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On The Controlled Chaos of S-Iteration According to Lyapunov Exponent and Certain Control Mechanisms in Discrete Dynamical Systems

Derya Sekman
Department of Mathematics,
Krehir Ahi Evran University, Krehir, Trkiye,
Email: deryasekman@gmail.com

Vatan Karakaya*
Department of Mathematical Engineering,
Yildiz Technical University, stanbul, Trkiye,
Email: vkkaya@yildiz.edu.tr

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Abstract. In this paper, our aim is to establish the control mechanism of the iteration process defined as S-iteration in the literature. First of all, it is to determine the dynamics of a system with chaotic property under S-iteration. Afterward, we investigate the stability state of the chaotic system under this iteration with the parameters of the S-iteration. In addition, using the chaotic logistic system, we will illustrate the theoretical results in this example in the MATLAB program. Finally, we will examine the periodic state of the S-iteration and investigate the cases in which the periodic behavior is stable or unstable of the S-iteration with the Lyapunov exponent.

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1. Introduction and Preliminaries

Chaos, which entered the literature as unpredictability, attracted the attention of many researchers and started to be studied as a theory. It should be emphasized that objects open to change in nature are expressed mathematically with a growth rate within the framework of a rule. In other words, let $x_{n+1} = h(x_n)$ be a linear or non-linear system, then $\frac{x_{n+1}-x_n}{x_n}$ is called a growth rate and sensitive to initial conditions. Changes in nature are generally studied as differential equations. However, we will consider the discrete dynamical system as the difference equations. While the growth rate takes place on any object, external factors affecting this growth rate must also be taken into account. A growth rate taken together with external factors is mathematically expressed as $\frac{x_{n+1}-x_n}{x_n} \cong \lambda\left(\frac{h(x_n)}{x_n}-1\right)$, where λ is a sufficiently small external effect. The mathematical notation representing the growth rate as a discrete dynamical system corresponds to an iteration with the growth part, which is the famous iteration method of fixed point theory with a wide field of study.

Fixed point iteration systems obtained by the above relationship can be considered discrete dynamical systems. Then, whether the fixed point is stable or unstable in the iteration system can be discussed as a factor affecting this system. Considering the fact that the change in the external effect λ in the growth rate will cause the chaotic behavior of the system, the appropriate selection of the external factor λ has led to the idea of control theory. Since λ is found as a parameter in fixed point iteration systems, control can be expressed as the appropriate parameter selection in chaotic iteration systems. Therefore, chaos control mechanisms have been developed by many researchers to control the chaotic structures given in [4, 5, 8, 9, 10, 17].

If a fixed point iteration method involves a chaotic structure as a dynamical system, then the control of this chaos becomes an important research area. The fixed point iteration method is widely used both in different branches of science and in technological applications as a useful method. For this reason, examining iteration methods from different aspects provides many benefits in the field of theory or practice. Therefore, the authors studied the chaos control of different iteration methods and some applications in the given list [18, 19]. In addition, fixed point and control applications related to robotics or machine learning can be found in [2, 3, 12, 13, 14].

In the literature, there are different iteration methods studied by many authors. These processes, which started with Picard [16], have been developed over time and have been replaced by many new faster iteration types. Recently, the S-iteration process was defined by Agarwal et al. [1]. This iteration process cannot be reduced to previous fixed point iteration processes that are Picard [16], Mann [15] and Ishikawa [11] iteration processes, but it is more useful as it converges faster than them.

The main purpose of the work is to introduce the chaos control mechanism of the S-iteration process, which is considered a dynamical system. In order to obtain the establishment of the control mechanism, it is to investigate how a function with chaotic behavior turns into chaos to stability by selecting appropriate parameter ranges under the S-iteration process. In addition, using the logistic system with chaotic behavior, the stability regions, periodic regions, and chaotic regions of the iteration will be decided by using the MATLAB program. Finally, it is to determine the stable or unstable states of the periodic behavior of the S-iteration process using the Lyapunov exponent.

Now let's give the basic definitions and theorems that we will use throughout the paper. The general structure of a dynamical system is expressed as a sequence of composite functions with

$$x_0, h(x_0), h^2(x_0), \ldots, h^n(x_0), \ldots,$$

where $x_0 \in \mathbb{R}$ is a initial point and $h : X \to X$ is a selfmap. In this iterative structure consisting of an initial point x_0, each element is evaluated as a dynamic and the sequences formed by the repeating composition of a function is called the orbit of the initial point x_0. Hence, $h^n(x_0)$ is the nth iterate of h evaluated at x_0.

Definition 1.1 (see [7]). *Given $x_0 \in \mathbb{R}$, then the orbit of a point x_0 under the mapping h is defined to be the sequence of points $x_0,\ x_1 = h(x_0),\ x_2 = h^2(x_0), \ldots,\ x_n = h^n(x_0), \ldots$. The point x_0 is called the seed of the orbit.*

Definition 1.2 (see [6]). *Let X be a nonempty set and $h : X \to X$ be a selfmap. A point $x^* \in X$ is said to be fixed point of h if $h(x^*) = x^*$.*

Definition 1.3 (see [7]). *Let $h : X \to X$ be a mapping. A point $x^* \in X$ is a periodic point of h with period p if $h^p(x^*) = x^*$. The point x^* has prime period p if $h^p(x^*) = x^*$ and $h^n(x^*) \neq x^*$ for $0 < n < p$.*

Definition 1.4 (see [7]). *Let $h : X \to X$ be a selfmap having a fixed point x^*, and let $h'(x^*)$ denote the first derivative of $h(x)$ at $x = x^*$. Then, the point x^* is called an attracting (stable) fixed point if $|h'(x^*)| < 1$. The point x^* is called a repelling (unstable) fixed point if $|h'(x^*)| > 1$. And, if $|h'(x^*)| = 1$, the fixed point x^* is called neutral or indifferent.*

Definition 1.5 (see [16]). *Let X be a nonempty set and $h : X \to X$ be a selfmap. Let (x_n) be iteration sequence for any $x_0 \in X$. If*

$$x_{n+1} = h(x_n) = h^{n+1}(x_0), \tag{1. 1}$$

$(n \in \mathbb{N})$, then it is called Picard iteration process.

Definition 1.6 (see [1]). *For C a convex subset of a linear space X and h a mapping of C into itself, the iterative sequence (x_n) of the iteration process is generated from $x_1 \in C$, and is defined by*

$$\begin{cases} x_{n+1} = (1-\lambda_n)\, h(x_n) + \lambda_n h(y_n), \\ y_n = (1-\mu_n)\, x_n + \mu_n h(x_n), \end{cases} \tag{1. 2}$$

$(n \in \mathbb{N})$, then it is called S-iteration process where $(\lambda_n), (\mu_n)$ are real sequences in $(0,1)$ satisfying the condition:

$$\sum_{n=1}^{\infty} \lambda_n \mu_n (1-\mu_n) = \infty. \tag{1. 3}$$

2. Fixed Point Type Control Mechanism of the S-iteration

The general form of one-dimensional discrete dynamical systems is given by

$$x_{n+1} = h(x_n), \tag{2. 4}$$

for all $n \in \mathbb{N}$, where $h : [a,b] \to [a,b]$ be a continuous function with $[a,b] \subset \mathbb{R}$. Since h is defined on a continuous and closed interval, the function h has at least one fixed point. If the original system (2. 4) is rearranged in light of Definition 1.6, then we get the S-iteration system given by

$$x_{n+1} = S(\lambda, \mu, h)(x_n) = (1-\lambda_n)\, h(x_n) + \lambda_n h\left[(1-\mu_n)\, x_n + \mu_n h(x_n)\right], \tag{2. 5}$$

where $(\lambda_n), (\mu_n) \subset (0,1)$ are control parameter sequences. Using Eq. (2. 5), we can construct the fixed point type control mechanism of the S-iteration. In the remainder of the article, fixed point type controlling mechanism of S-iteration will be briefly illustrated as $S - f.p.t.c.$ mechanism.

As it is known, the main purpose of establishing a control mechanism in iterations is the existence of an unstable fixed point and the unstability of the system accordingly. This state of unstability is tried to be stabilized with the control mechanism. For the S-iteration, this situation will be explained by the following theorem.

Theorem 2.1. *Let x^* be an unstable real fixed point of an original system h given by Eq. (2. 4). Given that if $h(x^*) = x^*$, then $S(\lambda, \mu, h)(x^*) = x^*$. Then there always exists an effective regime of the control parameters $(\lambda_n), (\mu_n)$ $(n \in \mathbb{N})$ for the $S - f.p.t.c.$ mechanism $S(\lambda, \mu, h)$ defined by Eq. (2. 5) such that*

$$|S'(\lambda, \mu, h)(x^*)| < 1,$$

for $|h'(x^*)| \neq 1$.

Proof. Suppose that $x^* \in [a, b]$ is a fixed point of h, that is $h(x^*) = x^*$, then

$$\begin{aligned} S(\lambda, \mu, h)(x^*) &= (1 - \lambda_n) h(x^*) + \lambda_n h[(1 - \mu_n) x^* + \mu_n h(x^*)] \\ &= (1 - \lambda_n) h(x^*) + \lambda_n h(x^*) \\ &= (1 - \lambda_n) x^* + \lambda_n x^* \\ &= x^*, \end{aligned}$$

that is, x^* is a fixed point for $S(\lambda, \mu, h)(x^*)$.

Let x^* be an unstable fixed point for the original system h, then it should be $|h'(x^*)| > 1$. In this case, there are two cases such that $h'(x^*) < -1$ and $h'(x^*) > 1$. However, since the stability range obtained in the case of $h'(x^*) > 1$ contradicts with the definition ranges of control parameters $(\lambda_n), (\mu_n)$ $(n \in \mathbb{N})$, we will not examine situation $h'(x^*) > 1$. Therefore, let's consider two cases of $S(\lambda, \mu, h)$ under condition $h'(x^*) < -1$. The change of the mechanism defined by Eq. (2. 5) at unstable fixed point x^* can be controlled by the derivative of the function $S(\lambda, \mu, h)$. Taking the derivative of $S(\lambda, \mu, h)$, then we obtain

$$\begin{aligned} S'(\lambda, \mu, h)(x_n) &= (1 - \lambda_n) h'(x_n) + \lambda_n (h[(1 - \mu_n) x_n + \mu_n h(x_n)])' \\ &= (1 - \lambda_n) h'(x_n) + \lambda_n [1 + \mu_n (h'(x_n) - 1)] h'[(1 - \mu_n) x_n + \mu_n h(x_n)]. \end{aligned}$$

So, the derivative of $S(\lambda, \mu, h)$ at the fixed point x^* is calculated as follows:

$$\begin{aligned} S'(\lambda, \mu, h)(x^*) &= (1 - \lambda_n) h'(x^*) \\ &\quad + \lambda_n [1 + \mu_n (h'(x^*) - 1)] h'[(1 - \mu_n) x^* + \mu_n h(x^*)] \\ &= (1 - \lambda_n) h'(x^*) + \lambda_n [1 + \mu_n (h'(x^*) - 1)] h'(x^*) \\ &= (1 - \lambda_n) h'(x^*) + \lambda_n h'(x^*) + \lambda_n \mu_n h'(x^*)(h'(x^*) - 1) \\ &= h'(x^*) + \lambda_n \mu_n h'(x^*)(h'(x^*) - 1) \\ &= h'(x^*)(1 + \lambda_n \mu_n (h'(x^*) - 1)). \end{aligned} \qquad (2.\ 6)$$

Under the condition $h'(x^*) < -1$, if $|S'(\lambda,\mu,h)(x^*)| < 1$, we have

$$\begin{aligned} \left|h'(x^*)\left(1+\lambda_n\mu_n\left(h'(x^*)-1\right)\right)\right| &< 1 \\ -1 < h'(x^*)\left(1+\lambda_n\mu_n\left(h'(x^*)-1\right)\right) &< 1 \\ \frac{1}{h'(x^*)} < 1+\lambda_n\mu_n\left(h'(x^*)-1\right) &< \frac{-1}{h'(x^*)} \\ \frac{1-h'(x^*)}{h'(x^*)} < \lambda_n\mu_n\left(h'(x^*)-1\right) &< \frac{-1-h'(x^*)}{h'(x^*)} \\ \frac{1+h'(x^*)}{h'(x^*)\left(1-h'(x^*)\right)} < \lambda_n\mu_n &< \frac{-1}{h'(x^*)}. \end{aligned} \tag{2. 7}$$

When the inequality (2. 7) is solved according to control parameters $(\lambda_n), (\mu_n)$ $(n\in\mathbb{N})$, two solution sets that are symmetrical with respect to the other are obtained. However, since these regions are symmetrical, it will be sufficient to take one of them.

$$\Lambda_\lambda^1 = \left(\frac{1+h'(x^*)}{h'(x^*)\left(1-h'(x^*)\right)}, \frac{-1}{h'(x^*)}\right] \text{ and } \Lambda_\mu^1 = \left(\frac{1+h'(x^*)}{h'(x^*)\left(1-h'(x^*)\right)(\lambda_n)}, 1\right),$$
$$\Lambda_\lambda^2 = \left(\frac{-1}{h'(x^*)}, 1\right) \text{ and } \Lambda_\mu^2 = \left(\frac{1+h'(x^*)}{h'(x^*)\left(1-h'(x^*)\right)(\lambda_n)}, \frac{-1}{h'(x^*)(\lambda_n)}\right).$$

Thus, we obtain the effective regime of the control parameters of the $S(\lambda,\mu,h)$ that stabilizes the system h. □

Let us give the following example with a logistic system with chaotic behavior in order to implement the control mechanism that we have proved analytically for the S-iteration.

Example 2.2. *Consider the logistic system defined by* $h:[0,1]\to[0,1]$

$$x_{n+1} = h(x_n) = 4x_n\left(1-x_n\right). \tag{2. 8}$$

This system is a chaotic system in terms of the parameter and it has two fixed points as $x_1^* = 0$ *and* $x_2^* = \frac{3}{4}$ *which are both unstable fixed points with* $h'(x_1^*) = 4 > 1$ *and* $h'(x_2^*) = -2 < -1$. *To find a suitable stability range that will stabilize the logistic system* h*, let us apply the* $S - f.p.t.c.$ *mechanism* $S(\lambda,\mu,h)$ *defined by Eq. (2. 5) to the logistic system given by Eq. (2. 8).*

Firstly, we obtained the controlling mechanism $S(\lambda,\mu,h)$ *modified with the logistic system as follow:*

$$\begin{aligned} x_{n+1} = S(\lambda,\mu,h)(x_n) &= (1-\lambda_n)h(x_n) + \lambda_n h\left[(1-\mu_n)x_n + \mu_n h(x_n)\right] \\ &= 4x_n(1-x_n)(1-\lambda_n) + \lambda_n h\left[(1-\mu_n)x_n + 4x_n(1-x_n)\mu_n\right] \\ &= 4x_n(1-x_n)(1-\lambda_n) + \lambda_n h\left[x_n\left(1+\mu_n(3-4x_n)\right)\right] \qquad (2.\ 9) \\ &= 4x_n(1-x_n)(1-\lambda_n) + 4x_n\lambda_n\left(1+\mu_n(3-4x_n)\right)\left(1-x_n\left(1+\mu_n(3-4x_n)\right)\right). \end{aligned}$$

If we construct the mechanism given by Eq. (2. 9) for an unstable fixed point x^*, *we have*

$$\begin{aligned} S(\lambda,\mu,h)(x^*) = 4x^*(1-x^*)(1-\lambda_n) &+ 4x^*\lambda_n\left(1+\mu_n(3-4x^*)\right) \\ &\times\left(1-x^*\left(1+\mu_n(3-4x^*)\right)\right). \end{aligned}$$

By solving the equation $S(\lambda,\mu,h)(x^*) = x^*$, we obtained the fixed points of the $S-f.p.t.c.$ mechanism

$$x_1^* = 0,\ x_2^* = \frac{3}{4},$$
$$x_{3,4}^* = \frac{8\lambda_n\mu_n + 12\lambda_n\mu_n^2 \pm \left(-64\lambda_n\mu_n^2(1+4\lambda_n\mu_n) + \left(-8\lambda_n\mu_n - 12\lambda_n\mu_n^2\right)^2\right)^{\frac{1}{2}}}{32\lambda_n\mu_n^2}.$$

It is clear that x_1^* and x_2^* are common real fixed points with the logistic system h, while the other fixed points x_3^* and x_4^* are complex.

Now, we will investigate the control parameter ranges at which the fixed point $x_2^* = \frac{3}{4}$, which is unstable for the logistic system h, becomes stable by the controlling mechanism $S(\lambda,\mu,h)$. For this, we will examine the stability condition $|S'(\lambda,\mu,h)(x_2^*)| < 1$. Using the Eq. (2. 6), we have

$$S'(\lambda,\mu,h)(\frac{3}{4}) = 6\lambda_n\mu_n - 2.$$

Hence, under the stability condition, the solution is

$$\frac{1}{6} < \lambda_n\mu_n < \frac{1}{2}.$$

As a result, the unstable fixed point $x_2^* = \frac{3}{4}$ of the logistic system h becomes stable after a processing order with the appropriate control parameters (λ_n),(μ_n) $(n \in \mathbb{N})$ choices of the $S-f.p.t.c.$ mechanism $S(\lambda,\mu,h)$ defined by Eq. (2. 9). This situation exhibits that there is an effective regime consisting of two regions where the unstable fixed point $x_2^* = \frac{3}{4}$ behaves completely stable as follow

$$\Lambda_\lambda = \frac{1}{6} < \lambda_n \leq \frac{1}{2} \Rightarrow (\lambda_n) \subset \left(\frac{1}{6}, \frac{1}{2}\right],$$
$$\Lambda_\mu = \frac{1}{6\lambda_n} < \mu_n < 1 \Rightarrow (\mu_n) \subset \left(\frac{1}{6\lambda_n}, 1\right),\ (n \in \mathbb{N}).$$

Here, we used the MATLAB software program because it would be very difficult to calculate how the system will behave in what range with analytical methods. Table 1 shows the control parameter ranges in which the stable, periodical and chaotic behaviors of the orbit of the controlling mechanism $S(\lambda,\mu,h)$ is determined for the initial value $x_0 = 0.7333$ and $10,000$th iterations. Then, a range of control parameters given in Table 1 is shown by graphical analysis. In Figure 1 and Figure 2, it is shown that the original function h, which has a chaotic structure, exhibits a stable behavior with the fixed point controlling mechanism. Also, in Figure 3, the function diagram of the chaotic state of the logistic system h given and in Figure 4, it is shown on the function diagram that the unstable fixed point x_2^* becomes repel for the same parameter values selected in Figure 2, while it becomes attractive with the $S-f.p.t.c.$ mechanism $S(\lambda,\mu,h)$. In addition, the bifurcation diagram is given in Figure 5, where all the behaviors of the control parameter ranges obtained in Table 1 are shown together.

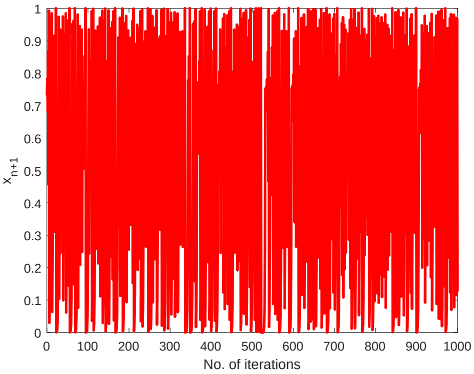

FIGURE 1. Chaotic state of original system $x_{n+1} = h(x_n)$

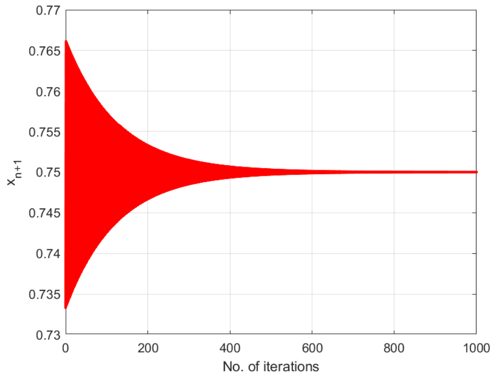

FIGURE 2. Stabilized orbits of $S(\lambda, \mu, h)$ for $\lambda_n = 0.35,\ 0.4762 < \mu_n < 1$

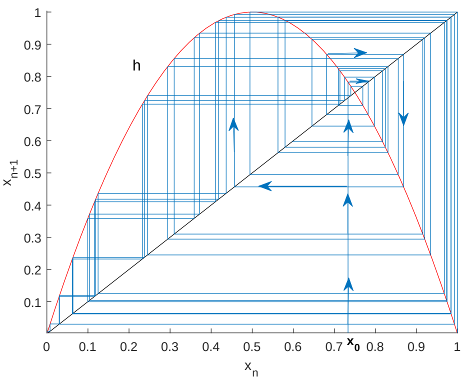

FIGURE 3. Function diagram of original system h

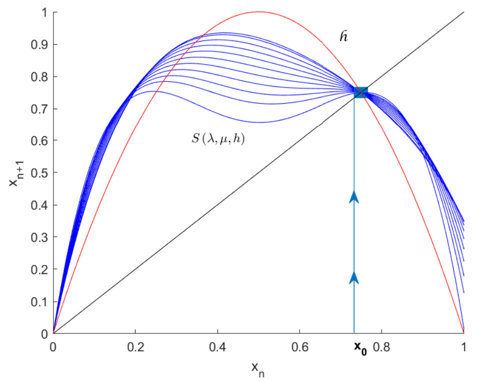

FIGURE 4. Function diagram of $S(\lambda, \mu, h)$

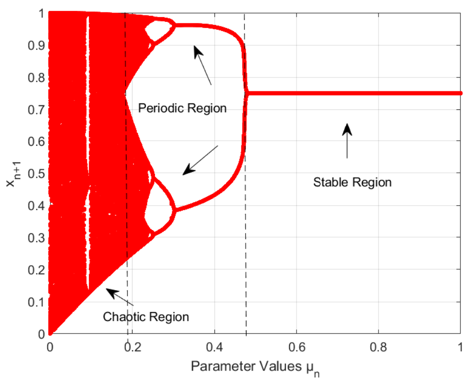

FIGURE 5. Bifurcation diagram of $S(\lambda, \mu, h)$ for $\lambda_n = 0.35,\ 0 < \mu_n < 1$

TABLE 1

Parameter values (λ_n)	*Parameter values* (μ_n)	*Orbit of* $S(\lambda,\mu,h)$
$0.17 \leq \lambda_n \leq 0.5$	$0 < \mu_n \leq 0.5099$	*Chaotic Behavior*
	$0.5099 < \mu_n \leq 0.9804$	*Periodical Behavior*
	$0.9804 < \mu_n < 1$	*Stable Behavior*
$0.25 \leq \lambda_n \leq 0.5$	$0 < \mu_n \leq 0.3430$	*Chaotic Behavior*
	$0.3430 < \mu_n \leq 0.6667$	*Periodical Behavior*
	$0.6667 < \mu_n < 1$	*Stable Behavior*
$0.35 \leq \lambda_n \leq 0.5$	$0 < \mu_n \leq 0.2366$	*Chaotic Behavior*
	$0.2366 < \mu_n \leq 0.4762$	*Periodical Behavior*
	$0.4762 < \mu_n < 1$	*Stable Behavior*
$0.48 \leq \lambda_n \leq 0.5$	$0 < \mu_n \leq 0.1672$	*Chaotic Behavior*
	$0.1672 < \mu_n \leq 0.3472$	*Periodical Behavior*
	$0.3472 < \mu_n < 1$	*Stable Behavior*

3. STABILIZATION OF THE UNSTABLE PERIODIC POINTS VIA S-ITERATION PROCESS

In this section, we firstly take the system $h^m = h \circ h \circ ... \circ h$ in place of original system h and then we will investigate that unstable fixed point in original system became to stable fixed point via $S-f.p.t.c.$ mechanism $S(\lambda,\mu,h)$. Let us redefine the $S-f.p.t.c.$ mechanism as

$$x_{n+1} = S_m(\lambda,\mu,h)(x_n) = (1-\lambda_n)h^m(x_n) + \lambda_n h^m[(1-\mu_n)x_n + \mu_n h^m(x_n)], \tag{3. 10}$$

where h^m denotes mth recurrent process of h.

In fixed point algorithms, it is necessary to advance the iteration m-steps to obtain periodic behaviors. To explain this situation, let's give the following theorem.

Theorem 3.1. *Let x^* be an unstable periodic fixed point of periods-m of an original system h defined by Eq. (2. 4). Given that if $h^m(x^*) = x^*$, then $S_m(\lambda,\mu,h)(x^*) = x^*$. Then there always exists an effective regime of the control parameters $(\lambda_n),(\mu_n)$ in the $S-f.p.t.c.$ mechanism $S_m(\lambda,\mu,h)$ defined by Eq. (3.10) such that*

$$|S'_m(\lambda,\mu,h)(x^*)| < 1,$$

for $|h^{m\prime}(x^*)| \neq 1$.

Proof. Given that $x^* \in [a,b]$ is a fixed point of periods-m of an original system h given by Eq. (2. 4), that is $h^m(x^*) = x^*$, then

$$\begin{aligned} S_m(\lambda,\mu,h)(x^*) &= (1-\lambda_n)h^m(x^*) + \lambda_n h^m[(1-\mu_n)x^* + \mu_n h^m(x^*)] \\ &= (1-\lambda_n)x^* + \lambda_n h^m(x^*) \\ &= x^*. \end{aligned}$$

Let x^* be an unstable fixed point of periods-m for the original system h given by Eq. (2. 4). The altering of the controlled mechanism defined by Eq. (3. 10) at unstable fixed point x^* can be obtained by the derivative of the function $S_m(\lambda, \mu, h)$. The derivative of $S_m(\lambda, \mu, h)$ at the fixed point x^* is calculated as follows:

$$S'_m(\lambda, \mu, h)(x^*) = h^{m\prime}(x^*)(1 + \lambda_n \mu_n (h^{m\prime}(x^*) - 1)).$$

Since $|S'_m(\lambda, \mu, h)(x^*)| < 1$, we obtain the following effective regime interval under the condition $h^{m\prime}(x^*) < -1$ as

$$\Lambda_\lambda = \left(\frac{1 + h^{m\prime}(x^*)}{h^{m\prime}(x^*)(1 - h^{m\prime}(x^*))}, \frac{-1}{h^{m\prime}(x^*)} \right],$$
$$\Lambda_\mu = \left(\frac{1 + h^{m\prime}(x^*)}{h^{m\prime}(x^*)(1 - h^{m\prime}(x^*))(\lambda_n)}, 1 \right). \quad (3.\ 11)$$

□

Now let's study the following example using the logistic function to illustrate the periodic behaviors that occur in the iteration h^m and its $S - f.p.t.c.$ mechanism.

Example 3.2. *Consider the logistic system defined by* $h : [0,1] \to [0,1]$

$$x_{n+1} = h(x_n) = 4x_n(1 - x_n). \quad (3.\ 12)$$

Let us determine the stability range of the unstable periods-2 *fixed points of the system* h^2 *for an effective regime of control parameters* $(\lambda_n), (\mu_n)$ $(n \in \mathbb{N})$ *in the* $S - f.p.t.c.$ *mechanism defined as Eq. (3. 10).*

First of all, in order to find the periods-2 *fixed points of the logistic system* h*, it is necessary to solve the fourth order equation* $h^2(x^*) = x^*$*. Therefore, we get*

$$h^2(x^*) = h(h(x^*)) = h(4x^*(1 - x^*)) = 16x^*(1 - x^*)\left(4x^{*2} - 4x^* + 1\right).$$

Solving the equation $h^2(x^*) = x^*$ *will find two trivial periods-*2 *fixed points* $x_1^* = 0$*,* $x_2^* = \frac{3}{4}$ *and a pair of non-trivial periods-*2 *fixed point* $x_1^{*(2)} = \frac{5-\sqrt{5}}{8}$*,* $x_2^{*(2)} = \frac{5+\sqrt{5}}{8}$*. Given derivative of* h^2*, we obtain*

$$h^{2'}(x^*) = 16(1 - 2x^*)\left(8x^{*2} - 8x^* + 1\right).$$

Therefore, we have $h^{2'}(x_1^{*(2)}) = h^{2'}(x_2^{*(2)}) = -4 < -1$ *which is unstable periods-*2 *fixed points.*

Now, consider the $S - f.p.t.c.$ *mechanism of* $S_2(\lambda, \mu, h)$ *defined by Eq. (3. 10) for* $m = 2$ *as follow*

$$x_{n+1} = S_2(\lambda, \mu, h) = (1 - \lambda_n) h^2(x_n) + \lambda_n h^2\left[(1 - \mu_n)x_n + \mu_n h^2(x_n)\right].$$

According to Eq. (3. 11), we obtain the stability interval for periods-2 *fixed points* $x_1^{*(2)}$ *and* $x_2^{*(2)}$ *as follows*

$$\Lambda_\lambda = \left(\frac{1 + h^{2\prime}(x^*)}{h^{2\prime}(x^*)(1 - h^{2\prime}(x^*))}, \frac{-1}{h^{2\prime}(x^*)} \right] = \frac{3}{20} < \lambda_n \leq \frac{1}{4} \Rightarrow (\lambda_n) \subset \left(\frac{3}{20}, \frac{1}{4} \right],$$
$$\Lambda_\mu = \left(\frac{1 + h^{2\prime}(x^*)}{h^{2\prime}(x^*)(1 - h^{2\prime}(x^*))\lambda_n}, 1 \right) = \frac{3}{20\lambda_n} < \mu_n < 1 \Rightarrow (\mu_n) \subset \left(\frac{3}{20\lambda_n}, 1 \right).$$

When control parameters $(\lambda_n), (\mu_n)$ $(n \in \mathbb{N})$ *are taken within* $\Lambda_\lambda, \Lambda_\mu$, *the controlling mechanism* $S_2(\lambda, \mu, h)$ *will converges to periodic fixed points either* $x_1^{*(2)}$ *or* $x_2^{*(2)}$ *depending on where the initial point start at. This situation is shown in Figure 6 for the initial values of* $x_0 = 0.3333$, $x_0 = 0.8333$ *and* $(\lambda_n) = 0.25$, $(\mu_n) = 0.75$.

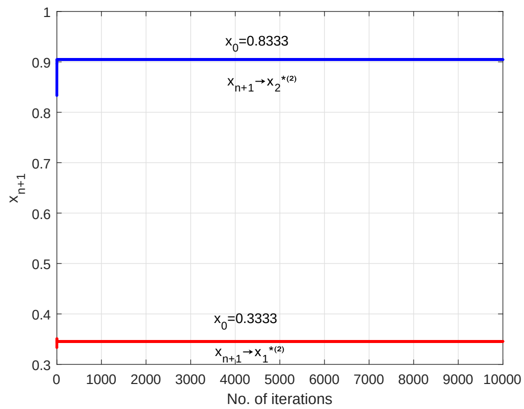

FIGURE 6. Stabilized orbits of the system $S_2(\lambda, \mu, h)$ for $(\lambda_n) = 0.25$, $(\mu_n) = 0.75$ and $x_0 = 0.3333, 0.8333$.

4. APPLICATION OF THE LYAPUNOV EXPONENTS FOR S-ITERATION PROCESS

Lyapunov exponent is a method used in nonlinear systems to measure the sensitive dependence between two orbits for very close starting points. For stable periodic behavior, this method measures the rate of convergence towards the stable fixed point. For chaotic behavior, it measures the rate of divergence between orbits.

Now, let's construct the Lyapunov exponent to use the S-iteration process. Let us denote the $S - f.p.t.c.$ mechanism $S(\lambda, \mu, h)$ defined by Eq. (2. 5). Also, let x and $x + \varepsilon$ $(0 < \varepsilon < 1)$ be two different points for orbits in the S-iteration process. In addition, assuming that the divergence between the two orbits measures δ and the exponential growth rate as $\varepsilon e^{m\rho}$, where ρ is the Lyapunov exponent, and m is a S-iteration of order m. Then, we can write

$$S(\lambda, \mu, h)(x + \varepsilon) - S(\lambda, \mu, h)(x) = \delta,$$

with $n \in \mathbb{N}$. That is,

$$S_m(\lambda, \mu, h)(x + \varepsilon) - S_m(\lambda, \mu, h)(x) = \varepsilon e^{m\rho},$$

and

$$\frac{S_m(\lambda, \mu, h)(x + \varepsilon) - S_m(\lambda, \mu, h)(x)}{\varepsilon} = e^{m\rho}. \tag{4. 13}$$

After taking the limit as $\varepsilon \to 0$ in Eq. (4. 13) , we get

$$\lim_{\varepsilon \to 0} \frac{S_m(\lambda, \mu, h)(x + \varepsilon) - S_m(\lambda, \mu, h)(x)}{\varepsilon} = e^{m\rho}.$$

So, we can write as follows

$$S'_m(\lambda,\mu,h)(x) = e^{m\rho}. \tag{4. 14}$$

If the logarithm is applied to both sides of the last equation again, we get

$$\rho = \frac{1}{m}\log\left|S'_m(\lambda,\mu,h)(x)\right|,$$

where $(\lambda_n),(\mu_n) \subset (0,1)$. Here, $S'_m(\lambda,\mu,h)(x)$ is the first derivative of $S_m(\lambda,\mu,h)$. We will use the chain rule to derivative the mth-degree polynomial. From the Eq. (4. 14), we can write that

$$\left|S'_m(\lambda,\mu,h)(x)\right| = S'(\lambda,\mu,h)(x_1).S'(\lambda,\mu,h)(x_2)\ldots S'(\lambda,\mu,h)(x_m) = e^{m\rho}. \tag{4. 15}$$

By the logarithm of the Eq. (4. 15), we get

$$\rho = \frac{1}{m}\sum_{k=1}^{m}\log\left|S'(\lambda,\mu,h)(x_k)\right|. \tag{4. 16}$$

By using Eq. (4. 16), we measure the convergence and divergence rates of an iteration, and we also decide whether the fixed points and periodic points of the given system are stable or unstable. In other words, we can explain when $\rho < 0$, the system is stable and when $\rho > 0$, it is also unstable.

Based on the above explanation, we will give the next two examples to decide on the stability and unstability of the periodic behavior of both S-iteration and S_m iterations with the Lyapunov exponent.

4.1. **Lyapunov exponent ρ for the fixed point $x_2^* = \frac{3}{4}$ when $\lambda_n = 0.35$, $\mu_n = 0.50$ for S-iteration.** Consider the $S - f.p.t.c.$ mechanism $S(\lambda,\mu,h)$ defined by Eq. (2. 5) such that

$$\begin{aligned} S'(\lambda,\mu,h)(x_n) =& (1-\lambda_n)(4-8x_n) + \lambda_n[1+\mu_n(3-8x_n)] \\ & \times [4-8((1-\mu_n)x_n + 4\mu_n x_n(1-x_n))]. \end{aligned} \tag{4. 17}$$

In (4. 17), taking $\lambda_n = 0.35$, $\mu_n = 0.50$ and $x_2^* = 0.75$, we get

$$\begin{aligned} S'(\lambda,\mu,h)(x_2^*) =& (1-(0.35)) \times (4 - 8 \times (0.75)) + (0.35) \\ & \times [1 + (0.5) \times (3 - 8 \times (0.75))] \\ & \times [4 - 8((1-(0.5)) \times (0.75) + 4 \times (0.5) \times (0.75) \times (1-(0.75)))] \\ =& -0.95. \end{aligned}$$

From the Eq. (4. 16) for the fixed point $x_2^* = 0.75$, we obtain

$$\rho = \log\left|S'(\lambda,\mu,h)(0.75)\right| = \log|-0.95| = -0.0222 < 0.$$

As a result, we obtain the $S - f.p.t.c.$ mechanism is stable according to value x_2^*, for $\lambda_n = 0.35$, $\mu_n = 0.50$.

4.2. **Lyapunov exponent ρ for the periodic points $x_1^{*(2)}$, $x_2^{*(2)}$ when $\lambda_n = 0.25$, $\mu_n = 0.75$.** Lyapunov exponent, given in Eq. (4. 16) for the $S - f.p.t.c.$ mechanism defined by Eq. (2. 5), is defined by taking $m = 2$ as follows

$$\rho = \frac{1}{2}\sum_{k=1}^{2}\log\left|S'\left(\lambda,\mu,h\right)(x_k^{(2)})\right|.$$

In (4. 17), taking periodic points $x_1^{*(2)} = \frac{5-\sqrt{5}}{8}$, $x_2^{*(2)} = \frac{5+\sqrt{5}}{8}$ and $\lambda_n = 0.25$, $\mu_n = 0.75$, we have

$$S'\left(\lambda,\mu,h\right)(x_1^{*(2)}) = (1-(0.25))\times(4-8\times\left(\frac{5-\sqrt{5}}{8}\right))+(0.25)$$
$$\times[1+(0.75)\times(3-8\times\left(\frac{5-\sqrt{5}}{8}\right))]$$
$$\times[4-8\times((1-(0.75))\times\left(\frac{5-\sqrt{5}}{8}\right)+4\times(0.75)\times\left(\frac{5-\sqrt{5}}{8}\right)\times(1-\left(\frac{5-\sqrt{5}}{8}\right)))]$$
$$= 0.303792.$$
$$S'\left(\lambda,\mu,h\right)(x_2^{*(2)}) = (1-(0.25))\times(4-8\times\left(\frac{5+\sqrt{5}}{8}\right))+(0.25)$$
$$\times[1+(0.75)\times(3-8\times\left(\frac{5+\sqrt{5}}{8}\right))]$$
$$\times[4-8\times((1-(0.75))\times\left(\frac{5+\sqrt{5}}{8}\right)+4\times(0.75)\times\left(\frac{5+\sqrt{5}}{8}\right)\times(1-\left(\frac{5+\sqrt{5}}{8}\right)))]$$
$$= -2.49129.$$

So, we get

$$\begin{aligned}\rho &= \frac{1}{2}\sum_{k=1}^{2}\log\left|S'\left(\lambda,\mu,h\right)(x_k^{(2)})\right| \\ &= \frac{1}{2}\left(\log\left|S'\left(\lambda,\mu,h\right)(x_1^{*(2)})\right|+\log\left|S'\left(\lambda,\mu,h\right)(x_2^{*(2)})\right|\right) \\ &= \frac{1}{2}\left(\log|0.303792|+\log|-2.49129|\right) \\ &= \frac{1}{2}\left(-1.19141+0.912801\right) \\ &= -0.139305 < 0.\end{aligned}$$

Since $\rho < 0$, the orbit of the logistic function h has a stable behavior. Thus, it is proved that the unstable periods-2 fixed points of the system h are stabilized by the $S - f.p.t.c.$ mechanism defined as Eq. (3. 10).

Remark 4.3. *Consider the iteration sequence obtained by the m-step iteration of the general logistic system $h(x) = rx(1-x)$. It is clear that the original logistic system has* *a chaotic behavior for $r = 4$. However, it has been shown in the present study that for $r = 4$, a control mechanism can be established in which an unstable fixed point becomes stable with the appropriate parameter selection of the $S-$iteration. For all that, how to develop a control mechanism for $r > 4$ of the coefficient r of the logistic the system without being dependent on an unstable fixed point is also an important problem. According to the results we obtained in the example we studied in Subsection 4.2, it has been shown that the chaotic logistic system can be stabilized for $r > 4$ using the λ_n, μ_n selected from the control interval. As can be seen from the Table 2, it has been shown that the behavior of the periodic $S-$iteration becomes stable with the help of the Lyapunov exponent when the r parameter in $rx(1-x)$ equation is $r_{\max} = 8.555$. Figure 7 shows the graphical representation of positive Lyapunov exponent ρ for $r \in [1, 4]$ of the original system $h(x)$. That is, the original system $h(x)$ behaves chaotic. Further, from Figure 8, it is observed that the Lyapunov exponent approaches to negative Lyapunov exponents ρ which means, for each initial point $x \in [0, 1]$ the orbit of the map converges to the stable attractor.*

TABLE 2

(λ_n)	(μ_n)	$r_{\max} \in [1, 10]$
0.16	0.94	6.922
0.2	0.76	7.806
0.25	0.61	8.555

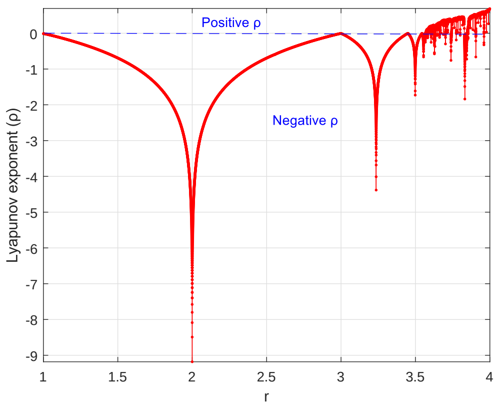

FIGURE 7. Lyapunov exponent diagram of original sytem $h(x)$ for $r \in [1, 4]$

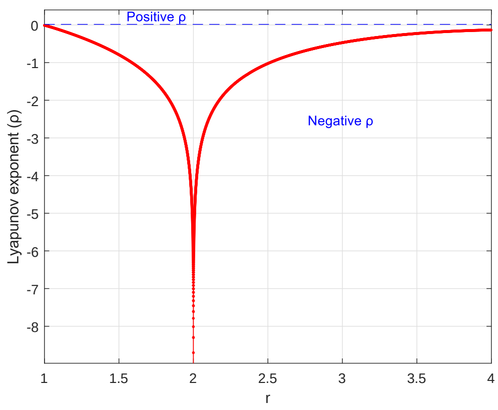

FIGURE 8. Lyapunov exponent diagram of the controlled system $S(\lambda, \mu, h)$ for $\lambda_n = 0.25$, $\mu_n = 0.75$ for $r \in [1, 4]$

5. CONCLUSION

We generally use two different methods to obtain the chaos control mechanisms of chaotic systems. The first method is to find the appropriate ranges of the parameters that are representative of the environmental factors affecting the system. Chaos control mechanisms related to fixed point iteration processes are possible with the appropriate selection of the parameters of the iteration. We should emphasize that the form $x_{n+1} - x_n = \lambda_n (h(x_n) - x_n)$ representing growth in the real world corresponds to the Mann iteration process mathematically represented as $x_{n+1} = (1 - \lambda_n) x_n + \lambda_n h(x_n)$. Since the mathematical modeling in which growthis represented corresponds to the fixed point iteration, the parameter (λ_n), which is determined as the environmental factor in the fixed point iteration process, causes chaos in certain situations. With the appropriate selection of the parameter that causes the chaos, the system is prevented from going into chaos and this process is called chaos control. The growth concept and its corresponding fixed point iteration process form the theoretical basis of algorithms used in contemporary scientific fields such as artificial intelligence, robotics, and machine learning. The use of chaotic functions in iteration processes creates chaos in the system and naturally, controlling mechanisms are required to prevent chaos. For example, the logistic function $rx(1-x)$ is chaotic for $r = 4$ and has unstable fixed points under the fixed point iteration system. Therefore, in the $4x(1-x)$ function, under fixed point iteration processes, the fixed point of the iteration is stabilized by choosing the appropriate interval. This method, which is used in the chaos control mechanism, guarantees the stability of the fixed points of the iteration processes and transforms the system from instability to stability. The second method provides the interval of r, which is the chaos coefficient of the $rx(1-x)$ function, for the parameters selected pointwise under the iteration processes of a function in the form of $rx(1-x)$. However, in this method, since the fixed point of the system changes for each r coefficient, it does not give the information that the fixed point of the iteration process is stable or unstable. Appropriate selection of iteration parameters used in the first method is a successful method in both transforming the system from unstable to stable and obtaining the maximum value of the r chaos coefficient in the function $rx(1-x)$.

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