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# A Modified Proximal Point Algorithm for Finite Families of Minimization Problems and Fixed Point Problems of Asymptotically Quasi-nonexpansive Multivalued Mappings

Imo Kalu Agwu
Department of Mathematics,
Micheal Okpara University of Agriculture,
Umudike, Umuahia Abia State, Nigeria
Email: agwuimo@gmail.com

Donatus Ikechi Igbokwe
Department of Mathematics,
Micheal Okpara University of Agriculture,
Umudike, Umuahia Abia State, Nigeria

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**Abstract.**:ln this paper, a new iterative algorithm for finding common elements of the set of fixed points for a finite family of asymptotically quasi-nonexpansive multivalued mappings and the set of minimizers for a finite family of minimization problem is constructed. Under mild conditions on the control sequences, strong convergence of our algorithm was achieved without necessarily imposing any compactness condition on the space or the operator by using an independent approach. Our results improve, extend and generalize many important results recently announced in current literature.

**Key Words:** Strong convergence, Variational inequality, Asymptotically quasi-nonexpansive multivalued mapping, Modified proximal point algorithm, Common fixed point, Hilbert space.

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## 1. Introduction

Let $H$ be a real Hilbert space. In this paper, we denote by $\langle ,.,\rangle$, $\|.\|$ and $K$, the inner product on $H$, norm on $H$ and a nonempty, closed and convex subset of $H$, respectively. Also, if $\{x_n\}$ is any sequence in $H$, then we denote the strong and weak convergence of $\{x_n\}$ by $\rightarrow$ and $\rightharpoonup$, respectively; a set of natural numbers will be represented with $\mathbb{N}$.

Let $V : K \longrightarrow K$ be a nonlinear mapping whose domain and range are $D(V)$ and $R(V)$, respectively. The set of fixed points of $V$ will be denoted by $F(V) = \{x \in K :$

$x = Vx\}$ while the set of common fixed points of the mapping $\{V_i\}_{i=1}^m : K \longrightarrow K$ will be denoted by $\cap_{i=1}^m F(V_i)$. Recall that $V$ is:

$(a)$ Lipschitizian if there exists a constant $\beta$ such that

$$\|Vx - Vy\| \leq \beta\|x - y\|, \forall x, y \in D(V), \tag{1. 1}$$

where $\beta \geq 0$ is the Lipschitizian constant of $V$. Note that if $\beta \in [0.1)$ in (1. 1 ), then $V$ is a contraction while $V$ is called nonexpansive if $\beta = 1$ in (1. 1 ).

$(b)$ asymptotically nonexpansive $(see$ $[17], [1])$ if for all $x, y \in D(V)$, there exists a sequence $\{\mu_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} \mu_n = 1$ such that

$$\|V^n x - V^n y\| \leq \mu_n\|x - y\|, \forall n \in \mathbb{N}. \tag{1. 2}$$

$(c)$ uniformly Lipschitizian if there exists a constant $\beta \geq 0$ such that

$$\|V^n x - V^n y\| \leq \beta\|x - y\|, \forall x, y \in D(V), \tag{1. 3}$$

$(d)$ asymptotically quasi-nonexpansive if $F(V) \neq \emptyset$ and $(b)$ is satisfied; that is, if $F(V) \neq \emptyset$ and for all $(x \times q) \in D(V) \times F(V)$, there exists a sequence $\{\mu_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} \mu_n = 1$ such that

$$\|V^n x - V^n q\| \leq \mu_n\|x - q\|, \forall n \in \mathbb{N}. \tag{1. 4}$$

Whereas the class of uniformly Lipschitizian mapping is a superclass of the classes of nonexpansive mapping and asymptotically nonexpansive mapping, the class of asymptotically quasi-nonexpansive mapping is a superclass of the classes of asymptotically nonexpansive mapping and quasi-nonexpansive mapping (Recall that a nonlinear mapping $V : K \longrightarrow K$ is called quasi-nonexpansive if $F(V) \neq \emptyset$ and $\forall(x \times q) \in D(V) \times F(V)$, we have $\|Vx - Vq\| \leq \beta\|x - q\|)(see$ $[45] :$ $Examples$ $[4.1, 4.3$ $and$ $4.9]$ $for$ $details)$ . A nonlinear mapping $V : H \longrightarrow H$ with domain $D(V) \subset H$ and range $R(V) \subset H$ is called:

$(e)$ strongly positive bounded linear operator if there exists a constant $\alpha > 0$ such that the inequality

$$\langle Vt, t\rangle_H \geq \alpha\|t\|^2, \forall t \in H \tag{1. 5}$$

holds.

$(f)$ monotone if

$$\langle Vx - Vy, x - y\rangle \geq 0, \forall x, y \in D(V). \tag{1. 6}$$

$(g)$ $\alpha$-strongly monotone if there exists $\alpha > 0$ such that

$$\langle Vx - Vy, x - y\rangle \geq \alpha\|x - y\|^2, \forall x, y \in D(V). \tag{1. 7}$$

$(h)$ $\delta$-inverse-strongly monotone (for short $\delta$-ism) such that

$$\langle Vx - Vy, x - y\rangle \geq \alpha\|Vx - Vy\|^2, \forall x, y \in D(V). \tag{1. 8}$$

If $V$ is nonexpansive, then the map $I - V$ is monotone. It is worthy to note that the projection operator $P_K$ is $1 - ism$. Inverse-strongly monotone, otherwise known as co-oercive, operators have been intensively employed in solving practical problems in such an important area as traffic assignment problems (see [3], [19] for further details) Fixed points for single-valued mappings have been a subject of major concern and different methods have been used to address this issue. Since exact solution is difficult to attain, approximation via iteration scheme becomes an indispensable tool for the solution of fixed point problems. Let $f$ be a contraction map on $H$. Starting from an initial point $x_1 \in H$, define the sequence $\{x_n\}$ iteratively as follows

$$x_{n+1} = \tau_n f(x_n) + (1 - \tau_n)Tx_n, n \geq 0, \tag{1. 9}$$

where $\{\tau_n\}$ is a sequence in $(0, 1)$ and $T$ is a nonexpansive mapping in $H$. The iteration sequence (1. 9 ) was first introduced by Moudafi [32]and has been gainfully used in approximating fixed points of different nonlinear mappings in recent times (see [49], [50] and the reference therein for further study).

Iteration scheme for the fixed point of nonexpansive mappings has been extensively investigated mainly because of the intimate connection between nonexpansive mappings and monotonicity methods. In this direction, Marino and Xu [34] and Xu [50] discovered that iterative method for nonexpansive mappings could be used to solve convex minimization problem. To be precise, it was shown in [50] that a typical minimization problem of a quadratic function of the form:

$$min_{z\in F(T)} \frac{1}{2}\langle Vz, z\rangle - \langle b, z\rangle \tag{1. 10}$$

over the set of fixed points for nonexpansive mappings in real Hilbert space could be solved using the iteration scheme

$$x_{n+1} = \alpha_n b + (1 - \alpha_n V)Tx_n, n \geq 0,$$

where $T$ is a nonexpansive mapping and $V$ is a strongly positive bounded linear operator (Recall from Definition 1.2 [(e) and (g)] that a strongly bounded linear operator is a $\|V\|$-Lipschitizian and $\alpha$-strongly monotone operator).

Inspired by the results in [32], Xu [49] generalized (1. 9 ) as follows: Let $f$ be a contraction on $H$, $T$ is a nonexpansive mapping on $H$ and $V : H \longrightarrow H$ be a strongly positive bounded linear operator. Let $\{s_n\}$ be the sequence generated from an arbitrary point $s_0 \in H$ such that

$$s_{n+1} = \alpha_n \gamma f(s_n) + (1 - \alpha_n V)Ts_n, n \geq 0, \tag{1. 11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$. He showed that (1. 11 ) converges strongly to the fixed point of $T$, which at the same time serves as an answer to the variational inequality problem below:

$$\langle Vs^\star - \gamma f(s^\star), s^\star - q\rangle \leq 0, \forall q \in F(T). \tag{1. 12}$$

Passing onto multivaued mapping, there has been a concentrated efforts in the evaluation of fixed points for nonlinear multivalued mappings. This special interest is believed to have come from the various practical applications of multivalued mappings. For instance, a monotonic operator in optimization theory is the multivalued mapping of the subdifferential of the function $g$, $\partial g : D(g) \subseteq H \longrightarrow 2^H$ and is defined by

$$\partial g = \{s \in H : g(z) \geq g(t) + \langle s, z - t\rangle, \forall z \in K,$$

and $0 \in \partial g(t)$ satisfies the condition

$$\langle t - z, 0\rangle = 0 \leq g(t) - g(z), \forall z \in K.$$

In particular, if $g : K \longrightarrow \mathbb{R}$ is a convex, continuously differentiable function, then $A = \bigtriangledown g$, the gradient is a subdifferential which is single-valued mapping and the condition $\bigtriangledown g(t) = 0$ is an operator equation, $\langle \bigtriangledown g(t), t - z\rangle \geq 0$ is variational inequalities and both conditions are closely related to optimality conditions. Hence, finding fixed points or common fixed points for multivalued mapping is an important area in applications. However, we have noticed (with concern) that fewer iteration schemes, especially in the direction of asymptotically nonlinear multivalued mappings, does exists.

Let $(X, \rho)$ be a metric space, $D$ a nonempty subset of $X$ and $V : D \longrightarrow 2^D$ be a multivalued mapping. A point $s \in D$ is said to be a fixed point of $V$ if $s \in Vs$. The fixed point set of $V$ is denoted by $F(V) = \{s \in D : s \in Vs\}$. Let $CB(X), KC(X)$ and $P(X)$ represent the family of closed and bounded subset of $X$, the family of nonempty compact and convex subset of $X$ and the family of proximinal subset of $X$, respectively. A subset $D$ of $X$ is called proximinal if for each $s \in X$, there exists a point $k \in D$ for which (1. 13 ) holds.

$$\rho(s, k) = \inf\{\|s - t\| : t \in D\} = \rho(s, D), \tag{1. 13}$$

where $\rho(s, t) = \|s - t\|, \forall s, t \in X$. It is known that every nonempty closed and convex subset of a real Hilbert is proximinal.

Let $Q, W \in CB(D)$, the Hausdorff metric $H$ induced by the metric $\rho$ is defined as

$$H(Q, W) = max\{\sup_{s\in Q} \rho(s, W), \sup_{t\in W}(t, Q)\}$$

Recall that a multivalued mapping $V : D(V) \subseteq X \longrightarrow CB(X)$ is called:

$(1)$ $\beta$-Lipschitizian if there exists $\beta > 0$ such that

$$H(Vs, Vt) \leq \beta\|s - t\|, \forall s, t \in D(V). \tag{1. 14}$$

Note that in (1. 14 ), $V$ is a contraction if $\beta \in (0, 1)$ and nonexpansive if $\beta = 1$.

$(2)$ uniformly Lipschitizian if there exists $\beta > 0$ such that

$$H(V^n s, V^n t) \leq \beta\|s - t\|, \forall s, t \in D(V), \forall n \geq 1. \tag{1. 15}$$

$(3)$ asymptotically nonexpansive if for all $x, y \in D(V)$, there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \mu_n = 1$ such that

$$H(V^n s, V^n t) \leq \mu_n\|s - t\|, \forall n \geq 1. \tag{1. 16}$$

$(4)$ asymptotically quasi-nonexpansive [49] if $F(V) \neq \emptyset$ and $(3)$ holds; that is, if $F(V) \neq \emptyset$ and for all $(s \times q) \in D(V) \times F(V)$, there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \mu_n = 1$ such that

$$H(V^n s, V^n q) \leq \mu_n\|s - q\|, \forall n \geq 1. \tag{1. 17}$$

Every asymptotically nonexpansive multivalued mapping properly includes nonexpansive multivalued mappings and contraction multivalued mappings. Also, the class of asymptotically quasi-nonexpansive multivalued mapping is a superclass of the classes of asymptotically nonexpansive multivalued mappings and quasi-noexpansive multivalued mappings (Recall that a multivalued mapping $V : D(V) \subseteq E \longrightarrow CB(E)$ is called quasi-nonexpansive (a superclass of the class of nonspreading-type multivalued mapping) if $F(V) \neq \emptyset$ and for all $(x \times q) \in D(V) \times F(V)$, we have $H(Vx, Vq) \leq \|x - q\|$. Also, $V$ is called nonspreading-type if the inequality $2H(Vx, Vy)^2 \leq \rho(x, Vy)^2 + \rho(y, Vx)^2$ holds $\forall x, y \in D(V)$. Infact, every nonspreading-type multivalued mapping with nonempty fixed point set is quasi-nonexpansive). Minimization problem, an invaluable problem in application, especially in the area of optimization and nonlinear analysis, is defined as follows: find $t \in H$ such that

$$g(t) = min_{z\in H} g(z), \tag{1. 18}$$

where $g : H \longrightarrow (-\infty, +\infty)$ is a proper, convex and lower semicontinuous function. Note that problem (1. 18 ) is consistent if it has a solution. The set of all solutions (minimizers) of $g$ on $H$ is defined as $argmin_{z\in H} g(t)$.

Let $g : H \longrightarrow (-\infty, +\infty)$ be a proper, convex and lower semicontinuous function. Starting from an arbitrary point $x_1 \in H$, define the iteration scheme $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in H \\ x_{n+1} = argmin_{u\in H}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], n \in \mathbb{N}. \end{cases} \tag{1. 19}$$

The algorithm (1. 19 ) for solving problem (1. 18 ) was first introduced in 1970 by Martinet [39], and was called proximal point algorithm (for short, PPA). In recent times, many researchers have studied and generalised (1. 19 ) and many interesting results have been obtained for different classes of nonlinear single-valued and multivalued mappings: Rockfeller [41] solved problem (1. 18 ) using (1. 19 ); Marino and Xu [34], and subsequently Phuengrattan and Lerkchaiyaphum [33], obtained weak and strong convergence to the common solution of minimzation problem and fixed point problem using the modified version of (1. 19 ) in the setting of real Hilbert spaces

More recently, Chang,Wu and Wang [9], using the scheme

$$\begin{cases} x_1 \in H; \\ y_n = argmin_{u\in H}\Big[f(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big]; \\ z_n = (1-\beta_n)x_n + \beta_n w_n, w_n \in Ty_n; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n, v_n \in Tz_n, n \in \mathbb{N}, \end{cases} \tag{1. 20}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0, 1], K$ is a closed and convex subset of a real Hilbert space $H$ and $g : K \longrightarrow (-\infty, +\infty)$ is a proper convex and lower semicontinuous function, proved that (1. 20 ) converges weakly and strongly to the common solutions of the minimization problem of (1. 18 ) and fixed point problem of nonspreading-type multivalued mapping $T$ in the framework of a real Hilbert space $H$.

Most recently, EL-yekheir, Mendy and Sow [15] introduced and studied the following modified PPA: Let $K$ be a closed and convex subset of a real Hilbert space $H$, $f : K \longrightarrow (-\infty, +\infty)$ a proper convex and lower semicontinuous function, $g : K \longrightarrow H$ an $d$-Lipschitizian mapping, $B : K \longrightarrow H$ an $\alpha$-strongly monotone and $L$-Lipschitizian operator and $T : K \longrightarrow CK(K)$ a multivalued quasi-nonexpansive mapping. The PPA-Ishikawa iteration method is defined as follows:

$$\begin{cases} x_1 \in H; \\ z_n = argmin_{u\in H}\Big[f(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big]; \\ y_n = (1-\beta_n)z_n + \beta_n w_n, w_n \in Tz_n; \\ x_{n+1} = P_K(\alpha_n\gamma g(x_n) + (1-\eta\alpha_n B)y_n), n \in \mathbb{N}, \end{cases} \tag{1. 21}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Under mild conditions on the iteration parameters, they proved that the sequence defined by (1. 21 ) converges strongly to the fixed point of a quasi-nonexpansive multivalued mapping in the framework of a real Hilbert space.

Inspired and moltivated by the results in [15] and [34], it is natural to ask the following question:

**Question 1.1.** *Can we construct a modified proximal point algorithm that generalizes* (1. 21 )*? If yes, can the proposed modified PPA be used to achieve convergence results for a larger class of asymptotically quasi-nonexpansive mappings in the setting of real Hilbert spaces?*

It is our purpose in this paper to give an affirmative answer to Question 1.1. Let $K$ be a closed and convex subset of a real Hilbert space $H$, $\{g_i\}_{i=1}^m : K \longrightarrow (-\infty,+\infty)$ a finite family of a proper convex and lower semicontinuous function and $\{T_i\}_{i=1}^m : K \longrightarrow CB(K)$ be a finite family of asymptotically quasi-nonexpansive multivalued mappings. Then, the modified PPA iteration scheme generated by $\{x_n\}$ for the above mentioned mappings is as follows:

$$\begin{cases} x_0 \in K; \\ s_n = W_\lambda^m(x); \\ y_n = (1-\sum_{i=1}^m \beta_{n,i})s_n + \sum_{i=1}^m \beta_{n,i} z_{n,i}, z_{n,i} \in T_i^n s_n; \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n), \end{cases} \tag{1. 22}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $W_\lambda^l(x) = J_\lambda^{g_l} \circ J_\lambda^{g_{l-1}} \circ J_\lambda^{g_{l-2}} \circ \cdots \circ J_\lambda^{g_2} \circ J_\lambda^{g_1}(x), l = 1,2,\cdots,m.$ Observe that if :

$(I)$ i=1 in (1. 22 ), we get

$$\begin{cases} x_0 \in K; \\ s_n = argmin_{v\in H}\Big[g(v) + \|v - x_n\|^2\Big]; \\ y_n = (1-\beta_n)s_n + \beta_n z_n, z_n \in T^n s_n; \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n),. \end{cases} \tag{1. 23}$$

$(II)$ i=1, $\gamma_n = 0 = \beta_n$ and $T^n = T$ in (1. 22 ), we get

$$\begin{cases} x_0 \in K; \\ s_n = argmin_{v\in H}\Big[g(v) + \|v - x_n\|^2\Big]; \\ y_n = (1-\beta_n)s_n + \beta_n z_n, z_n \in T s_n; \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + (I - \eta\alpha_n A)y_n). \end{cases} \tag{1. 24}$$

$(III)$ $\alpha_n = 0$ in (1. 24 ), we get

$$\begin{cases} x_0 \in K; \\ s_n = argmin_{v\in H}\Big[g(v) + \|v - x_n\|^2\Big]; \\ x_{n+1} = (1-\beta_n)s_n + \beta_n z_n, z_n \in T s_n. \end{cases} \tag{1. 25}$$

$(IV)$ $\beta_n = 0$ in (1. 25 ), we get

$$\begin{cases} x_0 \in K; \\ x_{n+1} = argmin_{v\in H}\Big[g(v) + \|v - x_n\|^2\Big]; \\ \end{cases} \tag{1. 26}$$

$(V)$ i=1, $W_\lambda^{g_l} = I$ (where I is an identity map on $H$) and $\beta_n = 0$ in (1. 22 ), we get, starting from an arbitrary point $x_0 \in H$,

$$x_{n+1} = (\alpha_n \gamma f(x_n) + (I - \eta\alpha_n A)Tx_n \tag{1. 27}$$

Note that (1. 23 ) generalizes (1. 21 ), (1. 20 ) and many other iteration schemes in this direction; (1. 24 ) is the same as (1. 21 ), which in turn generalizes (1. 20 ) since $T$ is a multivalued quasi-nonexpansive mapping; (1. 25 ) generalizes (1. 19 ), (1. 11 ) and (1. 9 ) and finally, (1. 27 ) generalizes (1. 9 ).

## 2. Preliminary

**Assumptiom 2.1:**
Throughout the remaining sections, $H$, $K$, $g : H \longrightarrow (-\infty, \infty]$, the operator $A : K \longrightarrow H$ and the function $g : H \longrightarrow (-\infty, +\infty)$ shall represent, a real Hilbert space, a non-empty, closed and convex subset of $H$, $L$-Lipschitizian and $\alpha$-strongly monotone operator and proper, convex and lower semicontinuous function, respectively.

Also, for the sake of convenience, we restate the following concepts and results:
Let $H$ and $K$ be defined as in Assumption 2.1. For every $t \in H$, there exists a unique nearest point in $K$, represented as $P_K t$, such that

$$\|t - P_K t\| \leq \|t - s\|, \forall s \in K$$

It has been established that for every $t \in H$,

$$\langle t - P_K t, s - P_K t\rangle \leq 0, \forall s \in K. \tag{2. 28}$$

Let $E$ be a real Banach space and $T : D(T) \subseteq E \longrightarrow 2^E$ a multivalued mapping. Then, $I - T$ is said to be weakly demiclosed at the origin (see, e.g.,$[14, 26]$) if for any sequence $\{t_n\}_{n=0}^{\infty} \subseteq D(T)$ such that $\{t_n\}$ converges weakly to $q$ and a sequence $s_n$ with $s_n \in Tt_n$ for all $n \in \mathbb{N}$ such that $\{t_n - s_n\}$ converges strongly to zero, then $q \in Tq$. $(see$ [23]$)$ A multivalued mapping $V : D \longrightarrow 2^D$ is $H$-continuous if whenevever $\{s_n\}$ converges to $x$ in $D$, we have

$$\lim_{n\to\infty} d(q_n - Tx) = 0,$$

for any sequence $\{q_n\}$ such that $q_n \in Ts_n,\ for\ n \in \mathbb{N}$. Note that if $V : D \longrightarrow 2^D$ is asymptotically nonexpansive, then $V$ is $H$-continuous. Given a closed convex subset $K$ of a Hilbert space $H$, a mapping $T : K \longrightarrow H$ is firmly nonexpansive if for all $s, t \in K$,

$$\|Ts - Tt\|^2 \leq \langle s - t, Ts - Tt\rangle.$$

**Lemma 2.1.** *(see, e.g.,* [11] *) Let $H$ be defined as in Assumption 2.1. Let $U$ be a nonempty closed and convex subset of $H$. Let $G : U \longrightarrow K(U)$ be a nonspreading-type multivalued mapping. Let $\{x_n\}$ be a sequence on $U$ such that $x_n \rightharpoonup q$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Gx_n$. Then, $q \in Gq$.*

**Lemma 2.2.** *(see, e.g.,* [10], [15]*) Let $H$ be defined as in Assumption 2.1. Then, for every $s, t \in H$ and for every $\mu \in [0, 1]$, the following inequality holds*

$(i)$

$$\|s - t\|^2 \leq \|s\|^2 + 2\langle t, s + t\rangle$$

$(ii)$

$$\|\mu x + (1 - \mu)y\|^2 \leq \mu\|s\|^2 + (1 - \mu)\|t\|^2 - \mu(1 - \mu)\|s - t\|^2$$

**Lemma 2.3.** *(see, e.g.,* [47]*) Let $\{a_n\}$ be a sequence of nonnegative real numbers with $a_{n+1} = (1 - \alpha_n)a_n + b_n, n \geq 0$, where $\alpha_n$ is a sequence in $(0, 1)$ and $b_n$ is a sequence in $R$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \dfrac{b_n}{\alpha_n} \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

**Lemma 2.4.** *(see e.g.,* [15], [46] *) Let $H$, $A$, and $K$ be defined as in Assumption 2.1 with $\alpha, L > 0$. Assume that $\dfrac{2k}{L^2} > \eta > 0$ and $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. Then, we have*

$$\|(I - s\eta A)x - (I - s\eta A)y\| \leq (1 - t\tau)\|x - y\|, \forall x, y \in K.$$

**Lemma 2.5.** *(see ,e.g.,*[15], [36] *) Let $H$, $K$ and $g$ be defined as in Assumption 2.1. Then, for $0 < r$ and $0 < \mu$ , the following equality holds:*

$$J_r^g x = J_\mu^g x\Big(\frac{\mu}{r}x + (1 - \frac{\mu}{r})J_r^g x\Big)$$

**Lemma 2.6.** *( Subdiffential inequality, see* [2]*) Let $H$ and $g$ be defined as in Assumption 2.1. Then, for every $x, y \in H$ and $\lambda > 0$, the following subdifferential inequality hods*

$$\frac{1}{2\lambda}\|J_\lambda x - y\|^2 - \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{2\lambda}\|x - J_\lambda x\|^2 \leq g(y) - g(J_\lambda x). \quad (2.\ 29)$$

**Lemma 2.7.** *(see, e.g.,* [15], [37] *) Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $s_{n_k}$ of $\{s_n\}$ such that $s_{n_k} \leq s_{n_k} + 1$ for $k \geq 0$. For $n \in \mathbb{N}$, sufficiently large, define the sequence of integers $\tau(n)$ as follows*

$$\tau(n) = max\{j \leq n : s_j \leq s_j + 1\}$$

*Then, $\tau(n) \to \infty$ as $n \to \infty$ and*

$$max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)} + 1 \quad (2.\ 30)$$

**Proposition 2.8.** $(see$ [42]) *Let $D$ be a nonempty subset of a real Hilbert space $H$. For a mapping $T : D \longrightarrow H$, the following definitions are equivalent:*

$(a)$ *$T$ is firmly nonexpansive;*
$(b)$ *$2T - I$ is nonexpansive, where $I$ is the identity map on $H$;*
$(c)$ *$T = \dfrac{1}{2}(I + S)$ with $S$ nonexpansive;*
$(d)$ *$0 \leq \langle Tx - Ty, (I - Tx)x - (I - T)y\rangle, \forall x, y \in D$;*
$(e)$ *$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - Tx) - (y - T)y\|^2, \forall x, y \in D$.*

**Lemma 2.9.** *Let $g$ be defined as in Assumption 2.1. For any $\lambda > 0$, define the Morea-Yosida resolvent of $g$ in Hilbert space $H$ as*

$$J_\lambda^g(x) = argmin_{y\in H}\Big[g(y) + \frac{1}{2\lambda}\|y-x\|^2\Big], \forall x \in H. \tag{2. 31}$$

*Then,*

$(i)$ *The set $F(J_\lambda^g)$ of fixed points of the resolvent of $g$ coincides with the set $argmin_{y\in H}$ of minimizers of $g$ (see $[14]$ and $[22]$ for details), and for any $\lambda > 0$, the resolvent $J_\lambda^g$ of $g$ is firmly nonexpansive mapping, and hence nonexpansive $[21]$.*

$(ii)$ *Since $J_\lambda^g$ is firmly nonexpansive mapping, if $F(J_\lambda^g) \neq \emptyset$, then from Proposition 2.9, we have*

$$\|J_\lambda^g x - q\|^2 \le \|x-q\|^2 - \|J_\lambda^g x - x\|^2, \forall x \in H, q \in F(J_\lambda^g) \tag{2. 32}$$

**Proposition 2.10.** $(see\quad [48])$ *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $T : C \longrightarrow C$ be a nonexpasive mapping. The map $I - T$ is demiclosed if and only if the following implication holds:*

$$((x_n) \subset C, x_n \rightharpoonup x, x_n - Tx_n \to y) \Rightarrow x - Tx = y.$$

*This is called the demiclosedness principle for nonexpansive mappings, which holds in the following spaces:*

$(i)$ *uniformly convex Banach space;*

$(ii)$ *Banach spaces satisfying Opial's property.*

**Proposition 2.11.** *Let $K$ be a nonempty, close and convex subset of a real Hilbert space $H$ and let $W_\lambda^m$ be as defined by* (3. 38 )*. Then, $F(W_\lambda^m) = \cap_{i=1}^m F(J_\lambda^{g_i})$.*

*Proof.* It is clear that $\cap_{i=1}^m F(J_\lambda^{g_i}) \subseteq F(W_\lambda^m)$. So, it remains to show that

$$F(W_\lambda^m) \subseteq \cap_{i=1}^m F(J_\lambda^{g_i}). \tag{2. 33}$$

Let $x^\star \in F(W_\lambda^m)$ and $y^\star \in \cap_{i=1}^m F(J_\lambda^{g_i})$. Then, we have

$$\begin{aligned}
\|x^\star - y^\star\| &= \|W_\lambda^m x^\star - y^\star\| \\
&= \|J_\lambda^{g_m} W_\lambda^{m-1} x^\star - J_\lambda^{g_m} y^\star\| \\
&\le \|W_\lambda^{m-1} x^\star - y^\star\| \\
&= \|J_\lambda^{g_{m-1}} W_\lambda^{m-2} x^\star - J_\lambda^{g_{m-1}} y^\star\| \\
&\le \|W_\lambda^{m-2} x^\star - y^\star\| \\
&\vdots \\
&\le \|W_\lambda^1 x^\star - y^\star\| \\
&= \|J_\lambda^{g_1} x^\star - y^\star\| \\
&\le \|x^\star - y^\star\|.
\end{aligned}$$

The last inequality implies that

$$\|x^\star - y^\star\| = \|W_\lambda^m x^\star - y^\star\| = \|W_\lambda^{m-1} x^\star - y^\star\| = \cdots = \|W_\lambda^1 x^\star - y^\star\| = \|J_\lambda^{g_1} x^\star - y^\star\| \tag{2. 34}$$

It follows from (2. 34 ) and Lemma 2.1 that for each $i = 1, 2, \cdots, m,$ we have

$$\|W_\lambda^i x^\star - y^\star\| + \|W_\lambda^i x^\star - W_\lambda^{i-1} x^\star\| \leq \|W_\lambda^{i-1} x^\star - y^\star\| = \|x^\star - y^\star\|. \quad (2.\ 35)$$

Since $\|W_\lambda^i x^\star - y^\star\| = \|x^\star - y^\star\|$, it follows that, for each $i = 1, 2, \cdots, m,$ we have

$$\|W_\lambda^i x^\star - W_\lambda^{i-1} x^\star\| = 0, \quad \text{i.e.,} \quad W_\lambda^{i-1} x^\star \in F(J_\lambda^{g_i}). \quad (2.\ 36)$$

Now, for $i = 1$ in (2. 36 ), we have

$$x^\star = J_\lambda^{g_1} x^\star.$$

For $i = 2$ in (2. 36 ), we have

$$x^\star = J_\lambda^{g_1} x^\star = J_\lambda^{g_2} x^\star.$$

For $i = 3$ in (2. 36 ), we have

$$x^\star = J_\lambda^{g_1} x^\star = J_\lambda^{g_2} x^\star = J_\lambda^{g_3} x^\star.$$

Thus, for $i = 1, 2, \cdots, m$ in (2. 36 ), we can easily see that

$$x^\star = J_\lambda^{g_1} x^\star = J_\lambda^{g_2} x^\star = J_\lambda^{g_3} x^\star = \cdots = J_\lambda^{g_{m-1}} x^\star = J_\lambda^{g_m} x^\star.$$

That is, $x^\star \in \cap_{i=1}^m F)J_\lambda^{g_i})$, which is as desired. □

## 3. Main Results

**Assumption 3.1**

Throughout this section, we assume that:

(I) $K$ is a nonempty, closed and convex subset of a real Hilbert space $H$;

(II) $f : K \longrightarrow H$ is an $\rho$-Lipschitizian mapping;

(III) $g_i : K \longrightarrow R, i = 1, 2, \cdots, m,$ is a finite family of a proper convex and lower semicontinuous function. For a given $\lambda > 0$, we define the Moreau-Yosida resolvent of $g_i$ in $K$ by

$$s_n = J_\lambda^{g_i}(x) = argmin_{v \in K} \Big[ g_i(v) + \frac{1}{2\lambda} \|v - x\|^2 \Big], i = 1, 2, \cdots, m. \quad (3.\ 37)$$

Set

$$W_\lambda^l(x) = J_\lambda^{g_l} \circ J_\lambda^{g_{l-1}} \circ J_\lambda^{g_{l-2}} \circ \cdots \circ J_\lambda^{g_2} \circ J_\lambda^{g_1}(x), l = 1, 2, \cdots, m. \quad (3.\ 38)$$

(IV) $\{T_i\}_{i=1}^m : K \longrightarrow \mathcal{P}(K)$ is a finite family of uniformly $L_i$-Lipschitizian and asymptotically quasi-nonexpansive multivalued mapping with $\lim_{n\to\infty} k_n = 1$ and $T_i, i = 1, 2, \cdots, m$ is demiclosed at the origin (that is, for any bounded sequence $x_n$ i $K$ such that $\lim_{n\to\infty} x_n = q$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, then $T_i q = q, i = 1, 2, \cdots, m.$

Let $H$ and $K$ be defined as in Assumption 3.1. A mutivalurd mapping $T : K \longrightarrow CB(K)$ is called asymptotically $\beta$-nonspreading if there exists $\beta > 0$ such that

$$H(T^n x, T^n y)^2 \leq \beta(d(T^n x, y)^2 + d(x, T^n y)^2), \forall x, y \in K.$$

Note that a multivalued mapping $T$ is called asymptotically nonspreading-type if $\beta = \dfrac{1}{2}$; that is,

$$2H(T^n x, T^n y)^2 \leq d(T^n x, y)^2 + d(x, T^n y)^2, \forall x, y \in K.$$

Again, every asymptotically nonspreading-type multivalued mapping $T$ with a nonempty fixed point set is asymptotically quasi-nonexpansive. Indeed, $\forall x \in K$ and $q \in F(T)$, we have

$$\begin{aligned} 2H(T^nx, T^nq)^2 &\leq d(T^nx, y)^2 + d(x, T^nq)^2 \\ &\leq H(T^nx, T^nq)^2 + \|x - q\|^2. \end{aligned}$$

Thus, it follows that

$$H(T^nx, T^ny) \leq \|x - q\|.$$

Let $K$, $f$, $g_i : K \longrightarrow R$ and $T_i : K \longrightarrow C(K), i = 1, 2, \cdots, m$, be defined as in Assumption 3.1 with sequences $\{\{k_{in}\}_{n=1}^{\infty}\}_{i=1}^{m} \in [1, \infty)$ such that $\lim_{n\to\infty} k_{in} = 1$, where $z_{n,i} \in T_is_n \; with \; d(s_n, z_{n,i}) = d(s_n, T_is_n)$, for each $i = 1, 2, \cdots, m$. Suppose $\mathcal{F} = \cap_{i=1}^{m} F(T_i) \cap \cap_{i=1}^{m} argmin_{v\in K} g_i(v) \neq \emptyset$ and $T_iq = q, \forall q \in F(T_i), \; for \; each \; i = 1, 2, \cdots, m$. Let $A : K \longrightarrow H$ be an $L$-Lipschitizian and $\alpha$-strongly monotone mapping with $L, \alpha > 0$. Assume that
$0 < \gamma_n < \kappa = \Big(1 - \dfrac{\gamma(1+\rho)}{2\tau}\Big), 0 < \eta < \dfrac{2\alpha}{L^2}, 0 < \gamma\rho < \tau$, where $\tau = \eta\Big(\alpha - \dfrac{L^2\eta}{2}\Big)$ and $I - T_i$, is demiclosed at the origin for each $i = 1, 2, \cdots, m$. Let $\{x_n\}$ be a sequence generated iteratively by

$$\begin{cases} x_0 \in K; \\ s_n = W_{\lambda}^{m}(x); \\ y_n = (1 - \sum_{i=1}^{m} \beta_{n,i})s_n + \sum_{i=1}^{m} \beta_{n,i} z_{n,i} z_{n,i} \in T_is_n; \\ x_{n+1} = P_K(\alpha_n\gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n), \end{cases} \tag{3. 39}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$, where $k_n = max_{1\leq i\leq m}\{k_{in}\}$;

(ii) $\sum_{n=1}^{\infty} \beta_{in} = 1 \; and \; 0 < \liminf \beta_{in}(1 - \beta_{in}) \leq \limsup \beta_{in}(1 - \beta_{in}) < 1$, for each $i = 1, 2, \cdots, m$;

(iii) $\{\lambda_n\}$ is such that $\lambda_n \geq \lambda > 0, \forall n \geq 1$ and for some $\lambda$.

Then, the sequence defined by (3. 39 ) converges strongly to $x^{\star} \in \mathcal{F}$, which is also a unique solution of the variational inequality problem:

$$\langle \eta Ax^{\star} - \gamma f(x^{\star}), x^{\star} - q\rangle \leq 0, q \in \mathcal{F}. \tag{3. 40}$$

*Proof.* First, we show that the solution of the variational inequality defined by (3. 77 ) is unique. To do this, we assume for contradiction that there exists two points $x^{\star}, y^{\star} \in \mathcal{F}$ which are solutions of (3. 77 ) and $x^{\star} \neq y^{\star}$. Then, we get

$$\langle \eta Ax^{\star} - \gamma f(x^{\star}), x^{\star} - y^{\star}\rangle \leq 0 \tag{3. 41}$$

and

$$\langle \eta Ay^{\star} - \gamma f(y^{\star}), y^{\star} - x^{\star}\rangle \leq 0 \tag{3. 42}$$

(3. 41 ) and (3. 42 ) imply that

$$\langle \eta Ay^{\star} - \eta Ax^{\star} + \gamma f(x^{\star}) - \gamma f(y^{\star}), y^{\star} - x^{\star}\rangle \leq 0 \tag{3. 43}$$

Also, from the assumptions given, we get

$$\begin{aligned}\frac{L^2\eta}{2} > 0 \quad &\Leftrightarrow \quad \alpha - \frac{L^2\eta}{2} < \alpha \\ &\Leftrightarrow \quad \eta\Big(\alpha - \frac{L^2\eta}{2}\Big) < \alpha\eta \\ &\Leftrightarrow \quad \tau < \alpha\eta,\end{aligned}$$

so that $0 < \rho\gamma < \tau < \alpha\eta$.

ln addition, the inequality:

$$\begin{aligned}\langle \eta Ay^\star - \eta Ax^\star + \gamma f(x^\star) - \gamma f(y^\star), y^\star - x^\star\rangle &= \langle \eta Ay^\star - \eta Ax^\star, y^\star - x^\star\rangle \\ &\quad -\gamma\langle f(y^\star) - f(x^\star), y^\star - x^\star\rangle \\ &= \langle \eta Ay^\star - \eta Ax^\star, y^\star - x^\star\rangle \\ &\quad -\gamma\|f(y^\star) - f(x^\star)\|\|y^\star - x^\star\| \\ &\geq \eta\alpha\|x^\star - y^\star\|^2 - \gamma\rho\|x^\star - y^\star\|^2 \\ &= (\eta\alpha - \gamma\rho)\|x^\star - y^\star\|^2,\end{aligned}$$

this contradicts (3. 43 ) and hence $x^\star = y^\star$ which is as desired.

Again, we note that the operator $P_K[\gamma I + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)]$ is a contraction. Indeed, for any two fixed real numbers $\alpha, \gamma$ in $\Big(0, min\Big\{1, \frac{1}{\tau}\Big\}\Big)$ and $\forall x, y \in H$, we have, using Lemma 2.4, $X = [\gamma + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)]x$ and $Y = [\gamma + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)]y$, that

$$\begin{aligned}\|P_KX - P_KY\| &\leq \|\gamma x + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)x - (\gamma y + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)y)\| \\ &\leq \alpha\gamma\|f(x) - f(y)\| + \gamma\|x - y\| + \|((1-\gamma)I - \eta\alpha A)(x - y)\| \\ &\leq \alpha\gamma\rho\|x - y\| + \gamma\|x - y\| + ((1-\gamma)I - \alpha\tau)\|(x - y)\| \\ &\leq (1 - \alpha(\tau - \gamma\rho))\|x - y\|\end{aligned}$$

Thus, by Banach contraction principle, the mapping $P_K[\gamma I + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)]$ has a fixed point, say $\bar{x} = P_K[\gamma I + (\alpha\gamma f + ((1-\gamma)I - \eta\alpha A)]\bar{x}$; and as such, by (2. 28 ), similar in value to the variational inequality problem below:

$$\langle \eta A\bar{x} - \gamma f(\bar{x}), \bar{x} - q\rangle \leq 0, q \in \mathcal{F}.$$

The remaining proof of Theorem 3.1 will be presented in two stages:

Stage 1: We prove that the sequence $\{x_n\}$ and $\{y_n\}$ are bounded. Let $q \in \mathcal{F}$. Then, $T_iq = \{q\}$ and $g(q) \leq g(u)$, for all $u \in K$. Hence, $J^{g_i}_{\lambda_n}q = q$ for all $n \geq 1$, where $J^{g_i}_{\lambda_n}$ is the Moreau-Josida resolvent of $g$ in $K$.

By Lemma 2.9, for each $i = 1, 2, \cdots, m, J^{g_i}_{\lambda}$ is nonexpansive; therefore, $W^m_\lambda$ is also nonexpansive. Hence,

$$\|s_n - q\| = \|W^m_\lambda(x_n) - W^m_\lambda(q)\| \leq \|x_n - q\|. \tag{3. 44}$$

Also, from (3. 39 ), we have

$$\begin{aligned}
\|y_n - q\| &= \|(1-\sum_{i=1}^{m}\beta_{n,i})s_n + \sum_{i=1}^{m}\beta_{n,i})z_{n,i} - q\| \\
&\leq (1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - q\| + \sum_{i=1}^{m}\beta_{n,i}\|z_{n,i} - q\| \\
&\leq (1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - q\| + \sum_{i=1}^{m}\beta_{n,i}H(T_i^n s_n, T_i^n q) \\
&\leq (1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - q\| + \sum_{i=1}^{m}\beta_{n,i}k_n\|s_n - q\| \\
&= [1+\sum_{i=1}^{m}\beta_{n,i}(k_n - 1)]\|s_n - q\| \qquad (3.\ 45)
\end{aligned}$$

By condition (i), there exists a constant $\epsilon$ with $0 < \epsilon < 1-\delta$ and $\sum_{i=1}^{m}\beta_{n,i}(k_n-1) < \epsilon\alpha_n$ ,for each $i = 1, 2, \cdots, m$. Again, from (3. 39 ), (3. 44 ), (3. 45 ) and Lemma 2.4, we get

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P_K(\alpha_n\gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n) - P_K q\| \\
&\leq \|\alpha_n\gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n) - q\| \\
&= \|\alpha_n(\gamma f(x_n) - \eta A q) + \gamma_n(x_n - q) + ((1-\gamma_n)I - \eta\alpha_n A)(y_n - q)\| \\
&\leq \alpha_n\|\gamma f(x_n) - \eta A q\| + \gamma_n\|x_n - q\| + ((1-\gamma_n)I - \alpha_n\tau)\|y_n - q\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(q)\| + \alpha_n\|\gamma f(q) - \eta A q\| + \gamma_n\|x_n - q\| \\
&\quad +((1-\gamma_n)I - \alpha_n\tau)\|y_n - q\| \\
&\leq \alpha_n\gamma\rho\|x_n - q\| + \alpha_n\|\gamma f(q) - \eta A q\| + \gamma_n\|x_n - q\| \\
&\quad +((1-\gamma_n)I - \alpha_n\tau)[1+\sum_{i=1}^{m}\beta_{n,i}(k_n-1)]\|s_n - q\| \\
&\leq \alpha_n\gamma\rho\|x_n - q\| + \alpha_n\|\gamma f(q) - \eta A q\| + (1 - \alpha_n\tau + \sum_{i=1}^{m}\beta_{n,i}(k_n-1))\|x_n - q\| \\
&\leq [1-(\tau-\epsilon-\gamma\rho)\alpha_n]\|x_n - q\| + \alpha_n\|\gamma f(q) - \eta A q\| \\
&\leq max\Big\{\|x_n - q\|, \frac{\|\gamma f(q) - \eta A q\|}{\tau-\epsilon-\gamma\rho}\Big\}
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - q\| \leq max\Big\{\|x_0 - q\|, \frac{\|\gamma f(q) - \eta A q\|}{\tau-\epsilon-\gamma\rho}\Big\}, n \geq 1.$$

Hence, $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$,$\{f(x_n)\}$ and $\{Ax_n\}$.

Stage 2: We prove that the sequence $\{x_n\}$ converges strongly to $x^\star$.

From (3. 39 ) and Lemma 2.2 (ii), we obtain

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1-\sum_{i=1}^{m}\beta_{n,i})s_n + \sum_{i=1}^{m}\beta_{n,i})z_{n,i}\|^2 \\
&\leq (1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - q\|^2 + \sum_{i=1}^{m}\beta_{n,i}\|z_{n,i} - q\|^2 - \sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - z_{n,i}\|^2 \\
&\leq (1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - q\|^2 + \sum_{i=1}^{m}\beta_{n,i}H(T_i^n s_n, T_i^n q)^2 - \sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - z_{n,i}\|^2 \\
&\leq (1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - q\|^2 + \sum_{i=1}^{m}\beta_{n,i}k_n^2\|s_n - q\|^2 - \sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - z_{n,i}\|^2 \\
&= [1+\sum_{i=1}^{m}\beta_{n,i}(k_n^2 - 1)]\|s_n - q\|^2 - \sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - z_{n,i}\|^2 \qquad (3.\ 46)
\end{aligned}$$

Set $t_n = \alpha_n\gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n$. Then, from (3. 39 ), (3. 44 ), (3. 46 ), Lemma 2.2(i) and Lemma 2.4, we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|\alpha_n\gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n) - q\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - \eta Aq) + \gamma_n(x_n - q) + ((1-\gamma_n)I - \eta\alpha_n A)(y_n - q)\|^2 \\
&\leq \|((1-\gamma_n)I - \eta\alpha_n A)(y_n - q) + \gamma_n(x_n - q)\|^2 + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle \\
&= \|((1-\gamma_n)I - \eta\alpha_n A)(y_n - q)\|^2 + \gamma_n^2\|x_n - q\|^2 \\
&\quad + 2\gamma_n\langle((1-\gamma_n)I - \eta\alpha_n A)(y_n - q), x_n - q\rangle + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle \\
&\leq ((1-\gamma_n)I - \alpha_n\tau)^2\|y_n - q\|^2 + \gamma_n^2\|x_n - q\|^2 \\
&\quad + 2\gamma_n((1-\gamma_n)I - \alpha_n\tau)\|y_n - q\|\|x_n - q\| + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle \\
&\leq ((1-\gamma_n)I - \alpha_n\tau)^2\|y_n - q\|^2 + \gamma_n^2\|x_n - q\|^2 \\
&\quad + \gamma_n((1-\gamma_n)I - \alpha_n\tau)[\|y_n - q\|^2 + \|x_n - q\|^2] + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle \\
&= ((1-\gamma_n)I - \alpha_n\tau)((1-\gamma_n)I - \alpha_n\tau + \gamma_n)\|y_n - q\|^2 \\
&\quad + \gamma_n[\gamma_n + (1-\gamma_n)I - \alpha_n\tau)]\|x_n - q\|^2 + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle \\
&\leq ((1-\gamma_n)I - \alpha_n\tau)\{[1+\sum_{i=1}^{m}\beta_{n,i}(k_n^2 - 1)]\|s_n - q\|^2 \\
&\quad - \sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - z_{n,i}\|^2\} + \gamma_n\|x_n - q\|^2 + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle \\
&\leq (1-\alpha_n\tau)\|x_n - q\|^2 + ((1-\gamma_n)I - \alpha_n\tau)\sum_{i=1}^{m}\beta_{n,i}(k_n^2 - 1)\|x_n - q\|^2 \\
&\quad - ((1-\gamma_n)I - \alpha_n\tau)\sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n - z_{n,i}\|^2 + 2\alpha_n\langle\gamma f(x_n) - \eta Aq, t_n - q\rangle
\end{aligned}$$

Put $D_n = ((1-\gamma_n)I-\alpha_n\tau)\sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n-z_{n,i}\|^2$. Since the sequence $\{x_n\}$ is bounded, there exists a positive constant $M$ such that the last inequality becomes:

$$\begin{aligned} D_n &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+((1-\gamma_n)I-\alpha_n\tau)\sum_{i=1}^{m}\beta_{n,i}(k_n^2-1)M \\ &\quad +2\alpha_n\langle\gamma f(x_n)-\eta Aq, t_n-q\rangle \end{aligned} \tag{3. 47}$$

Now, to show that $\{x_n\}$ is convergent, we consider the following two cases:
Case A : Assume that the sequence $\{\|x_n-q\|\}$ is monotonically decreasing sequence. Then, $\{\|x_n-q\|\}$ is convergent. Indeed, we have

$$\lim_{n\to\infty}[\|x_n-q\|-\|x_{n+1}-q\|=0] \tag{3. 48}$$

Thus, by (3. 47 ), condition [(i) and (ii)] and the fact that $\lim_{n\to\infty}k_n=1$, we have

$$\lim_{n\to\infty}D_n=\lim_{n\to\infty}((1-\gamma_n)I-\alpha_n\tau)\sum_{i=1}^{m}\beta_{n,i}(1-\sum_{i=1}^{m}\beta_{n,i})\|s_n-z_{n,i}\|^2=0 \tag{3. 49}$$

Since, $\beta_n\in[a,b]\subset(0,1)$, for each $i=1,2,\cdots,m$, we get

$$\lim_{n\to\infty}\|s_n-z_{n,i}\|=0 \tag{3. 50}$$

Applying $H$-continuity $(see$ [50]$)$, we have

$$\lim_{n\to\infty}\|s_n-T_i^n s_n\|=0 \tag{3. 51}$$

Also, using (3. 39 ), we get

$$\begin{aligned} \|x_{n+1}-y_n\| &\leq \|\alpha_n\gamma f(x_n)+\gamma_n x_n+((1-\gamma_n)I-\eta\alpha_n A)y_n-y_n\| \\ &\leq \alpha_n\|\gamma f(x_n)-\eta\alpha_n A\|+\gamma_n\|x_n-y_n\|. \end{aligned} \tag{3. 52}$$

(3. 52 ) and condition (i) imply that

$$\lim_{n\to\infty}(\|x_{n+1}-y_n\|-\|x_n-y_n\|)=0. \tag{3. 53}$$

It follows from (3. 53 ) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3. 54}$$

Again, from the nonexpansivity of $J_\lambda^{g_i}, i=1,2,\cdots,m,$ (3. 39 ) and Lemma 2.9, we get, for any $q\in\mathcal{F}$, that

$$\begin{aligned} \|W_\lambda^m x_n-W_\lambda^m q\|^2 &= \|J_\lambda^{g_m}\circ J_\lambda^{g_{m-1}}\circ J_\lambda^{g_{m-2}}\circ\cdots\circ J_\lambda^{g_2}\circ J_\lambda^{g_1}x_n \\ &\quad -J_\lambda^{g_m}\circ J_\lambda^{g_{m-1}}\circ J_\lambda^{g_{m-2}}\cdots\circ J_\lambda^{g_2}\circ J_\lambda^{g_1}q\|^2 \\ &\leq \|J_\lambda^{g_{m-1}}\circ J_\lambda^{g_{m-2}}\circ J_\lambda^{g_{m-3}}\circ\cdots\circ J_\lambda^{g_2}\circ J_\lambda^{g_1}x_n \\ &\quad -J_\lambda^{g_{m-1}}\circ J_\lambda^{g_{m-2}}\circ J_\lambda^{g_{m-3}}\cdots\circ J_\lambda^{g_2}\circ J_\lambda^{g_1}q\|^2 \\ &\quad -\|J_\lambda^{g_m}\circ J_\lambda^{g_{m-1}}\circ J_\lambda^{g_{m-2}}\circ\cdots\circ J_\lambda^{g_2}\circ J_\lambda^{g_1}x_n \\ &\quad -J_\lambda^{g_{m-1}}\circ J_\lambda^{g_{m-2}}\circ J_\lambda^{g_{m-3}}\cdots\circ J_\lambda^{g_2}\circ J_\lambda^{g_1}q\|^2 \\ &= \|W_\lambda^{m-1}x_n-W_\lambda^{m-1}q\|^2-\|W_\lambda^m x_n-W_\lambda^{m-1}x_n\|^2, \end{aligned}$$

so that

$$\|s_n-W_\lambda^{m-1}x_n\|^2=\|W_\lambda^m x_n-W_\lambda^{m-1}x_n\|^2\leq\|x_n-q\|^2-\|s_n-q\|^2 \tag{3. 55}$$

Furthermore, from (3. 39 ), (3. 44 ),(3. 46 ) and (3. 55 ), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \|\alpha_n\gamma f(x_n)\gamma_n x_n+((1-\gamma_n)I-\eta\alpha_n A)y_n)-q\|^2\\
&= \|\alpha_n(\gamma f(x_n)-\eta Aq)-\gamma_n(q-x_n)+((1-\gamma_n)I-\eta\alpha_n A)(y_n-q)\|^2\\
&\leq \|((1-\gamma_n)I-\eta\alpha_n A)(y_n-q)-\gamma_n(q-x_n)\|^2+2\alpha_n\langle\gamma f(x_n)-\eta Aq,x_{n+1}-q\rangle\\
&\leq \|((1-\gamma_n)I-\eta\alpha_n A)(y_n-q)\|^2+\gamma_n^2\|q-x_n\|^2+2\alpha_n\gamma\langle f(x_n)-f(q),x_{n+1}-q\rangle\\
&\quad +2\alpha_n\langle\gamma f(q)-\eta Aq,x_{n+1}-q\rangle\\
&\leq ((1-\gamma_n)I-\alpha_n\tau)^2\|y_n-q\|^2+\gamma_n^2\|q-x_n\|^2+2\alpha_n\gamma\|f(x_n)-f(q)\|\|x_{n+1}-q\|\\
&\quad +2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|\\
&\leq ((1-\gamma_n)I-\alpha_n\tau)^2[1+\sum_{i=1}^m\beta_{n,i}(k_n^2-1)]\|s_n-q\|^2+\gamma_n^2\|q-x_n\|^2\\
&\quad +2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|\\
&= ((1-\gamma_n)I-\alpha_n\tau)^2\|s_n-q\|^2+((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^m\beta_{n,i}(k_n^2-1)\|s_n-q\|^2\\
&\quad +\gamma_n^2\|q-x_n\|^2+2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|\\
&\leq ((1-\gamma_n)I-\alpha_n\tau)^2[\|x_n-q\|^2-\|s_n-W_\lambda^{m-1}x_n\|^2]\\
&\quad +((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^m\beta_{n,i}(k_n^2-1)\|x_n-q\|^2+\gamma_n^2\|q-x_n\|^2\\
&\quad +2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|\\
&= [1-2\gamma_n+\gamma_n^2+\alpha_n^2\tau^2-2(1-\gamma_n)\alpha_n\tau]\|x_n-q\|^2\\
&\quad -((1-\gamma_n)I-\alpha_n\tau)^2\|s_n-W_\lambda^{m-1}x_n\|^2\\
&\quad +((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^m\beta_{n,i}(k_n^2-1)\|x_n-q\|^2+\gamma_n^2\|q-x_n\|^2\\
&\quad +2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|\\
&= [1+2\gamma_n(\gamma_n-1)+\alpha_n^2\tau^2-2(1-\gamma_n)\alpha_n\tau+((1-\gamma_n)I-\alpha_n\tau)\epsilon\alpha_n]\|x_n-q\|^2\\
&\quad -((1-\gamma_n)I-\alpha_n\tau)^2\|s_n-W_\lambda^{m-1}x_n\|^2+2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|\\
&\quad +2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|\\
&\leq [1+\alpha_n^2\tau^2+((1-\gamma_n)I-\alpha_n\tau)\epsilon\alpha_n]\|x_n-q\|^2\\
&\quad -((1-\gamma_n)I-\alpha_n\tau)^2\|s_n-W_\lambda^{m-1}x_n\|^2+2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|\\
&\quad +2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\| \qquad (3.\ 56)
\end{aligned}$$

(3. 56 ) implies that

$$\begin{aligned}
((1-\gamma_n)I-\alpha_n\tau)^2\|s_n-W_\lambda^{m-1}x_n\|^2 &\leq \|x_n-x_{n+1}\|^2+[\alpha_n^2\tau^2+((1-\gamma_n)I-\alpha_n\tau)\epsilon\alpha_n]\|x_n-q\|^2\\
&\quad +2\alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|\\
&\quad +2\alpha_n\|\gamma f(q)-\eta Aq\|\|x_{n+1}-q\|.
\end{aligned}$$

The last inequality and condition (i) yields

$$\lim_{n\to\infty} \|s_n - W_\lambda^{m-1}x_n\| = 0. \tag{3. 57}$$

Similarly, using the same approach as above, it can easily be seen that

$$\lim_{n\to\infty} \|W_\lambda^{m-i}x_n - W_\lambda^{m-(i+1)}x_n\| = 0, i = 1, 2, \cdots, m-1. \tag{3. 58}$$

From (3. 57 ) and (3. 58 ), we obtain

$$\begin{aligned} \|s_n - x_n\| &= \|W_\lambda^m x_n - x_n\| \\ &\leq \|W_\lambda^m x_n - W_\lambda^{m-1}x_n\| + \|W_\lambda^{m-1}x_n - W_\lambda^{m-2}x_n\| + \cdots + \|W_\lambda^1 x_n - W_\lambda^2 x_n\| \\ &\quad + \|W_\lambda^1 x_n - x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3. 59}$$

Since

$$\|x_{n+1} - s_n\| \leq \|x_{n+1} - x_n\| + \|x_n - s_n\|,$$

it follows from (3. 54 ) and (3. 59 ) that

$$\lim_{n\to\infty} \|x_{n+1} - s_n\| = 0. \tag{3. 60}$$

Letting $u_{n,i} \in T_i^n s_n,\ for\ each\ i = 1, 2, \cdots, m$, we have

$$\|u_{n,i} - z_{n,i}\| \leq H(T_i^n s_n, T_i^n s_n) \leq L_i\|s_n - s_n\| \to 0\ as\ n \to \infty. \tag{3. 61}$$

Using (3. 50 ) and (3. 61 ), we obtain

$$\|s_n - u_{n,i}\| \leq \|s_n - z_{n,i}\| + \|z_{n,i} - u_{n,i}\| \to 0\ as\ n \to \infty. \tag{3. 62}$$

Since $u_{n,i} \subset T_i^n s_n,\ for\ each\ i = 1, 2, \cdots, m$, it follows that $Tu_{n,i} \subset T_i^{n+1}s_n$. Also, let $r_{n+1,i} \in T_i^n u_{n,i}$ so that $r_{n+1,i} \in T^{n+1}s_{n..}$ Since $T_i$ is uniformly Lipschitizian, for each $i = 1, 2, \cdots, m$, we get

$$\|r_{n+1,i} - u_{n,i}\| \leq \|r_{n+1,i} - u_{n+1,i}\| + \|u_{n+1,i} - s_{n+1}\| + \|s_{n+1} - s_n\| + \|s_n - u_{n,i}\|. \tag{3. 63}$$

Since from (3. 54 ) and (3. 59 )

$$\|s_{n+1} - s_n\| \leq \|s_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - s_n\| \to 0\ as\ n \to \infty, \tag{3. 64}$$

it follows from (3. 62 ) that

$$\|u_{n,i} - s_{n+1}\| \leq \|u_{n,i} - s_n\| + \|s_n - s_{n+1}\| \to 0\ as\ n \to \infty. \tag{3. 65}$$

Letting $z_{n,i} \in T_i s_n$, for each $i = 1, 2, \cdots, m$, we obtain, using (3. 46 ), (3. 62 ) and (3. 63 ), that

$$\begin{aligned} \|z_{n,i}, s_n\| &\leq \|z_{n,i} - r_{n+1,i}\| + \|r_{n+1,i} - u_{n,i}\| + \|u_{n,i} - s_n\| \\ &\leq H(T_i^n s_n, T_i^{n+1}s_n) + \|r_{n+1,i} - u_{n,i}\| + \|u_{n,i} - s_n\| \\ &\leq L_i\|s_n - T_i^n s_n\| + \|r_{n+1,i} - u_{n,i}\| + \|u_{n,i} - s_n\| \to 0\ as\ n \to \infty. \end{aligned} \tag{3. 66}$$

Now, we show that $\limsup_{n\to\infty}\langle \eta Ax^\star - \gamma f(x^\star), x^\star - x_n\rangle \leq 0$. By the nature of $H$, and from the boundedness of the sequence $\{x_n\}_{n=0}^\infty$, we can find a subsequence $\{x_{n_j}\}_{j=0}^\infty$ of $\{x_n\}$ that converges weakly to a point $\omega \in K$. With this fact, it is not hard to see that

$$\limsup_{n\to\infty}\langle \eta Ax^\star - \gamma f(x^\star), x^\star - x_n\rangle = \limsup_{j\to\infty}\langle \eta Ax^\star - \gamma f(x^\star), x^\star - x_{n_j}\rangle. \tag{3. 67}$$

Using (3. 59 ), (3. 66 ) and the fact that $I - T_i$ is demiclosed at the origin for each $i = 1, 2, \cdots, m,$ we have $\omega \in \cap_{i=1}^m F(T_i)$. Also, since $W_\lambda^m$ is single-valued and nonexpansive, using (3. 59 ), Proposition 2.11 and using Proposition 2.10, we get $\omega \in F(W_\lambda^m) = argmin_{u\in K} g_i(v)$. Consequently, $\omega \in \mathcal{F}$. Hence, using (3. 67 ) , we get

$$\begin{aligned} \limsup_{n\to\infty}\langle \eta Ax^\star - \gamma f(x^\star), x^\star - x_n\rangle &\leq \langle \eta Ax^\star - \gamma f(x^\star), x^\star - \omega\rangle \\ &\leq 0. \end{aligned} \tag{3. 68}$$

Lastly, we prove that $x_n \to x^\star(n \to \infty)$. From (3. 39 ), (3. 46 ) and Lemma 2.5, we have

$$\begin{aligned} \|x_{n+1} - x^\star\|^2 &\leq \|\alpha_n\gamma f(x_n)\gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n) - x^\star\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - \eta Ax^\star) - \gamma_n(x^\star - x_n) + ((1-\gamma_n)I - \eta\alpha_n A)(y_n - x^\star)\|^2 \\ &\leq \|((1-\gamma_n)I - \eta\alpha_n A)(y_n - x^\star) - \gamma_n(x^\star - x_n)\|^2 \\ &\quad +2\alpha_n\langle\gamma f(x_n) - \eta Ax^\star, x_{n+1} - x^\star\rangle \\ &\leq \|((1-\gamma_n)I - \eta\alpha_n A)(y_n - x^\star)\|^2 + \gamma_n^2\|x^\star - x_n\|^2 \\ &\quad +2\alpha_n\gamma\langle f(x_n) - f(x^\star), x_{n+1} - x^\star\rangle + 2\alpha_n\langle\gamma f(x^\star) - \eta Ax^\star, x_{n+1} - x^\star\rangle \end{aligned}$$

$$\begin{aligned} &\leq ((1-\gamma_n)I - \alpha_n\tau)^2\|y_n - x^\star\|^2 + \gamma_n^2\|x^\star - x_n\|^2 \\ &\quad +2\alpha_n\gamma\|f(x_n) - f(x^\star)\|\|x_{n+1} - x^\star\| + 2\alpha_n\langle\gamma f(x^\star) - \eta Ax^\star, x_{n+1} - x^\star\rangle \\ &\leq ((1-\gamma_n)I - \alpha_n\tau)^2[1 + \sum_{i=1}^m \beta_{n,i}(k_n^2 - 1)]\|s_n - x^\star\|^2 + \gamma_n^2\|x^\star - x_n\|^2 \\ &\quad +\alpha_n\gamma[\|f(x_n) - f(x^\star)\|^2 + \|x_{n+1} - x^\star\|^2] + 2\alpha_n\langle\gamma f(x^\star) - \eta Ax^\star, x_{n+1} - x^\star\rangle \\ &= [1 - 2\gamma_n + \gamma_n^2 - 2(1-\gamma_n)\alpha_n\tau + \alpha_n^2\tau^2]\|x_n - x^\star\|^2 \\ &\quad +((1-\gamma_n)I - \alpha_n\tau)^2\sum_{i=1}^m \beta_{n,i}(k_n^2 - 1)\|x_n - x^\star\|^2 + \gamma_n^2\|x^\star - x_n\|^2 \\ &\quad +\alpha_n\gamma[\|f(x_n) - f(x^\star)\|^2 + \|x_{n+1} - x^\star\|^2] + 2\alpha_n\langle\gamma f(x^\star) - \eta Ax^\star, x_{n+1} - x^\star\rangle \\ &= [1 + \gamma_n(\gamma_n - 2) - 2(1-\gamma_n)\alpha_n\tau]\|x_n - x^\star\|^2 \\ &\quad +[\alpha_n^2\tau^2 + ((1-\gamma_n)I - \alpha_n\tau)^2\sum_{i=1}^m \beta_{n,i}(k_n^2 - 1)]M + \alpha_n\gamma\rho\|x_n - x^\star\|^2 + \alpha_n\gamma\|x_{n+1} - x^\star\|^2 \\ &\quad +2\alpha_n\langle\gamma f(x^\star) - \eta Ax^\star, x_{n+1} - x^\star\rangle \\ &\leq [1 - (2(1-\gamma_n)\tau - \gamma\rho)\alpha_n]\|x_n - x^\star\|^2 + [\alpha_n^2\tau^2 + ((1-\gamma_n)I - \alpha_n\tau)^2\sum_{i=1}^m \beta_{n,i}(k_n^2 - 1)]M \\ &\quad +\alpha_n\gamma\|x_{n+1} - x^\star\|^2 + 2\alpha_n\langle\gamma f(x^\star) - \eta Ax^\star, x_{n+1} - x^\star\rangle \end{aligned}$$

The last inequality implies that

$$\begin{aligned}(1-\alpha_{n\gamma})\|x_{n+1}-x^\star\|^2 &\leq [1-(2(1-\gamma_n)\tau-\gamma\rho)\alpha_n]\|x_n-x^\star\|^2\\ &+\alpha_n\Big\{[\alpha_n\tau^2+((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^{m}\beta_{n,i}\frac{(k_n^2-1)}{\alpha_n}]M\\ &+2\langle\gamma f(x^\star)-\eta Ax^\star,x_{n+1}-x^\star\rangle\Big\}.\end{aligned}$$

$$\begin{aligned}\Rightarrow\|x_{n+1}-x^\star\|^2 &\leq \Big[1-\frac{(2(1-\gamma_n)\tau-\gamma-\gamma\rho)\alpha_n}{1-\alpha_n\gamma}\Big]\|x_n-x^\star\|^2\\ &+\frac{\alpha_n}{1-\alpha_n\gamma}\Big\{[\alpha_n\tau^2+((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^{m}\beta_{n,i}\frac{(k_n^2-1)}{\alpha_n}]M\\ &+2\langle\gamma f(x^\star)-\eta Ax^\star,x_{n+1}-x^\star\rangle\Big\},\end{aligned}$$

where $M=\sup_{n\geq1}\|x_n-x^\star\|^2$.

Put

$$\begin{aligned}\sigma_n &= \frac{(2(1-\gamma_n)\tau-\gamma-\gamma\rho)\alpha_n}{1-\alpha_n\gamma},\\ \tau_n &= \frac{\alpha_n}{1-\alpha_n\gamma}\Big\{[\alpha_n\tau^2+((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^{m}\beta_{n,i}\frac{(k_n^2-1)}{\alpha_n}]M\\ &\quad+2\langle\gamma f(x^\star)-\eta Ax^\star,x_{n+1}-x^\star\rangle\Big\}\end{aligned}$$

and

$$\begin{aligned}w_n &= \frac{\tau_n}{\sigma_n}\\ &= \frac{1}{2(1-\gamma_n)\tau-\gamma-\gamma\rho}\Big\{[\alpha_n\tau^2+((1-\gamma_n)I-\alpha_n\tau)^2\sum_{i=1}^{m}\beta_{n,i}\frac{(k_n^2-1)}{\alpha_n}]M\\ &\quad+2\langle\gamma f(x^\star)-\eta Ax^\star,x_{n+1}-x^\star\rangle\Big\}.\end{aligned}$$

Then, from condition (i) and (3. 68 ), we get

$$\sigma_n\to0\quad\text{as}\quad n\to\infty,\sum_{n=1}^{\infty}\sigma_n=\infty\quad\text{and}\quad\limsup_{n\to\infty}w_n\leq0.$$

Thus, using Lemma 2.3, the result follows as required (i.e., $x_n\to x^\star$ as $n\to\infty$).

Case B:

Suppose $\{\|x_n-x^\star\|\}$ is monotonically increasing sequence. Set $G_n=\|x_n-x^\star\|$ and the mapping $\tau:\mathbb{N}\longrightarrow\mathbb{N}$, for all $n\geq n_0$ (for some $n_0$ large enough), by $\tau_n=max\{k\in\mathbb{N}:k\leq n,G_k\leq G_{k+1}\}$. Then $\tau$ is a nondecreasing sequence such that $\tau_n\to\infty$ as $n\to\infty$

and $G_{\tau(n)} \leq G_{\tau_n+1}$ for $n \geq n_0$. From (3. 47 ), we have

$$\begin{aligned} D_{\tau_n} &\leq \|x_{\tau_n} - x^\star\|^2 - \|x_{\tau_n+1} - x^\star\|^2 + ((1-\gamma_{\tau_n})I - \alpha_{\tau_n}\tau)\sum_{i=1}^{m}\beta_{\tau_n,i}(k_{\tau_n}^2 - 1)M \\ &\quad +2\alpha_{\tau_n}\langle\gamma f(x_{\tau_n}) - \eta A x^\star, t_{\tau_n} - x^\star\rangle \to 0(n\to\infty), \end{aligned}$$

where $D_{\tau_n} = ((1-\gamma_{\tau_n})I - \alpha_{\tau_n}\tau)\sum_{i=1}^{m}\beta_{\tau_n,i}(1-\sum_{i=1}^{m}\beta_{\tau_n,i})\|s_{\tau_n} - z_{\tau_n,i}\|^2$. In addition, we have

$$\lim_{n\to\infty}\|s_{\tau_n} - z_{\tau_n,i}\|^2 = 0$$

Therefore,

$$\lim_{n\to\infty} d(s_{\tau_{(n)}}, T_i^{\tau(n)} s_{\tau(n)}) = 0. \tag{3. 69}$$

Using similar approach as in Case A above, it is easy to see that

$$\limsup_{\tau(n)\to\infty}\langle\eta A x^\star - \gamma f(x^\star), x^\star - x_{\tau(n)+1}\rangle \leq 0.$$

Hence, for all $n \geq n_0$, we have

$$\begin{aligned} 0 \leq \|x_{\tau(n)+1} - x^\star\|^2 - \|x_{\tau(n)} - x^\star\|^2 &\leq \alpha_{\tau(n)}\Big[-\frac{(2(1-\gamma_n)\tau - \gamma - \gamma\rho)\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\gamma}\Big]\|x_{\tau(n)} - x^\star\|^2 \\ &\quad + \frac{\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\gamma}\Big\{[\alpha_{\tau(n)}\tau^2 + ((1-\gamma_{\tau(n)})I - \alpha_{\tau(n)}\tau)^2 \\ &\quad \times\sum_{i=1}^{m}\beta_{\tau(n),i}\frac{(k_{\tau(n)}^2 - 1)}{\alpha_{\tau(n)}}]M + 2\langle\gamma f(x^\star) - \eta A x^\star, x_{n+1} - x^\star\rangle\Big\}, \end{aligned}$$

which follows that

$$\begin{aligned} \|x_{\tau(n)} - x^\star\|^2 &\leq \frac{1}{2(1-\gamma_n)\tau - \gamma - \gamma\rho}\Big\{[\alpha_{\tau(n)}\tau^2 + ((1-\gamma_{\tau(n)})I - \alpha_{\tau(n)}\tau)^2 \\ &\quad \times\sum_{i=1}^{m}\beta_{\tau(n),i}\frac{(k_{\tau(n)}^2 - 1)}{\alpha_{\tau(n)}}]M + 2\langle\gamma f(x^\star) - \eta A x^\star, x_{n+1} - x^\star\rangle\Big\}, \end{aligned}$$

Consequently, we get

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^\star\| = 0.$$

and

$$\lim_{n\to\infty} G_{\tau(n)} = \lim_{n\to\infty} G_{\tau(n)+1} = 0. \tag{3. 70}$$

Thus, by Lemma 2.7, we conclude that

$$0 \leq G_{\tau(n)} \leq \max\{G_{\tau(n)}, G_{\tau(n)+1}\} = G_{\tau(n)+1.} \tag{3. 71}$$

Hence, $\lim_{n\to\infty} G_n = 0$; that is, $\{x_n\}$ converges strongly to $x^\star$. The proof is complete.

□

It is worth to mention at this point that if $T_i : K \longrightarrow C(K), i = 1, 2, \cdots, m$, is a finite family of asymptotically nonspreading-type multivalued mapping, then the following theorem is a direct consequence of Theorem 3.1.

Let $K$, $f$ and $g_i : K \longrightarrow R, i = 1, 2, \cdots, m$, be defined as in Assumption 3.1 and $T_i : K \longrightarrow C(K), i = 1, 2, \cdots, m$, a finite family of $L_i$-Lipschitizian and asymptotically nonspreading-type multivalued mappings with sequences $\{\{k_{in}\}_{n=1}^{\infty}\}_{i=1}^{m} \in [1, \infty)$ such that $\lim_{n\to\infty} k_{in} = 1$, where $z_{n,i} \in T_i s_n \; with \; d(s_n, z_{n,i}) = d(s_n, T_i s_n)$, for each $i = 1, 2, \cdots, m$. Suppose $\mathcal{F} = \cap_{i=1}^{m} F(T_i) \cap \cap_{i=1}^{m} argmin_{v\in K} g_i(v) \neq \emptyset$ and $T_i q = q, \forall q \in F(T_i), \; for\, each\, i = 1, 2, \cdots, m$. Let $A : K \longrightarrow H$ be an $L$-Lipschitizian and $\alpha$-strongly monotone mapping with $L, \alpha > 0$. Assume that
$0 < \gamma_n < \kappa = \Big(1 - \dfrac{\gamma(1+\rho)}{2\tau}\Big), 0 < \eta < \dfrac{2\alpha}{L^2}, 0 < \gamma\rho < \tau$, where $\tau = \eta\Big(\alpha - \dfrac{L^2\eta}{2}\Big)$ and $I - T_i$, is demiclosed at the origin for each $i = 1, 2, \cdots, m$. Let $\{x_n\}$ be a sequence generated iteratively by

$$\begin{cases} x_0 \in K; \\ s_n = W_{\lambda}^{m}(x); \\ y_n = (1 - \sum_{i=1}^{m} \beta_{n,i}) s_n + \sum_{i=1}^{m} \beta_{n,i} z_{n,i}, z_{n,i} \in T_i s_n; \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A) y_n), \end{cases} \tag{3. 72}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$, where $k_n = max_{1\le i\le m}\{k_{in}\}$;

(ii) $\sum_{n=1}^{\infty} \beta_{in} = 1 \; and \; 0 < \liminf \beta_{in}(1 - \beta_{in}) \le \limsup \beta_{in}(1 - \beta_{in}) < 1$, for each $i = 1, 2, \cdots, m$;

(iii) $\{\lambda_n\}$ is such that $\lambda_n \ge \lambda > 0, \forall n \ge 1$ and for some $\lambda$.

Then, the sequence defined by (3. 76 ) converges strongly to $x^\star \in \mathcal{F}$, which is also a unique solution of the variational inequality problem:

$$\langle \eta A x^\star - \gamma f(x^\star), x^\star - q \rangle \le 0, q \in \mathcal{F}. \tag{3. 73}$$

*Proof.* Since every asymptotically nonspreading-type mapping with nonempty fixed point set is asymptotically quasinonexpansive multivalued mapping, the proof of Theorem 3.2 immediately follows from Lemma 2.1 and Theorem 3.1. □

Again, if $\{T_i\}_{i=1}^{m}$ is asymptotically quasi-nonexpansive single-valued mapping and $A$ is a strongly positive bounded linear operator, then the following theorem can be obtained from Theorem 3.1. Let $K$, $f$ and $g_i : K \longrightarrow R, i = 1, 2, \cdots, m$, be defined as in Assumption 3.1 and $T_i : K \longrightarrow K, i = 1, 2, \cdots, m$, a finite family of asymptotically quasi-nonexpansive single-valued mappings with sequences $\{\{k_{in}\}_{n=1}^{\infty}\}_{i=1}^{m} \in [1, \infty)$ such that $\lim_{n\to\infty} k_{in} = 1$ for each $i = 1, 2, \cdots, m$. Suppose $\mathcal{F} = \cap_{i=1}^{m} F(T_i) \cap \cap_{i=1}^{m} argmin_{v\in K} g_i(v) \neq \emptyset$ and $T_i q = q, \forall q \in F(T_i), \; for \; each \; i = 1, 2, \cdots, m$. Let $A : K \longrightarrow H$ be an $L$-Lipschitizian and $\alpha$-strongly monotone mapping with $L, \alpha > 0$. Assume that
$0 < \gamma_n < \kappa = \Big(1 - \dfrac{\gamma(1+\rho)}{2\tau}\Big), 0 < \eta < \dfrac{2\alpha}{L^2}, 0 < \gamma\rho < \tau$, where $\tau = \eta\Big(\alpha - \dfrac{L^2\eta}{2}\Big)$ and $I - T_i$, is demiclosed at the origin for each $i = 1, 2, \cdots, m$. Let $\{x_n\}$ be a sequence

generated iteratively by

$$\begin{cases} x_0 \in K; \\ s_n = W_\lambda^m(x); \\ y_n = (1 - \sum_{i=1}^m \beta_{n,i})s_n + \sum_{i=1}^m \beta_{n,i}T_i^n s_n; \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n), \end{cases} \tag{3. 74}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$, where $k_n = max_{1\le i\le m}\{k_{in}\}$;

(ii) $\sum_{n=1}^\infty \beta_{in} = 1$ $and$ $0 < \liminf \beta_{in}(1-\beta_{in}) \le \limsup \beta_{in}(1-\beta_{in}) < 1$, for each $i = 1, 2, \cdots, m$;

(iii) $\{\lambda_n\}$ is such that $\lambda_n \ge \lambda > 0, \forall n \ge 1$ and for some $\lambda$.

Then, the sequence defined by (3. 74 ) converges strongly to $x^\star \in \mathcal{F}$, which satisfies the optimality condition of the minimization problem

$$min_{x\in H} \frac{\eta}{2}\langle Ax, x\rangle - h(x), \tag{3. 75}$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ on $H$. Let $K$, $f$ and $g_i : K \longrightarrow R, i = 1, 2, \cdots, m$, be defined as in Assumption 3.1. Let $T_i : K \longrightarrow K, i = 1, 2, \cdots, m$, a finite family of asymptotically nonspreading-type single-valued mappings with sequences $\{\{k_{in}\}_{n=1}^\infty\}_{i=1}^m \in [1, \infty)$ such that $\lim_{n\to\infty} k_{in} = 1$ for each $i = 1, 2, \cdots, m$. Suppose $\mathcal{F} = \cap_{i=1}^m F(T_i) \cap \cap_{i=1}^m argmin_{v\in K} g_i(v) \ne \emptyset$ and $T_i q = q, \forall q \in F(T_i),\ for\ each\ i = 1, 2, \cdots, m$. Let $A : K \longrightarrow H$ be an $L$-Lipschitizian and $\alpha$-strongly monotone mapping with $L, \alpha > 0$. Assume that
$0 < \gamma_n < \kappa = \Big(1 - \dfrac{\gamma(1+\rho)}{2\tau}\Big), 0 < \eta < \dfrac{2\alpha}{L^2}, 0 < \gamma\rho < \tau$, where $\tau = \eta\Big(\alpha - \dfrac{L^2\eta}{2}\Big)$
and $I - T_i$, is demiclosed at the origin for each $i = 1, 2, \cdots, m$. Let $\{x_n\}$ be a sequence generated iteratively by

$$\begin{cases} x_0 \in K; \\ s_n = W_\lambda^m(x); \\ y_n = (1 - \sum_{i=1}^m \beta_{n,i})s_n + \sum_{i=1}^m \beta_{n,i}T_i^n s_n; \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \eta\alpha_n A)y_n), \end{cases} \tag{3. 76}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$, where $k_n = max_{1\le i\le m}\{k_{in}\}$;

(ii) $\sum_{n=1}^\infty \beta_{in} = 1$ $and$ $0 < \liminf \beta_{in}(1-\beta_{in}) \le \limsup \beta_{in}(1-\beta_{in}) < 1$, for each $i = 1, 2, \cdots, m$;

(iii) $\{\lambda_n\}$ is such that $\lambda_n \ge \lambda > 0, \forall n \ge 1$ and for some $\lambda$.

Then, the sequence defined by (3. 76 ) converges strongly to $x^\star \in \mathcal{F}$, which is also a unique solution of the variational inequality problem:

$$\langle \eta Ax^\star - \gamma f(x^\star), x^\star - q\rangle \le 0, q \in \mathcal{F}. \tag{3. 77}$$

## 4. NUMERICAL EXAMPLE

In this section, we give a numerical example to show that our proposed algorithm can be implementable.
Let $H = \mathbb{R}$ be endowed with the uasual metric and $K = [0, 1]$. Then,

$$P_K(x) = \begin{cases} 0, \ if \ x < 0; \\ x, \ if \ x \in K; \\ 1, \ if \ x > 1 \end{cases}$$

is a metric projection onto $K$. For, $i = 1, 2, \cdots, m$, define $T_i : \mathbb{R}^+ \longrightarrow P(H)$ by

$$T_i x = \left[ -\frac{1}{3^i}x, -\frac{1}{5^i}x \right], \forall x \in \mathbb{R}^+.$$

Then, for each $i = 1, 2, \cdots, m$, $T_i$ is an asymptotically quasinonexpansive multivalued mapping with $F(T) = 0$. Indeed, for each $i = 1, 2, \cdots, m$ and for all $x \in \mathbb{R}^+$, we have

$$\begin{aligned} H(T_i^n x, 0) &= max_{1 \leq i \leq m} \left\{ |\frac{1}{3^{in}}x|, |\frac{1}{5^{in}}x| \right\} \\ &= |\frac{1}{3^{in}}x| \\ &\leq \left(1 + \frac{1}{3^{in}}\right)|x - 0|. \end{aligned}$$

Thus, $T_i$ is an asymptotically quasinonexpansive multivalued mapping for each $i = 1, 2, \cdots, m$. Again, we check that $T_i$ is uniformly Lipschitizian for each $i = 1, 2, \cdots, m$. Indeed, for each $i = 1, 2, \cdots, m$. and for each $x, y \in \mathbb{R}^+$, we get

$$\begin{aligned} H(T_i^n x, 0) &= max_{1 \leq i \leq m} \left\{ | -\frac{1}{5^{in}}x + \frac{1}{5^{in}}y|, | -\frac{1}{3^{in}}x + \frac{1}{3^{in}}y| \right\} \\ &= \frac{1}{3^{in}}|x - y| \\ &\leq \frac{1}{3^i}|x - y| \end{aligned}$$

Hence, $T_i$ is uniformly $\frac{1}{3^i}$-Lipschitizian for each $i = 1, 2, \cdots, m$.

Now, define $g_i : \mathbb{R} \longrightarrow (-\infty, \infty]$ by

$$g_i(x) = \frac{1}{3}|D_i(x) - d_i|^2,$$

where $D_i(x) = 3ix$ and $d_i = 0, i = 1, 2, \cdots, m$. Since $D_i$ is continuous and linear for each $i = 1, 2, \cdots, m$, then we get that $g_i$ is proper, convex and lower semicontinuous function. Let $\lambda_n = 1 \ for \ all \ n \geq 1$, then for each $i = 1, 2, \cdots, m$,

$$\begin{aligned} J_1^{g_i}(x) &= Proxg_i(x) \\ &= argmin_{v \in K} \left[ g_i(v) + \frac{1}{2}\|v - x\|^2 \right] \\ &= (I + D_i^T D_i)^{-1}(x + D_i^T d_i). \end{aligned} \quad (4.\ 78)$$

Furthermore, for each $i = 1, 2$, take $\alpha_n = \frac{1}{2n+1}, \beta_{n,i} = \frac{in-1}{4in}, \gamma_n = \frac{1}{n^2}, f(x) = \frac{1}{2}x, \eta = \gamma = \frac{1}{4}, \alpha = 1, \lambda = \frac{3n+1}{4n}, L = 2$ so that all the conditions desired for the validity of Theorem 3.1 is satisfied. Hence, for $x_1 \in \mathbb{R}$, after applying our algorithm, (3. 39 ) becomes

$$\begin{cases} x_1 \in K; \\ s_n = W_\lambda^m(x) = J_1^{g(1)}(J_1^{g_1}(x)); \\ y_n = \left[1 - \left(\frac{n-1}{4n} + \frac{2n-1}{8n}\right)\right] s_n + \frac{n-1}{4n} z_{n,1} + \frac{2n-1}{8n} z_{n,2}; \\ z_{n,i} \in \left[\frac{1}{3^{in}} x_n, \frac{1}{5^{in}} x_n\right]; \\ x_{n+1} = P_K\left[\left(\frac{1}{2n+1}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}x_n\right) + \frac{1}{n^2}x_n + \left(\left(1 - \frac{1}{n^2}\right)I - \frac{1}{12(2n+1)}x_n\right)y_n\right], \end{cases} \tag{4. 79}$$

It follows from (4. 78 ) and (4. 79 ) that for $i = 1, 2$, we have

$$\begin{cases} x_1 \in K; \\ s_n = (I + D_1^T D_1)^{-1}(x_n + D_1^T d_1)[(I + D_2^T D_2)^{-1}(x_n + D_2^T d_2)]; \\ y_n = \left(1 - \frac{4n-3}{8n}\right) s_n + \frac{n-1}{4n} z_{n,1} + \frac{2n-1}{8n} z_{n,2}; \\ z_{n,i} \in \left[\frac{1}{3^{in}} x_n, \frac{1}{5^{in}} x_n\right], \ for\ i = 1, 2; \\ x_{n+1} = P_K\left[\frac{n^2 + 8(2n+1)}{8n^2(2n+1)} x_n + \left(\left(1 - \frac{1}{n^2}\right)I - \frac{1}{12(2n+1)}x_n\right)y_n\right], \end{cases} \tag{4. 80}$$

The table below shows the numerical experiment of algorithm (3. 39 ). **Conclusion**

In this paper, a modified proximinal point algorithm to approximate a common element of the set of solutin of fixed point problem for finite families of asymptotically quasi-nonexpansive multivalued mapping and minimization problem of (1. 18 ) is introduced and studied. Under mild conditions on the iteration parameters, strong convergence results were obtained using the algorithm so introduced and hence provides an affirmative answer to Question 1.1 raised in the paper. Since asymptotically quasi-nonexpansive multivalued mapping is much more general than asymptotically nonexpansive multivalued mapping , asymptotically nonspreading-type multivalued mapping, quasi-nonexpansive multivalued mapping, nonexpansive multivalued mapping, nonspreading-type multivalued mapping, the problem studied in our paper is quite general and includes in it problems in optimisation, varirtional inequality and fixed point as its special cases. Again, since (1. 23 ) generalizes (1. 21 ), (1. 20 ) and many other iteration schemes in this direction; (1. 24 ) is the same as (1. 21 ), which in turn generalizes (1. 20 ) since $T$ is a multivalued quasi-nonexpansive mapping; (1. 25 ) generalizes (1. 19 ), (1. 11 ) and (1. 9 ) and (1. 27 ) generalizes (1. 9 ), it follows that Theorem 3.2 in our paper improves, extends and generalizes the results obtained in $[14, 39, 40, 41, 43, 46, 47]$ and many more others currently existing in literature.

**Abbreviations Used**

Not applicable

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**Donatus Ikechi Igbokwe:** Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

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TABLE 1. Convergence of our new iterative algorithm.

| IV /Step | 0.4 | 0.6 | 0.8 | 0.11 | 0.13 |
|---|---|---|---|---|---|
| 1 | 0.40000000 | 0.60000000 | 0.80000000 | 0.11000000 | 0.13000000 |
| 2 | 0.05037507 | 0.11462787 | 0.20203119 | 0.00345377 | 0.00485356 |
| 3 | 0.00071686 | 0.00375542 | 0.01206665 | 0.00000337 | 0.00000665 |
| 4 | 0.00000014 | 0.00000398 | 0.00004108 | red!1000.00000000 | red!1000.00000000 |
| 5 | red!1000.00000000 | red!1000.00000000 | red!1000.00000000 | 0.00000000 | 0.00000000 |
| 6 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 7 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 8 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |