Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol.52(12)2020 pp. 9-22

Bicomplex Version of Some Well Known Results in Complex Analysis

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Received: 29 April, 2020 / Accepted: 20 November, 2020 / Published online:22 December, 2020

Abstract.: In this paper, we explore for the bicomplex version of the well known Hadamard's three circles theorem in complex analysis and also deduce its convex form. Also, the relation between zeros and poles of a bicomplex valued function is established. Moreover the Jenson's Inequality as well as some results on univalent functions are proved here in the bicomplex context.

AMS (MOS) Subject Classification Codes: 30D30, 30G35.
Key Words: Bicomplex Valued Function, Zero, Pole, Hadamard's Three Circles Theorem, Jenson's Inequality, Univalent Function.

1. Introduction

he theory of bicomplex numbers is a matter of active research for quite a long time since seminal work as carried in [12] and [1] in search of special algebra. The algebra of bicomplex numbers are widely used in the literature as it becomes viable commutative alternative {cf. [13]} to the non skew field of quaternions introduced by Hamilton {cf. [5]} (both are four dimensional and generalization of complex numbers).

2. Preliminaries

2.1. **The Bicomplex Numbers**{**cf.**[10]}**.** A bicomplex number is defined as

$$\begin{aligned} z &= x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \\ &= (x_1 + i_1 x_2) + i_2 (x_3 + i_1 x_4) \\ &= z_1 + i_2 z_2, \end{aligned}$$

where $x_i, i = 1, 2, 3, 4$ are all real numbers with $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1, (i_1 i_2)^2 = 1$ and z_1, z_2 are complex numbers.

The set of all bicomplex numbers, complex numbers and real numbers are respectively denoted by $\mathbb{C}_2, \mathbb{C}_1$ and $\mathbb{C}_0$.

2.2. **Algebra of Bicomplex Numbers** {**cf.**[10]}**.** Addition is the operation on $\mathbb{C}_2$ defined by the function $\oplus : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$\begin{aligned} & (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) \\ = \ & (x_1 + y_1) + i_1 (x_2 + y_2) + i_2 (x_3 + y_3) + i_1 i_2 (x_4 + y_4) . \end{aligned}$$

Scalar multiplication is the operation on $\mathbb{C}_2$ defined by the function $\odot : \mathbb{C}_0 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(a, x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) = (a x_1 + i_1 a x_2 + i_2 a x_3 + i_1 i_2 a x_4) .$$

The system $(\mathbb{C}_2, \oplus, \odot)$ is a linear space. Here the norm is defined as

$$\begin{aligned} ||\ || &: \mathbb{C}_2 \to \mathbb{R}_{\geq 0}, \\ \|x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4\| &= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{\frac{1}{2}} . \end{aligned}$$

So the system $(\mathbb{C}_2, \oplus, \odot, ||\ ||)$ is a normed linear space.

The space $\mathbb{C}_0^4$ with the Euclidean norm is known to be a complete space. As $\mathbb{C}_2$ is embedded in $\mathbb{C}_0^4$ so that $x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ corresponds to (x_1, x_2, x_3, x_4) and for this reason the norm on $\mathbb{C}_2$ is the same as the norm of $\mathbb{C}_0^4$, then the normed linear space $(\mathbb{C}_2, \oplus, \odot, ||\ ||)$ is a complete Space. Hence $(\mathbb{C}_2, \oplus, \odot, ||\ ||)$ is a Banach Space.

The product on $\mathbb{C}_2$ is defined as

$$\otimes : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2\ by,$$

$$(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) = \begin{pmatrix} x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4 \\ + i_1 (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3) \\ + i_2 (x_1 y_3 - x_2 y_4 + x_3 y_1 - x_4 y_2) \\ + i_1 i_2 (x_1 y_4 + x_2 y_3 + x_3 y_2 + x_4 y_1) \end{pmatrix} .$$

As,

$$\begin{aligned}(i)\,\|z\,(z_1+i_2z_2)\| &= |z|\cdot\|z_1+i_2z_2\|\,and\\ (ii)\,\|(z_1+i_2z_2)\,(w_1+i_2w_2)\| &\leq \sqrt[2]{2}\,\|z_1+i_2z_2\|\cdot\|w_1+i_2w_2\|\,,\end{aligned}$$

where $z \in \mathbb{C}_1, (z_1+i_2z_2) \in \mathbb{C}_2$ and $(w_1+i_2w_2) \in \mathbb{C}_2$. So, $(\mathbb{C}_2, \oplus, \odot, ||\ \ ||, \otimes)$ is a Banach Algebra.

2.3. **Idempotent Representation of Bicomplex Numbers** $\{$**cf.**[10]$\}$**.** There are four idempotent elements in $\mathbb{C}_2$. They are

$$0, 1, \frac{1+i_1i_2}{2}, \frac{1-i_1i_2}{2}.$$

We now denote two non trivial idempotent elements by

$$e_1 = \frac{1+i_1i_2}{2} \text{ and } e_2 = \frac{1-i_1i_2}{2} \text{ in } \mathbb{C}_2.$$

where

$$e_1^2 = e_1, e_2^2 = e_2, e_1e_2 = e_2e_1 = 0, e_1 + e_2 = 1.$$

So, e_1 and e_2 are alternatively called orthogonal idempotents.

Every element $\xi : (z_1+i_2z_2) \in \mathbb{C}_2$ has the following unique representation

$$\begin{aligned}\xi &= (z_1-i_1z_2)\,e_1 + (z_1+i_1z_2)\,e_2\\ &= \xi_1e_1 + \xi_2e_2, \text{ where } \xi_1, \xi_2 \text{ are complex numbers.}\end{aligned}$$

This is known as idempotent representation of the element $\xi : (z_1+i_2z_2) \in \mathbb{C}_2$.An element $\xi : (z_1+i_2z_2) \in \mathbb{C}_2$ is non-singular iff $\left|z_1^2+z_2^2\right| \neq 0$ and it is singular iff $\left|z_1^2+z_2^2\right| = 0$. The set of all singular elements is denoted by θ_2.

2.4. **Topological Aspects of Bicomplex Space**$\{$**cf.**[11]$\}$**.** The topological concepts employed for sets of bi complex numbers will be those of four dimensional euclidean and space. For example a set of points S will be called open if for every z_0 in S. An open connected set will be called a region. The set of all bicomplex numbers with this topology will be called the bicomplex space. If T is a region, and if each z on T is written in the form $z = z_1e_1 + z_2e_2$, $\left(\text{where } e_1 = \frac{1}{2}(1+ij)\,,\ e_2 = \frac{1}{2}(1-ij)\right)$, then the set T_1 of values of z_1 is a region in the z_1-plane (in the topology of that plane) and the set T_2 of values of z_2-plane. These region T_1 and T_2 will be termed the component regions of T. If the regions T_1 and T_2 are given, the largest region T whose component regions are T_1 and T_2 will be termed the product-region of T_1 and T_2.

It should be observed that for convenience the regions T_1 and T_2 have been chosen in the complex z_1- and z_2-planes, which are not planes of the bicomplex spaces. If component-regions in the space itself are desired, the components z_1e_1 and z_2e_2 of the number z, located in the first and second nil-planes, respectively, should be considered.

2.5. **The Discuss**$\{$**cf.**[10]$\}$**.** Let $a:(a_1+i_1a_2+i_2a_3+i_1i_2a_4)$ be a fixed point in $\mathbb{C}_2$. Set $\alpha=\alpha_1+i_1\alpha_2$ and $\beta=\beta_1+i_1\beta_2$. Then $a=(a_1+i_1a_2+i_2a_3+i_1i_2a_4)=\alpha+i_2\beta$. Let r,r_1,r_2 denote the numbers in $\mathbb{C}_0$ such that $r>0,\quad r_1>0$ and $\quad r_2>0$. Also let $A_1=\{z_1-i_1z_2 : z_1,z_2 \text{ in } \mathbb{C}_1\}$, $A_2=\{z_1+i_1z_2 : z_1,z_2 \text{ in } \mathbb{C}_1\}$. And w_1 and w_2 respectively denote the numbers in A_1 and A_2.We observe here that w_1 and w_2 are in fact complex numbers in $\mathbb{C}_1$. We should recall that the open ball $B(a,r)$ and the closed ball $\overline{B}(a,r)$ with centre a and radius r are repectively defined as follows:

$$\begin{aligned} B(a,r) &= \{z_1+i_2z_2 \text{ in } \mathbb{C}_2 : \|(z_1+i_2z_2-(\alpha+i_2\beta))\|<r\}\ and \\ \overline{B}(a,r) &= \{z_1+i_2z_2 \text{ in } \mathbb{C}_2 : \|(z_1+i_2z_2-(\alpha+i_2\beta))\|\le r\}. \end{aligned}$$

Then the open and closed discuss with centre a and radii r_1 , r_2 respectively denoted by $D(a;r_1,r_2)\ and\ \overline{D}(a;r_1,r_2)$ are defined as

$$D(a;r_1,r_2)=\left\{\begin{array}{c} z_1+i_2z_2 \text{ in } \mathbb{C}_2 : z_1+i_2z_2=w_1e_1+w_2e_2, \\ |w_1-(\alpha-i_1\beta)|<r_1, |w_2-(\alpha-i_1\beta)|<r_2 \end{array}\right\} and$$

$$\overline{D}(a;r_1,r_2)=\left\{\begin{array}{c} z_1+i_2z_2 \text{ in } \mathbb{C}_2 : z_1+i_2z_2=w_1e_1+w_2e_2, \\ |w_1-(\alpha-i_1\beta)|\le r_1, |w_2-(\alpha-i_1\beta)|\le r_2 \end{array}\right\}.$$

2.6. **Compact Set** $\{$**cf.**[3]$\}$**.** A set $K\subset\mathbb{C}_2$ is compact if for a collection ζ of open sets in $\mathbb{C}_2$ with the property $K\subset\cup\{G : G\in\zeta\}$, there is a finite number of sets $G_1,G_2,....,G_n$ in ζ such that $K\subset G_1\cup G_2\cup....\cup G_n$.

2.7. **Uniformly Convergence in** $\mathbb{C}_2\{$**cf.**[3]$\}$**.** A sequence $\{f_n\}$ of bicomplex holomorphic functions defined on domain $S\subseteq\mathbb{C}_2$ is said to be converge uniformly on a compact subset of S to a bicomplex function f if for any compact subset K of S and for $\epsilon>0$, there is a positive integer n_0 such that

$$\|f_n(n)-f(w)\|<\epsilon$$

for all $n\ge n_0$ and $w\in K$. By the notation $f_n\to f$, we consider that $\{f_n\}$ converges to f in $\mathbb{C}_2$.

2.8. **Bicomplex Holomorphic Function**$\{$**cf.**[10]$\}$**.** We start with a bicomplex valued function

$$f:\Omega\subset\mathbb{C}_2\to\mathbb{C}_2.$$

The derivative of f at a point $\omega_0\in\Omega$ is defined by

$$f'(\omega)=\lim_{h\to 0}\frac{f(\omega_0+h)-f(\omega_0)}{h}$$

provided the limit exists and the domain is so chosen that

$$h=h_0+i_1h_1+i_2h_2+i_1i_2h_3$$

is invertible. It is easy to prove that h is not invertible only for $h_0=-h_3, h_1=h_2$ or $h_0=h_3, h_1=-h_2. i.e. h\notin\theta_2.$

If the bicomplex derivative of f exists at each point of its domain then in similar to complex function, f will be a bicomplex holomorphic function in Ω. Indeed if f can be expressed as

$$\begin{aligned} f(\omega) &= g_1(z_1, z_2) + i_2 g_2(z_1, z_2) \\ \omega &= z_1 + i_2 z_2 \in \Omega \end{aligned}$$

then f will be holomorphic if and only if g_1, g_2 are both complex holomorphic in z_1, z_2 and

$$\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2}, \frac{\partial g_1}{\partial z_2} = -\frac{\partial g_2}{\partial z_1}.$$

Moreover,

$$f'(\omega) = \frac{\partial g_1}{\partial z_2} + i_2 \frac{\partial g_2}{\partial z_1}.$$

Alternatively, Let $f(z)$ be a bicomplex valued function of the bicomplex variable $z = x + jy$, defined in a region T. Let z_0 be a point in T. Then $f(z)$ will be termed analytic at z_0 if and only if there exists a bicomplex number $f'(z_0)$ such that for any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\| < \epsilon$$

whenever $\|z - z_0\| < \delta_\epsilon$ and $|z - z_0| \neq 0$.

A function $f(z)$ will be termed analytic in aregion T if it is analytic at each point of T.

2.9. **Bicomplex Entire Function{cf.[10]}.** A function f is said to be a bicomplex entire function if f is analytic in the whole bicomplex plane $\mathbb{C}_2$.

2.10. **Decomposition Theorem of Ringleb{cf.[11]}.** Let $f(z)$ be analytic in a region T, and let T_1 and T_2 be the component regions of T, in the z_1- and z_2-planes, respectively. Then there exists a unique pair of complex-valued analytic functions, $g(z_1)$ and $h(z_2)$, defined in T_1 and T_2, respectively, such that

$$f(z) = g(z_1) e_1 + h(z_2) e_2$$

for all z in T. Conversely, if $g(z_1)$ is any complex-valued analytic function in a region T_1 and $h(z_2)$ is any complex-valued analytic function in a region T_2, then the bicomplex valued function $f(z)$ defined by the formula is an analytic function of the bicomplex variable z in the product-region T of T_1 and T_2.

2.11. **Zero and Pole of a Bicomplex Function{cf.[11]}.** By Decomposition Theorem of Ringleb, if $f(z)$ is analytic in a neighbourhood of the origin then $f(z)$ will said to have a zero of order at least n, where n is a positive integer, at the origin if and only if both $g(z_1)$ has a zero of order at least n at $z_1 = 0$ and $h(z_2)$ has a zero of order at least n at $z_2 = 0$.

By Decomposition Theorem of Ringleb, if $f(z)$ is analytic in a deleted neighbourhood of $z = z_0 = z_1^0 e_1 + z_2^0 e_2$.Then $f(z)$ will be said to have a pole of order at most n, where n is a non negative integer, in the nil plane with respect to z_0 if both $g(z_1)$ has a pole of order at most n at $z_1 = z_1^0$and $h(z_2)$ has a zero of order at least n at $z_2 = z_2^0$. If both $g(z_1)$ and $h(z_2)$ have a pole of order n at these points, $f(z)$ will be said to have a pole of order n.

Example Let $f(z) = \frac{z-1}{z}$.Then $z = 1$is the zero of f in $\mathbb{C}_2$ and on the other hand, $z = 0$ is the pole of order 1 of f in $\mathbb{C}_2$.

2.12. **Convex Function**$\{$**cf.**[2]$\}$**.** Let us now recall the definition of convex function.

A function $f(x)$ of real variable x is said to be convex downwards or simple convex if the curve $y = f(x)$ between x_1 and x_2 always lies below the chord joining the points $(x_1, y_1), (x_2, y_2)$ where $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

The equation of the chord is

$$\begin{aligned} y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\ i.e., y &= \frac{x - x_1}{x_2 - x_1} y_2 + \left(1 + \frac{x - x_1}{x_2 - x_1}\right) y_1 \\ i.e., y &= \left(\frac{x - x_1}{x_2 - x_1}\right) y_2 + \left(\frac{x_2 - x}{x_2 - x_1}\right) y_1. \end{aligned}$$

Therefore $y = f(x)$ is convex iff analytically the following condition is satisfied $i.e.$,

$$\begin{aligned} y &< \left(\frac{x - x_1}{x_2 - x_1}\right) y_2 + \left(\frac{x_2 - x}{x_2 - x_1}\right) y_1 \\ i.e., f(x) &< \left(\frac{x - x_1}{x_2 - x_1}\right) f(x_2) + \left(\frac{x_2 - x}{x_2 - x_1}\right) f(x_1). \end{aligned}$$

2.13. **Univalent Function or Simple Function** $\{$**cf.**[11]$\}$**.** A bicomplex valued function f is univalent (i.e. simple) in a region D if it is regular, one valued and does not take any value more than once in D i.e. $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$, $z_1, z_2 \in D$.

In this paper our prime concern is to derive the bicomplex analog of some well known results, especially Hadamard's three circles theorem with its convex form, Jenson's Inequality on univalent (i.e. simple) functions in $\mathbb{C}_1$. We do not explain the standard definitions and notatios of the theories of bicomplex valued entire functions as those are available in $\{$cf.[10], [2], [7] and [8]$\}$.

3. Lemmas

In this section we present some relevant lemmas which will be needed in the sequel.

Lemma 3.1. [2] *Limit of a uniformly convergent sequence of univalent function in $\mathbb{C}_1$ is either simple or constant.*

Lemma 3.2. [14][11] *Let f be analytic in the deleted neighbourhood of α and has a pole of order $m \geqslant 1$ at α iff f can be expressed in the form $f(z) = \frac{\psi(z)}{(z-\alpha)^m}$ is some neighbourhood of α where ψ is analytic at $z = \alpha$ and $\psi(\alpha) \neq 0$,where α is zero of f.*

Proof. Let $\alpha = \alpha_1 e_1 + \alpha_2 e_2$ be a pole of f of order m and let $f(z) = f_1(z_1) e_1 + f_2(z_2) e_2$ [Ringleb Decomposition].

Then from the definition of pole we can say tha α_1 *and* α_2 is a pole of order m for $f_1(z_1)$ *and* $f_2(z_2)$ respectively, where $\alpha_1, \alpha_2, f_1(z_1), f_2(z_2) \in \mathbb{C}_1$.

Then in some neighbourhood of $\alpha_i, f_i(\alpha_i)$ for $i = 1, 2$ has Laurent seies expansion $\{$cf.[3]$\}$,

$$f_1(z_1) = \sum_{n=0}^{\infty} a_n^{'} (z_1 - \alpha_1)^n + \sum_{j=1}^{m} b_j^{'} (z_1 - \alpha_1)^{-j}, \text{ where } b_m^{'} \neq 0,$$

and

$$f_2(z_2)=\sum_{n=0}^{\infty}a_n^{''}(z_2-\alpha_2)^n+\sum_{j=1}^{m}b_j^{''}(z_2-\alpha_2)^{-j}\ ,\ \text{where } b_m^{''}\neq 0.$$

Therefore by Ringleb decomposition of f in $\mathbb{C}_2$ it follows that,

$$f(z)=\left\{\sum_{n=0}^{\infty}a_n^{'}(z_1-\alpha_1)^n+\sum_{j=1}^{m}b_j^{'}(z_1-\alpha_1)^{-j}\right\}e_1+\left\{\sum_{n=0}^{\infty}a_n^{''}(z_2-\alpha_2)^n+\sum_{j=1}^{m}b_j^{''}(z_2-\alpha_2)^{-j}\right\}e_2.$$

$$=\sum_{n=0}^{\infty}\left(a_n^{'}e_1+a_n^{''}e_2\right)\left[(z_1e_1+z_2e_2)-(\alpha_1e_1+\alpha_2e_2)\right]^n$$
$$+\sum_{n=0}^{\infty}\left(b_j^{'}e_1+b_j^{''}e_2\right)\left[(z_1e_1+z_2e_2)-(\alpha_1e_1+\alpha_2e_2)\right]^{-j}$$

Since $b_n^{'}\neq 0, b_n^{''}\neq 0$ we have $b_m\neq 0$ and also in view of $a_n=a_n^{'}e_1+a_n^{''}e_2, b_j=b_j^{'}e_1+b_j^{''}e_2$ we may write that,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{j=1}^{m}b_j(z-\alpha)^{-j}$$

$$=\phi(z)+\sum_{j=1}^{m}b_j(z-\alpha)^{-j}$$

$$=\frac{(z-\alpha)^m\phi(z)+b_m+\sum_{j=1}^{m-1}b_j(z-\alpha)^{m-j}}{(z-\alpha)^m}$$

$$=\frac{\psi(z)}{(z-\alpha)^m},$$

where $\psi(z)=(z-\alpha)^m\phi(z)+b_m+\sum_{j=1}^{m-1}b_j(z-\alpha)^{m-j}$. Clearly $\psi(z)$ is bicomplex valued analytic at $z=\alpha$ and $\psi(\alpha)=b_m\neq 0$.

Next let us suppose that in some neighbourhood of $z=\alpha$, $f(z)=\frac{\psi(z)}{(z-\alpha)^m}$ where ψ is analytic at $z=\alpha$ and $\psi(\alpha)\neq 0$.

Now, we expand ψ in Taylor's series around α in order to get $\psi(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n$ where $a_0\neq 0$ becomes $\psi(\alpha)\neq 0\{cf.[10]\}$. Hence,

$$f(z)=\frac{\sum_{n=0}^{\infty}a_n(z-\alpha)^n}{(z-\alpha)^m}$$

$$=\frac{a_0}{(z-\alpha)^m}+\frac{a_1}{(z-\alpha)^{m-1}}+.......+\frac{a_{m-1}}{(z-\alpha)}+\sum_{n=m}^{\infty}a_n(z-\alpha)^{n-m},$$

which is the bicomplex analog of Laurent Series expanssion of f around α. Since in the principal part of the above series the coefficient of $(z-\alpha)^{-m}$ is non vanishing, it therefore follows that α is a pole of f of order $m \geqslant 1$.Thus the theorem is established. □

Lemma 3.3. [14][11]*The point α is a zero of order $m(\geqslant 1)$ of a bicomplex valued analytic function f iff f can be expressed in the form $f(z) = (z-\alpha)^m \phi(z)$, where ϕ is analytic at α and $\phi(\alpha) \neq 0$ and the representation is valid in some neighbourhood of α, where α is zero of f.*

Lemma 3.4. *The point α is a pole of order $m(\geqslant 1)$ of a bicomplex valued function f iff it is a zero of order m of $\frac{1}{f}$.*

Proof. Since f has a pole of order $m\ at\ z = \alpha$. By Lemma3.2 we may write, in some neighbourhood of $z = \alpha$, $f(z) = \phi(z)(z-\alpha)^{-m}$where ϕ is analytic at $z = \alpha$ and $\phi(\alpha) \neq 0$. Therefore

$$\frac{1}{f(z)} = (z-\alpha)^m \psi(z)$$

where $\psi(z) = \frac{1}{\phi(z)}$ is analytic at α and $\psi(\alpha) = \frac{1}{\phi(\alpha)} \neq 0$. So, α is a zero of $\frac{1}{f(z)}$ of order m.

Conversely let, $z = \alpha$ be a zero of order m of $\frac{1}{f}$. Then we can write $\frac{1}{f(z)} = (z-\alpha)^m g(z)$ in some neighbourhood of α where g is analytic at α and $g(\alpha) \neq 0$.

So,

$$f(z) = \frac{1}{(z-\alpha)^m g(z)} = \frac{h(z)}{(z-\alpha)^m}$$

where $h(z) = \frac{1}{g(z)}$ is analytic at $z = \alpha$ and $h(\alpha) = \frac{1}{g(\alpha)} \neq 0$. Hence h is a pole of order $m \geqslant 1$. □

4. Theorems

In this section we present the main results of our paper.

Theorem 4.1 is the bicomplex analig of Hadamard's Three Circle Theorem.

Theorem 4.1. *Let $f(z)$ be bicomplex valued analytic in $r_1 \leq \|z\| \leq r_3$ and let $r_1 < r_2 < r_3$. If M_i be the maximum value of $\|f(z)\|$ on the circles $\|z\| = r_i$ for $i = 1, 2, 3$, then*

$$M_2^{\log\ \frac{r_2}{r_3}} \leq M_1^{\log\ \frac{r_3}{r_2}} \cdot M_3^{\log\ \frac{r_2}{r_1}} .$$

Proof. Let us consider the function $z^{-k} f(z)$ where k is a constant to be detemined. Since

$$\max_{\|z\|=r_i} \|f(z)\| = M_i, \ \ i = 1, 2, 3,$$

The maximum value of $\left\|z^{-k} f(z)\right\|$ on $\|z\| = r_1$ and $\|z\| = r_3$ are given by $\frac{M_1}{r_1^k}$ and $\frac{M_3}{r_3^k}$ respectively.

Let k be determined by the condition that

$$\frac{M_1}{r_1^k} = \frac{M_3}{r_3^k} \tag{4. 1}$$

and λ by the condition

$$r_1^{\lambda} \cdot r_3^{1-\lambda} = r_2 \tag{4. 2}$$

Taking logarithm on both sides of (4. 2) we get that,

$$\lambda \log r_1 + (1-\lambda) \log r_3 = \log r_2$$

$$\begin{aligned} i.e., \log r_3 - \log r_2 &= \lambda (\log r_3 - \log r_1) \\ i.e., \lambda &= \frac{\log \left(\frac{r_3}{r_2}\right)}{\log \left(\frac{r_3}{r_1}\right)} \end{aligned} \tag{4. 3}$$

So,

$$1 - \lambda = 1 - \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} = \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1}. \tag{4. 4}$$

Now any quantity a can be written as

$$a = a^{\lambda} \cdot a^{1-\lambda}.$$

Therefore from (4. 2) we have,

$$a = \frac{M_1}{r_1^k} = \frac{M_3}{r_3^k} = \frac{M_1^{\lambda}}{r_1^{k\lambda}} \cdot \frac{M_3^{1-\lambda}}{r_3^{k(1-\lambda)}}.$$

It follows that

$$\frac{M_1}{r_1^k} = \frac{M_3}{r_3^k} = \frac{M_1^{\lambda} \cdot M_3^{1-\lambda}}{r_2^k}.$$

From above we observe that the maximum of $\left\|z^{-k} f(z)\right\|$ on circles $\|z\| = r_1$ and $\|z\| = r_3$ are respectively $\frac{M_1}{r_1^k}$ and $\frac{M_3}{r_3^k}$ and these are equal, each being equal to $\frac{M_1^{\lambda} \cdot M_3^{1-\lambda}}{r_2^k}$.

Now let us consider the maximum of $\left\|z^{-k} f(z)\right\|$ on circles $\|z\| = r_2$ and we get that

$$\frac{M_2}{r_2^k} \leq \frac{M_1^{\lambda} \cdot M_3^{1-\lambda}}{r_2^k}$$

$$i.e., M_2 \leq M_1^{\lambda} \cdot M_3^{1-\lambda}$$

Putting the value of λ and $1-\lambda$ from (4. 3) and (4. 4) in the above we obtain that

$$M_2 \leq M_1^{\log(r_3/r_2)/\log(r_3/r_1)} \cdot M_3^{\log(r_2/r_1)/\log(r_3/r_1)}$$

Raising both sides to the power $\log(r_3/r_1)$ it follows that

$$M_2^{\log \frac{r_2}{r_3}} \leq M_1^{\log \frac{r_3}{r_2}} \cdot M_3^{\log \frac{r_2}{r_1}}.$$

This completes the proof of the theorem. □

Remark 4.2. *Hadamard's Three Circle Theorem in $\mathbb{C}_2$ may be expressed by saying that $M(r)$ is a convex function of $\log r$ as we see in the following theorem.*

Theorem 4.3. *If $f(z)$ is a bicomplex valued analytic in the closed ring $r_1 \leq \|z\| \leq r_3$, and if $M(r_i)$ denotes the maximum value of $\|f(z)\|$ on the circles $\|z\| = r_i$ with $r_1 \leq r_2 \leq r_3$then*

$$\log M(r_2) \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3) \tag{4. 5}$$

Proof. In view of Theorem 4.1 we have,

$$\{M(r_2)\}^{\log \frac{r_2}{r_3}} \leq \{M(r_1)\}^{\log \frac{r_3}{r_2}} \cdot \{M(r_3)\}^{\log \frac{r_2}{r_1}} . \tag{4. 6}$$

Therefore we obtain (4. 5) . □

Remark 4.4. *The sign of equality holds when $f(z)$ is of the form az^n, where a is a constant in $\mathbb{C}_2$ as we see below.*

Considering $f(z) = az^n$ where $a, z \in \mathbb{C}_2$.Then $M(r_i) = \max\{\|f(z)\| : \|z\| = r_i \; for \; i = 1,2,3\} = ||a||r_i^n$ on the circles $\|z\| = r_i$ for $i = 1,2,3$. Now

$$\begin{aligned}
\log M(r_2) &= \log ||a|| + n \log r_2 \\
&\quad \text{and } \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3) \\
&= \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1}(\log ||a|| + n \log r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1}(\log ||a|| + n \log r_3) \\
&= \log ||a|| + n \log r_2 .
\end{aligned}$$

The equality in Theorem 3.6 holds for az^n.

Theorem 4.5. *Let $f(z)$ be a bicomplex valued entire function which does not vanish at the origin. Also let $r_1, r_2, r_3, \ldots.., r_n$ be the moduli of zeros $z_1, z_2, z_3, \ldots.., z_n$ of $f(z)$ arranged as a non decreasing sequence, multiple zero being repeated. Then*

$$R^n \|f(0)\| \leq M(R) \cdot r_1 \cdot r_2 \cdot r_3 \cdot \cdots \cdot r_n \quad \textit{when } r_n < R < r_{n+1}.$$

Proof. Let us consider the function

$$F(z) = f(z) \prod_{m=1}^{n} \frac{\left(R^2 - z \cdot \overline{z}_m\right)}{R(z - z_m)}, \tag{4. 7}$$

where $\overline{z}_m$ is i_2 conjugate of z_m{cf.[10]}

Since $f(z)$ is entire in $\mathbb{C}_2$, it follows that $F(z)$ is also entire in $\mathbb{C}_2$.

Now, by idempotent decompositions of both f, F we can write (4. 7) as

$$F_1\left(z'\right) e_1 + F_2\left(z''\right) e_2 = \left(f_1\left(z'\right) e_1 + f_2\left(z''\right) e_2\right) \prod_{m=1}^{n} \frac{\left\{R^2 - \left(z' e_1 + z'' e_2\right)\left(\overline{z'} e_1 + \overline{z''} e_2\right)\right\}}{R\left\{(z' e_1 + z'' e_2) - (z'_m e_1 + z''_m e_2)\right\}}, \tag{4. 8}$$

where $z = z' e_1 + z'' e_2$ and $z', z''$are all in $\mathbb{C}_1$.

Also, if

$$z = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2 = z' e_1 + z'' e_2$$

then,

$$\overline{z} = z_1 - i_2 z_2 = \overline{(z_1 - i_1 z_2)} e_1 + \overline{(z_1 + i_1 z_2)} e_2 = \overline{z'} e_1 + \overline{z''} e_2.$$

Now from (4. 8) we can write that,

$$F_1\left(z'\right) = f_1\left(z'\right) \prod_{m=1}^{n} \frac{\left(R^2 - z' \cdot z'_m\right)}{R\left(z' - z'_m\right)} \tag{4. 9}$$

and

$$F_2\left(z'\right) = f_2\left(z''\right) \prod_{m=1}^{n} \frac{\left(R^2 - z'' \cdot z''_m\right)}{R\left(z'' - z''_m\right)}. \tag{4. 10}$$

Now as

$$R_1^2\left(z' - a'\right)\left(\overline{z'} - \overline{a'}\right) = R_1^2\left(z' \cdot \overline{z'} - \left(a' \cdot \overline{z'} + \overline{a'} \cdot z'\right) + a' \cdot \overline{a'}\right),$$

and on the circle $\left|z'\right| = R_1$, $z' \cdot \overline{z'} = \left|z'\right|^2 = R_1^2$ therefore it follows that

$$\begin{aligned}
& R_1\left(z' - a'\right) R_1\left(\overline{z' - a'}\right) \\
= \;& R_1^2\left[R_1^2 - \left(a'\overline{z'} + \overline{a'}z'\right) + a'\overline{a'}\right] \\
= \;& \left(R_1^2 - \overline{a'}z'\right)\left(R_1^2 - a\overline{z'}\right) \\
= \;& \left(R_1^2 - \overline{a'}z'\right)\overline{\left(R_1^2 - \overline{a'}z'\right)} \\
i.e., \left|R_1\left(z' - a'\right)\right|^2 = \;& \left|R_1^2 - \overline{a'}z'\right|^2
\end{aligned}$$

$$i.e., \left|\frac{R_1^2 - \overline{a'}z'}{R_1\left(z' - a'\right)}\right| = 1. \tag{4. 11}$$

Replacing a' by z'_m to(4. 11) we have

$$\left|\frac{R_1^2 - \overline{z'_m}z'}{R_1\left(z' - z'_m\right)}\right| = 1 \quad \text{on } \left|z'\right| = R_1. \tag{4. 12}$$

Similarly we can write that

$$\left|\frac{R_2^2 - \overline{z''_m}z''}{R_2\left(z'' - z''_m\right)}\right| = 1 \quad \text{on } \left|z''\right| = R_2. \tag{4. 13}$$

By (4. 12 and (4. 9) we obtain that

$$\left|F_1\left(z'\right)\right| = \left|f_1\left(z'\right)\right| \quad \text{on } \left|z'\right| = R_1. \tag{4. 14}$$

Similarly by (4. 13) and in view of (4. 10) ,

$$\left|F_2\left(z''\right)\right| = \left|f_2\left(z''\right)\right| \quad \text{on } \left|z''\right| = R_2. \tag{4. 15}$$

Considering $R=\sqrt{\frac{R_1^2+R_2^2}{2}}$,combining (4. 14) and (4. 15) it follows that$\|F(z)\| = \|f(z)\|$ on $\|z\| = R$. So by the maximum modulus principle in $\mathbb{C}_2$ {cf.[6]},

$$\|F(z)\| \leq \max_{\|z\|=R} \|F(z)\| = \max_{\|z\|=R} \|f(z)\|.$$

Choosing $M(R)$ to be the maximum modulus of $\|f(z)\|$ on $\|z\| = R$, we get that

$$\|f(z)\| \leq M(R).$$

Putting $z = 0$in (4. 7) we obtain that

$$\begin{aligned} \left\| f(0) \prod_{m=1}^{n} \frac{R^2}{R(0-z_m)} \right\| &= \|F(0)\| \leq M(R) \\ i.e., \|f(0)\| \cdot \prod_{m=1}^{n} \frac{R}{\|z_m\|} &\leq M(R) \\ i.e., \|f(0)\| \cdot \frac{R\cdot R\cdot R\cdot\cdots\cdot R\,(n \text{ factors})}{\|z_1\|\cdot\|z_2\|\cdot\cdots\cdot\|z_n\|} &\leq M(R) \\ i.e., \frac{R^n\|f(0)\|}{r_1\cdot r_2\cdot r_3\cdot\cdots\cdot r_n} &\leq M(R) \end{aligned}$$

$$i.e., R^n\|f(0)\| \leq M(R)\cdot r_1\cdot r_2\cdot r_3\cdot\cdots\cdot r_n.$$

Thus the theorem is established. □

Remark 4.6. *Theorem 4.3 is analogous to Jenson's Inequality in* $\mathbb{C}_1$*{cf.*[2]}.

The next two theorems show some light on the bicomplex analog of result connected with univalent functions.

Theorem 4.7. *Let* $\{f_n\}$ *be a sequence in* $\mathbb{C}_2$*.Also let* $f_n = f_n^{'}e_1 + f_n^{''}e_2$*,where* $\left\{f_n^{'}\right\}$ *and* $\left\{f_n^{''}\right\}$ *are sequence in* $\mathbb{C}_1$*. Then* $\{f_n\}$ *is a uniformly convergent sequence of univalent functions in* $\mathbb{C}_2$ *iff* $\left\{f_n^{'}\right\}$ *and* $\left\{f_n^{''}\right\}$ *are uniformly convergent sequences of univalent functions in* $\mathbb{C}_1$*.*

Proof. Let $\{f_n\}$ be uniformly convergent sequence of univalent functions in $\mathbb{C}_2$. So $\{f_n\}$ is a uniformly convergent sequence of analytic functions and $f_n = f_n^{'}e_1 + f_n^{''}e_2$, where $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ are uniformly convergent sequences of analytic functions {cf. [3]}.

Now, as $\{f_n\}$ is a sequence of univalent functions, we have, $z \neq w$ implies that $f_n(z) \neq f_n(w)$ where $z, w \in \mathbb{C}_2$.

Let $z = z_1e_1 + z_2e_2$, $w = w_1e_1 + w_2e_2$ where $z_1, z_2, w_1, w_2 \in \mathbb{C}_1$. Since, $z \neq w$, we get

$$\begin{aligned} z_1e_1 + z_2e_2 &\neq w_1e_1 + w_2e_2 \\ i.e. z_1 &\neq w_1 \text{ and } z_2 \neq w_2. \end{aligned} \quad (4.\ 16)$$

$$Hence,\ f_n(z) \neq f_n(w)$$

$$i.e., f_n^{'}(z_1)\,e_1 + f_n^{''}(z_2)\,e_2 \neq f_n^{'}(w_1)\,e_1 + f_n^{''}(w_2)\,e_2.$$

$$i.e., f_n^{'}(z_1) \neq f_n^{'}(w_1) \text{ and } f_n^{''}(z_2) \neq f_n^{''}(w_2). \quad (4.\ 17)$$

Hence combining (4. 16) and (4. 17) we can write that $z_1 \neq w_1 \Rightarrow f_n^{'}(z_1) \neq f_n^{'}(w_1)$ and $z_2 \neq w_2 \Rightarrow f_n^{''}(z_2) \neq f_n^{''}(w_2)$. So, $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ are both uniformly convergent sequences of univalent functions in $\mathbb{C}_1$.

Conversely let, $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ be both uniformly convergent sequence of univalent functions in $\mathbb{C}_1$. Therefore, $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ are uniformly convergent sequences of analytic function. Since, $f_n = f_n^{'}e_1 + f_n^{''}e_2$the sequence $\{f_n\}$ is uniformly convergent sequence of analytic functions {cf. [3]}. Now let $z \neq w, i.e., z_1 \neq w_1$ and $z_2 \neq w_2$. Since $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ are univalent functions. So as $z_1 \neq w_1$ and $z_2 \neq w_2$ we have respectively that $f_n^{'}(z_1) \neq f_n^{'}(w_1)$ and $f_n^{''}(z_2) \neq f_n^{''}(w_2)$.

Hence it follows that

$$f_n^{'}(z_1)\, e_1 + f_n^{''}(z_2)\, e_2 \neq f_n^{'}(w_1)\, e_1 + f_n^{''}(w_2)\, e_2$$

$$i.e., f_n(z) \neq f_n(w).$$

Thus $\{f_n\}$ is a uniformly convergent sequence of univalent functions in $\mathbb{C}_2$. □

Theorem 4.8. *Limit of a uniformly convergent sequence of univalent functions in* $\mathbb{C}_2$ *is either simple or constant.*

Proof. Let $\{f_n\}$ be uniformly convergent sequence of univalent functions in $\mathbb{C}_2$.

$$\text{Then, } f_n = f_n^{'}e_1 + f_n^{''}e_2, \text{ where } \left\{f_n^{'}\right\}\ and \left\{f_n^{''}\right\} are \text{ sequences in } \mathbb{C}_1. \quad (4.\ 18)$$

So, in view of Theorem 3.8 we can say that $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ are uniformly convergent sequences of univalent functions in $\mathbb{C}_1$.Thus by Lemma 2.1 the respective limits of a uniformly convergent sequence of univalent functions in $\mathbb{C}_1$ is either simple or constant. So, limits of $\left\{f_n^{'}\right\}$ and $\left\{f_n^{''}\right\}$ are either simple or constant.

Hence from (4. 18), the limit of $\{f_n\}$ in $\mathbb{C}_2$ is either simple or constant. □

5. Future Prospect

HIn the line of the works as carried out in the paper one may think of the formation of Hadamard's three circles theorem with its convex form, Jenson's Inequality and the notion of univalency(i.e., simplicity) of functions with the help of the idempotents 0, 1, $\frac{1+i_1i_2}{2}$, $\frac{1-i_1i_2}{2}$, $\frac{1+i_1i_3}{2}$, $\frac{1-i_1i_3}{2}$, $\frac{1+i_2i_3}{2}$, $\frac{1-i_2i_3}{2}$,...., $\frac{1+i_{n-1}i_n}{2}$ and $\frac{1-i_{n-1}i_n}{2}$ in $\mathbb{C}_n$ and these derivations may be posed as open problems for the future workers in this area.

6. Acknowledgment

Authors are thankful to the referees for their valuable suggestions towards the improvement of the paper. The first and fourth authors sincerely acknowledge the financial support as rendered by DST-FIST 2019-2020.

Authors are grateful to the reviewers for their valuable suggestions.

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