 k_n \text{+} c_1(bt_r \sin(bs) + k_g k_n \cos(bs)) \\
&\quad \text{+} c_2(k_g k_n \sin(bs) - bt_r \cos(bs)) \text{-} \frac{(k_g - sk_n t_r)}{b^2}, \\
r_2(s) &= c_0 k_g + c_1 \left(k_n^2 - t_r^2\right)\cos(bs) + c_2\left(k_n^2 - t_r^2\right)\sin(bs) - \frac{(k_n + sk_g t_r)}{b^2},
\end{aligned}
$$

where $k_g^2 + k_n^2 - t_r^2 = -b^2 < 0$, b is a nonzero real constant, c_0, c_1 and c_2 are arbitrary constants.

Proof. Since $k_g^2 + k_n^2 - t_r^2 = -b^2 < 0$, then the eigenvalues of the matrix in Equation 4 are complex numbers. In this case, we obtain the eigenvalues of the matrix as $\lambda_1 = 0$, $\lambda_2 = bi$ and $\lambda_3 = -bi$. Then the corresponding eigenvectors of the matrix are found as

$$
V_1 = \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix},\ V_2 = \begin{bmatrix} k_g t_r - k_n bi \\ k_g k_n - t_r bi \\ k_n^2 - t_r^2 \end{bmatrix},\ V_3 = \begin{bmatrix} k_g t_r + k_n bi \\ k_g k_n + t_r bi \\ k_n^2 - t_r^2 \end{bmatrix},
$$

respectively. The homogenous solution is obtained as follows:

$$
\begin{aligned}
X_h(s) &= c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + d_1 e^{bsi} \begin{bmatrix} k_g t_r - k_n bi \\ k_g k_n - t_r bi \\ k_n^2 - t_r^2 \end{bmatrix} + d_2 e^{-bsi} \begin{bmatrix} k_g t_r + k_n bi \\ k_g k_n + t_r bi \\ k_n^2 - t_r^2 \end{bmatrix} \\
&= c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + d_1(\cos(bs) + i\sin(bs)) \begin{bmatrix} k_g t_r - k_n bi \\ k_g k_n - t_r bi \\ k_n^2 - t_r^2 \end{bmatrix} \\
&\quad + d_2(\cos(bs) - i\sin(bs)) \begin{bmatrix} k_g t_r + k_n bi \\ k_g k_n + t_r bi \\ k_n^2 - t_r^2 \end{bmatrix}
\end{aligned}
$$

where c_0, d_1, d_2 are arbitrary constants. By rewriting the constants as follows

$$
d_1 + d_2 = c_1,\ d_1 i - d_2 i = c_2,
$$

the homogenous solution is given as

$$
\begin{aligned}
X_h(s) = c_0 &\begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} \text{+} c_1 \begin{bmatrix} bk_n \sin(bs) + k_g t_r \cos(bs) \\ bt_r \sin(bs) + k_g k_n \cos(bs) \\ \left(k_n^2 - t_r^2\right)\cos(bs) \end{bmatrix} \\
&\text{+} c_2 \begin{bmatrix} k_g t_r \sin(bs) - bk_n \cos(bs) \\ k_g k_n \sin(bs) - bt_r \cos(bs) \\ \left(k_n^2 - t_r^2\right)\sin(bs) \end{bmatrix}.
\end{aligned}
$$

The fundamental matrix is obtained as follows:

$$\varphi(s)=\begin{bmatrix} -t_r & bk_n\sin(bs)+k_gt_r\cos(bs) & k_gt_r\sin(bs)-bk_n\cos(bs) \\ -k_n & bt_r\sin(bs)+k_gk_n\cos(bs) & k_gk_n\sin(bs)-bt_r\cos(bs) \\ k_g & \left(k_n^2-t_r^2\right)\cos(bs) & \left(k_n^2-t_r^2\right)\sin(bs) \end{bmatrix}.$$

By using the same method, which is given in the proof of Theorem 1, we obtain the particular solution of Equation 4 as follows:

$$X_p(s)=\frac{1}{b^2}\begin{bmatrix} st_r^2 \\ -\left(k_g-sk_nt_r\right) \\ -\left(k_n+sk_gt_r\right) \end{bmatrix}.$$

Thus, the general solution of the system of linear differential equation is found as follows

$$\begin{aligned} r_0(s) &= -c_0t_r+c_1(bk_n\sin(bs)+k_gt_r\cos(bs)) \\ &\quad +c_2(k_gt_r\sin(bs)-bk_n\cos(bs))+\frac{1}{b^2}st_r^2, \\ r_1(s) &= \text{-}c_0k_n\text{+}c_1(bt_r\sin(bs)+k_gk_n\cos(bs)) \\ &\quad \text{+}c_2(k_gk_n\sin(bs)-bt_r\cos(bs))\text{-}\frac{1}{b^2}\left(k_g-sk_nt_r\right), \\ r_2(s) &= c_0k_g+c_1\left(k_n^2-t_r^2\right)\cos(bs) \\ &\quad +c_2\left(k_n^2-t_r^2\right)\sin(bs)-\frac{1}{b^2}\left(k_n+sk_gt_r\right). \end{aligned}$$

□

3. Characterizations of Spacelike Curve Lying on Timelike Surface

In this section, characterization of spacelike curve on a given timelike surface M is investigated. Let the position vector of unit speed spacelike curve be given by

$$\alpha(s)=p_0(s)\,t(s)+p_1(s)\,y(s)+p_2(s)\,n(s) \tag{5}$$

for some differentiable functions p_0, p_1 and p_2 of $s\in I\subset\mathbb{R}$. If we take the derivative of Equation 5 with respect to the arc length parameter and use derivative formulas of darboux frame, then we obtain the following equations:

$$\begin{aligned} p_0'(s) &= -k_g(s)p_1(s)-k_n(s)p_2(s)+1, \\ p_1'(s) &= -k_g(s)p_0(s)-t_r(s)p_2(s), \\ p_2'(s) &= k_n(s)p_0(s)-t_r(s)p_1(s) \end{aligned} \tag{6}$$

with the use of the equality $\alpha'(s)=t(s)$.

Theorem 3. Let M be a given timelike surface and $\alpha: I\subset\mathbb{R}\to M$ be a unit speed spacelike curve on M. Then the position vector of the curve α is given by the differentiable

functions as follows:

$$\begin{aligned}
p_0(s) &= -c_0 t_r + c_1(a k_n \sinh(as) - k_g t_r \cosh(as)) \\
&\quad + c_2(a k_n \cosh(as) - k_g t_r \sinh(as)) + \frac{t_r^2 s}{a^2}, \\
p_1(s) &= \text{-}c_0 k_n \text{+} c_1(a t_r \sinh(as) - k_n k_g \cosh(as)) \\
&\quad \text{+} c_2(-k_n k_g \sinh(as) + a t_r \cosh(as)) \text{+} \frac{k_n t_r s + k_g}{a^2}, \\
p_2(s) &= c_0 k_g + c_1(k_n^2 - t_r^2)\cosh(as) \\
&\quad + c_2(k_n^2 - t_r^2)\sinh(as) - \frac{k_g t_r s + k_n}{a^2},
\end{aligned}$$

where $k_g^2 - k_n^2 + t_r^2 = a^2 > 0$, a is is a nonzero real constant, c_0, c_1 and c_2 are arbitrary constants.

Proof. First of all, equations in 6 are written as follows:

$$\begin{bmatrix} p_0'(s) \\ p_1'(s) \\ p_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & -k_g & -k_n \\ -k_g & 0 & -t_r \\ k_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} p_0(s) \\ p_1(s) \\ p_2(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

The eigenvalues are obtained as $\lambda_1 = 0$, $\lambda_2 = a$, $\lambda_2 = -a$ and the corresponding eigenvectors of coefficient matrix in Equation 7 as follows:

$$V_1 = \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix},\ V_2 = \begin{bmatrix} -(k_g t_r - k_n a) \\ -(k_g k_n - t_r a) \\ k_n^2 - t_r^2 \end{bmatrix}, V_3 = \begin{bmatrix} -(k_g t_r + k_n a) \\ -(k_g k_n + t_r a) \\ k_n^2 - t_r^2 \end{bmatrix}.$$

where $k_g^2 - k_n^2 + t_r^2 = a^2 > 0$ and a is a nonzero real constant. Thus, the homogenous solution of Equation 7 is obtained as follows:

$$\begin{aligned}
X_h(s) &= c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + d_1 e^{as} \begin{bmatrix} -(k_g t_r - k_n a) \\ -(k_g k_n - t_r a) \\ k_n^2 - t_r^2 \end{bmatrix} + d_2 e^{-as} \begin{bmatrix} -(k_g t_r - k_n a) \\ -(k_g k_n - t_r a) \\ k_n^2 - t_r^2 \end{bmatrix} \\
&= c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + d_1(\cosh(as) + \sinh(as)) \begin{bmatrix} -(k_g t_r - k_n a) \\ -(k_g k_n - t_r a) \\ k_n^2 - t_r^2 \end{bmatrix} \\
&\quad + d_2(\cosh(as) - \sinh(as)) \begin{bmatrix} -(k_g t_r - k_n a) \\ -(k_g k_n - t_r a) \\ k_n^2 - t_r^2 \end{bmatrix}
\end{aligned}$$

where c_0, d_1, d_2 are arbitrary real constants. To express the coefficients more simply, let's assume the following equations:

$$d_1 + d_2 = c_1,\ d_1 - d_2 = c_2.$$

By rearranging the constants, the homogenous solution of Equation 7 is found as follows:

$$X_h(s) = c_0 \begin{bmatrix} -t_r \\ -k_n \\ k_g \end{bmatrix} + c_1 \begin{bmatrix} ak_n \sinh(as) - k_g t_r \cosh(as) \\ at_r \sinh(as) - k_n k_g \cosh(as) \\ (k_n^2 - t_r^2)\cosh(as) \end{bmatrix} + c_2 \begin{bmatrix} ak_n \cosh(as) - k_g t_r \sinh(as) \\ -k_n k_g \sinh(as) + at_r \cosh(as) \\ (k_n^2 - t_r^2)\sinh(as) \end{bmatrix}.$$

It can be seen that, the fundamental matrix of the nonhomogeneous of Equation 7 can be expressed as

$$\varphi(s) = \begin{bmatrix} -t_r & ak_n \sinh(as) - k_g t_r \cosh(as) & ak_n \cosh(as) - k_g t_r \sinh(as) \\ -k_n & at_r \sinh(as) - k_n k_g \cosh(as) & -k_n k_g \sinh(as) + at_r \cosh(as) \\ k_g & (k_n^2 - t_r^2)\cosh(as) & (k_n^2 - t_r^2)\sinh(as) \end{bmatrix}.$$

The particular solution of Equation 7 can be found by using the equality

$$\varphi(s)\, u'(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then we may use Crammer's method to find the derivative of the vector field $u(s)$. The particular solution of Equation 7 is found as

$$X_p(s) = \frac{1}{a^2} \begin{bmatrix} t_r^2 s \\ (k_n t_r s + k_g) \\ -(k_g t_r s + k_n) \end{bmatrix}$$

by making all the necessary calculations. Therefore, the general solution of Equation 7 is obtained as follows

$$\begin{aligned} p_0(s) &= -c_0 t_r + c_1(ak_n \sinh(as) - k_g t_r \cosh(as)) \\ &\quad + c_2(ak_n \cosh(as) - k_g t_r \sinh(as)) + \frac{t_r^2 s}{a^2}, \\ p_1(s) &= -c_0 k_n + c_1(at_r \sinh(as) - k_n k_g \cosh(as)) \\ &\quad + c_2(-k_n k_g \sinh(as) + at_r \cosh(as)) + \frac{k_n t_r s + k_g}{a^2}, \\ p_2(s) &= c_0 k_g + c_1(k_n^2 - t_r^2)\cosh(as) \\ &\quad + c_2(k_n^2 - t_r^2)\sinh(as) - \frac{k_g t_r s + k_n}{a^2}. \end{aligned}$$

□

Remark: Since

$$a^2 = k_g^2 - k_n^2 + t_r^2 = \kappa^2 + t_r^2$$

is always greater than zero, the position vector of the curve is obtained in only one case.

4. Characterizations of Spacelike Curve Lying on Spacelike Surface

In this section, we give characterization of spacelike curve on a given spacelike surface M. Assume that the position vector of unit speed spacelike curve α is given by

$$\alpha\left(s\right)=m_0\left(s\right)t\left(s\right)+m_1\left(s\right)y\left(s\right)+m_2\left(s\right)n\left(s\right)$$

for some differentiable functions m_0, m_1 and m_2 of $s\in I\subset\mathbb{R}$. By taking the derivative of the above equation with respect to the parameter s and using derivative formulas of darboux frame, we get the following equations:

$$\begin{aligned} m_0'\left(s\right)&=k_g(s)m_1\left(s\right)-k_n(s)m_2(s)+1,\\ m_1'\left(s\right)&=-k_g(s)m_0\left(s\right)-t_r(s)m_2(s),\\ m_2'\left(s\right)&=-k_n(s)m_0(s)-t_r(s)m_1(s).\end{aligned} \tag{8}$$

There are two cases $-k_g^2+k_n^2+t_r^2>0$ and $-k_g^2+k_n^2+t_r^2<0$ depending on the eigenvalues of the coefficients matrix of the nonhomogeneous linear system of differential Equation 8.

Theorem 4. Let M be a given spacelike surface and $\alpha:I\subset\mathbb{R}\to M$ be a unit speed spacelike curve on M. Then the position vector of the curve α is given by the differentiable functions as follows:

$$\begin{aligned} m_0(s)&=-c_0t_r-c_1\left(qk_n\sinh\left(qs\right)+k_gt_r\cosh\left(qs\right)\right)\\ &\quad-c_2\left(qk_n\cosh\left(qs\right)+k_gt_r\sinh\left(qs\right)\right)+\frac{t_r^2s}{q^2},\\ m_1(s)&=c_0k_n\text{-}c_2\left(qt_r\cosh\left(qs\right)-k_gk_n\sinh\left(qs\right)\right)\\ &\quad\text{-}c_1\left(qt_r\sinh\left(qs\right)-k_gk_n\cosh\left(qs\right)\right)+\frac{k_g-q^2sk_nt_r}{q^2},\\ m_2(s)&=c_0k_g+c_1\cosh\left(qs\right)\left(k_n^2+t_r^2\right)\\ &\quad+c_2\sinh\left(qs\right)\left(k_n^2+t_r^2\right)+\frac{k_n-q^2sk_gt_r}{q^2},\end{aligned}$$

where $-k_g^2+k_n^2+t_r^2=q^2>0$, q is a nonzero real constant, c_0,c_1 and c_2 are arbitrary constants.

Proof. First of all, we rewrite the nonhomogeneous linear differential system of equations in 8 as a matrix equation:

$$\begin{bmatrix} m_0'\left(s\right)\\ m_1'\left(s\right)\\ m_2'\left(s\right)\end{bmatrix}=\begin{bmatrix} 0 & k_g & -k_n\\ -k_g & 0 & -t_r\\ -k_n & -t_r & 0\end{bmatrix}\begin{bmatrix} m_0\left(s\right)\\ m_1\left(s\right)\\ m_2\left(s\right)\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Similarly to proof of previous theorem, the homogenous solution of Equation 8 is obtained as follows:

$$
\begin{aligned}
X_h = c_0 \begin{bmatrix} -t_r \\ k_n \\ k_g \end{bmatrix} &+ c_1 \begin{bmatrix} -\cosh(qs)\, k_g t_r - \sinh(qs)\, q k_n \\ \cosh(qs)\, k_g k_n - \sinh(qs)\, q t_r \\ \cosh(qs)\left(k_n^2 + t_r^2\right) \end{bmatrix} \\
&+ c_2 \begin{bmatrix} -\cosh(qs)\, q k_n - \sinh(qs)\, k_g t_r \\ -q t_r \cosh(qs) + \sinh(qs)\, k_g k_n \\ \sinh(qs)\left(k_n^2 + t_r^2\right) \end{bmatrix}
\end{aligned}
$$

where c_0, c_1, c_2 are arbitrary constants. It can be seen that, the fundamental matrix of the nonhomogeneous of Equation 8 can be expressed as

$$
\varphi(s) = \begin{bmatrix} -t_r & -\cosh(qs)\, k_g t_r - \sinh(qs)\, q k_n & -\cosh(qs)\, q k_n - \sinh(qs)\, k_g t_r \\ k_n & \cosh(qs)\, k_g k_n - \sinh(qs)\, q t_r & -q t_r \cosh(qs) + \sinh(qs)\, k_g k_n \\ k_g & \cosh(qs)\left(k_n^2 + t_r^2\right) & \sinh(qs)\left(k_n^2 + t_r^2\right) \end{bmatrix}.
$$

The particular solution of Equation 8 is found as follows:

$$
X_p(s) = \frac{1}{q^2} \begin{bmatrix} s t_r^2 \\ k_g - s k_n t_r \\ k_n - s k_g t_r \end{bmatrix}
$$

by making all the necessary calculations. Therefore, the general solution of Equation 8 is obtained as follows:

$$
\begin{aligned}
m_0(s) &= -c_0 t_r - c_1\left(q k_n \sinh(qs) + k_g t_r \cosh(qs)\right) \\
&\quad - c_2\left(q k_n \cosh(qs) + k_g t_r \sinh(qs)\right) + \frac{t_r^2 s}{q^2}, \\
m_1(s) &= c_0 k_n - c_2\left(q t_r \cosh(qs) - k_g k_n \sinh(qs)\right) \\
&\quad - c_1\left(q t_r \sinh(qs) - k_g k_n \cosh(qs)\right) + \frac{k_g - q^2 s k_n t_r}{q^2}, \\
m_2(s) &= c_0 k_g + c_1 \cosh(qs)\left(k_n^2 + t_r^2\right) \\
&\quad + c_2 \sinh(qs)\left(k_n^2 + t_r^2\right) + \frac{\left(k_n - q^2 s k_g t_r\right)}{q^2}.
\end{aligned}
$$

□

Theorem 5. Let M be a given spacelike surface and $\alpha : I \subset \mathbb{R} \to M$ be a unit speed spacelike curve on M. Then the position vector of the curve α is given by the differentiable

functions as follows:

$$\begin{aligned}
m_0(s) &= -c_0 t_r + c_1 \left(q k_n \sin(qs) - k_g t_r \cos(qs)\right) \\
&\quad + c_2 \left(q k_n \cos(qs) + k_g t_r \sin(qs)\right) - \frac{t_r^2 s}{q^2}, \\
m_1(s) &= c_0 k_n + c_1 \left(q t_r \sin(qs) + k_g k_n \cos(qs)\right) \\
&\quad + c_2 \left(q t_r \cos(qs) - k_g k_n \sin(qs)\right) + \frac{k_g + s k_n t_r}{q^2}, \\
m_2(s) &= c_0 k_g + c_1 \cos(qs) \left(k_n^2 + t_r^2\right) \\
&\quad - c_2 \sin(qs) \left(k_n^2 + t_r^2\right) + \frac{k_n + s k_g t_r}{q^2}
\end{aligned}$$

where $-k_g^2 + k_n^2 + t_r^2 = -q^2 < 0$, q is a nonzero real constant, c_0, c_1 and c_2 are arbitrary constants.

Proof. The proof can be done by similarly to other proofs. □

5. Conclusion

There are many studies in the literature dealing with the position vectors of curves. In most of these studies, the position vector is expressed as a linear combination of Serret Frenet frame. Recently, the curve whose position vector can be expressed with the help of Serret Frenet frame in n dimensional Euclidean space in [1]. Then similar discussions are done with use of parallel transport frame of Euclidean 3-space in [2]. Not only in Euclidean space, but also in Minkowski space, the curves are studied with position vectors. In this case, different studies have been carried out depending on the character (spacelike, timelike or null) of the curves. For example, the position vector of a spacelike curve is expressed by a linear combination of its Serret Frenet frame with differentiable functions in [21]. Since this study deals with the differential geometric structure of the curves on the surface, the more suitable darboux frame is used. Examination of the position vector of the curves on the surface with the darboux frame is a new method, and its applications in Euclidean space are discussed in study [24]. This study will accompany the scientists who will conduct new studies on curves on higher dimensional Minkowski space as a main source since it is one of the first studies on this subject.

According to all findings of this paper, we can summarize the results on the characterization of nonlightlike curves on nonlightlike surface. In the homogeneous solution of the differential equation systems in Equations 2, 6 and 8, coefficients c_0, c_1 and c_2 appear which are arbitrary coefficients. Of course, there is no requirement that all coefficients are zero. But we can consider the case, where the homogeneous solution is trivial. This means that we may consider only particular solutions of Equations in 3, 6 and 8. Furthermore, following table shows the differences that will occur in each different casual character of the curve and surface:

Character of the surface M	Character of the curve $\alpha: I \to M$	t(s) component of position vector	$y(s)$ component of position vector	n(s) component of position vector	Values of k_g, k_n and t_r
Timelike	Timelike	$-\frac{t_r^2 s}{b^2}$	$\frac{k_g - k_n t_r s}{b^2}$	$\frac{k_g t_r s + k_n}{b^2}$	$k_g^2(s) + k_n^2(s) - t_r^2(s) = b^2 > 0$
		$\frac{t_r^2 s}{b^2}$	$-\frac{(k_g - s k_n t_r)}{b^2}$	$-\frac{(k_n + s k_g t_r)}{b^2}$	$k_g^2(s) + k_n^2(s) - t_r^2(s) = -b^2 < 0$
	Spacelike	$\frac{t_r^2 s}{a^2}$	$\frac{k_n t_r s + k_g}{a^2}$	$\frac{k_n t_r s + k_g}{a^2}$	$k_g^2(s) - k_n^2(s) + t_r^2(s) = a^2 > 0$
Spacelike	Spacelike	$\frac{t_r^2 s}{q^2}$	$\frac{k_g - q^2 s k_n t_r}{q^2}$	$\frac{k_n - q^2 s k_g t_r}{q^2}$	$-k_g^2(s) + k_n^2(s) + t_r^2(s) = q^2 > 0$
		$-\frac{t_r^2 s}{q^2}$	$\frac{k_g + s k_n t_r}{q^2}$	$\frac{k_n + s k_g t_r}{q^2}$	$-k_g^2(s) + k_n^2(s) + t_r^2(s) = -q^2 < 0$

FIGURE 1. Characterizations of curves on surface with components of position vectors according to darboux frame

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