Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(6)(2020) pp. 19-29

Possible Heights of Alexandroff Square Transformation Groups

Fatemah Ayatollah Zadeh Shirazi
Faculty of Mathematics, Statistics and Computer Science,
College of Science, University of Tehran, Enghelab Ave., Tehran, Iran
E–mail: fatemah@khayam.ut.ac.ir
:
Fatemeh Ebrahimifar
Faculty of Mathematics, Statistics and Computer Science,
College of Science, University of Tehran , Enghelab Ave., Tehran, Iran
E–mail: ebrahimifar64@ut.ac.ir
:
Reza Yaghmaeian
Faculty of Mathematics, Statistics and Computer Science,
College of Science, University of Tehran, Enghelab Ave., Tehran, Iran
E–mail: rezayaghma@yahoo.com
:
Hamed Yahyaoghli
Department of Mathematics,
Tarbiat Modares University, Modiriat Bridge, Tehran, Iran
E–mail: yahyaoghli@gmail.com

Received: 08 November, 2018 / Accepted: 19 March, 2020 / Published online: 01 June, 2020

Abstract.: In the following text we compute possible heights of $\mathbb{A}$ (Alexandroff square), $\mathbb{O}$ (unit square $[0,1]\times[0,1]$ with lexicographic order topology) and $\mathbb{U}$ (unit square $[0,1]\times[0,1]$ with induced topology of Euclidean plane). We prove $P_h(\mathbb{A}) = \{n : n \geq 5\} \cup \{+\infty\}$, $P_h(\mathbb{O}) = \{n : n \geq 4\} \cup \{+\infty\}$, $P_h(\mathbb{U}) = \{n : n \geq 1\} \cup \{+\infty\}$ (where for topological space X, by $P_h(X)$ we mean the collection of heights of transformation groups with phase space X. Additionally we show that there is no topological transitive (resp. Devaney chaotic) transformation group $(G, \mathbb{A})$.

AMS (MOS) Subject Classification Codes: 54H15, 54H20
Key Words: Alexandroff square, height, orbit space, transformation group.

1. INTRODUCTION

Studying closed unit ball $\{< x, y >\in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ with induced topology of Euclidean plane $\mathbb{R}^2$ is one of the main purposes of numerous texts (old and new) (see e.g., [8, 10]. Let us mention that unit disk and unit square $[0, 1] \times [0, 1]$ with induced topology of Euclidean plane, are homeomorphic.

On the other hand, many texts deal with dynamical properties of special topological spaces [2, 7]. In the following text we have a comparative study on dynamical properties of unit square transformation groups with emphasis on their heights (and obit spaces), where unit square $[0, 1] \times [0, 1]$ is equipped with Euclidean topology, lexicographic order topology, Alexandroff square topology. For convenience suppose (by $< x, y >$ we mean the ordered pair $\{x, \{x, y\}\}$):

- $\mathbb{A}$ is $[0, 1] \times [0, 1]$ as Alexandroff square.
- $\mathbb{O}$ is $[0, 1] \times [0, 1]$ equipped with lexicographic order topology,
- $\mathbb{U}$ is $[0, 1] \times [0, 1]$ equipped with Euclidean plane $\mathbb{R}^2$,

where for $< x, y >, < s, t >\in [0, 1] \times [0, 1]$ we define lexicographic order $\preceq_\ell$ with $< x, y >\preceq_\ell< s, t >$ if and only if "$x < s$" or "$x = s$ and $y \leq t$". Alexandroff square $\mathbb{A} = [0, 1] \times [0, 1]$ equipped with topological basis generated by the following sets, see [9]:

• $\{x\} \times U$ where $x \in [0, 1]$ and U is an open subset of $[0, 1]$ (with induced topology of Euclidean line $\mathbb{R}$) and $x \notin U$,

• $([0, 1] \times U) \setminus (\{x_1, \ldots, x_n\} \times [0, 1])$ where U is an open subset of $[0, 1]$ (with induced topology of Euclidean line $\mathbb{R}$).

As it has been mentioned in [9], $\mathbb{A}$ and $\mathbb{O}$ are compact Hausdorff non–metrizable spaces. Consider the following notations and sets (for $x, y \in \mathbb{R}$ let $(x, y) = \{z \in \mathbb{R} : x < z < y\}$):

$\Delta := \{< x, x >: x \in [0, 1]\}$;

$\mathsf{P}_1 :=< 0, 0 >, \mathsf{P}_2 :=< 0, 1 >, \mathsf{P}_3 :=< 1, 1 >, \mathsf{P}_4 :=< 1, 0 >$;

$\mathsf{L}_1 := \{0\} \times (0, 1), \mathsf{L}_2 := (0, 1) \times \{1\}, \mathsf{L}_3 := \{1\} \times (0, 1), \mathsf{L}_4 := (0, 1) \times \{0\}$.

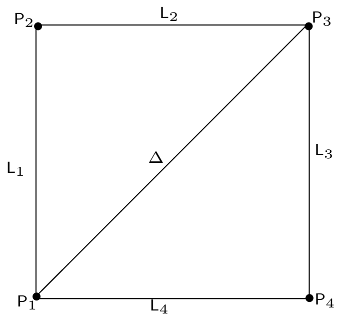

Fig. 1

Background on transformation groups. By a (topological) transformation group (G, X, ρ) or simply (G, X) we mean a compact Hausdorff topological space X (phase space), discrete topological group G (phase group) with identity e and continuous map $\rho : G \times X \to X$, $\rho(g, x) = gx$ ($g \in G, x \in X$) such that for all $x \in X$ and $g, h \in G$ we have $ex = x$ and $g(hx) = (gh)x$. Note that for all $g \in G$, $\rho_g : X \to X$, where $\rho_g(x) = gx$ is a homeomorphism of X, and $\rho_g\rho_h = \rho_{gh}$. Thus we may consider G as a

group of self–homeomorphisms of X with composition as a binary operation. In transformation group (G, X) for $x \in X$ we call $Gx := \{gx : g \in G\}$ the orbit of x (under G) and $\frac{X}{G} := \{Gy : y \in X\}$ the orbit space of (G, X). A nonempty subset D of X is invariant ($G-$invariant) if $GD := \{gy : g \in G, y \in D\} \subseteq D$, for more details on transformation groups (and orbit spaces) see [4, 6].
For a topological space X suppose that $\mathcal{G}_X$ is the collection of all homeomorphisms $h : X \to X$ ($\mathcal{G}_X$ is equipped with discrete topology).
Closed and open invariant subsets of a transformation group play important role in studying its dynamical properties (see e.g. [5] for transitivity in transformation groups). The height of transformation group (G, X) is $h(G, X) := \sup\{n \geq 0 :$ there exist closed invariant subsets $D_0, \dots, D_n$ of X with $\varnothing \neq D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_n = X\}$, i,e., $h(G, X) = +\infty$ if $\{\overline{Gx} : x \in X\}$ is infinite and $h(G, X) = card(\{\overline{Gx} : x \in X\}) - 1$ otherwise [1]. We also call $P_h(X) := \{h(G, X) : G$ is a subgroup of $\mathcal{G}_X\}$ the collection of all possible heights of X. In transformation group (G, X) the map $\varphi : \{\overline{Gy} : y \in X\} \to \{\overline{\mathcal{G}_X y} : y \in X\}$ with $\varphi(\overline{Gy}) = \overline{\mathcal{G}_X y}$ (for $y \in X$) is onto, so $h(\mathcal{G}_X, X) \leq h(G, X)$ therefore $\min P_h(X) = h(\mathcal{G}_X, X)$.

2. Computing $\frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}, \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$ and $\frac{\mathbb{U}}{\mathcal{G}_{\mathbb{U}}}$

Considering the definition of height of transformation group (G, X) it's evident that for computing $h(G, X)$ one may compute $\{\overline{Gy} : y \in X\}$, and begin with $\frac{X}{G} = \{Gy : y \in X\}$. Since $\min P_h(X) = h(\mathcal{G}_X, X)$, a first step towards finding $P_h(X)$ is to work out $\frac{X}{\mathcal{G}_X}$ and thus to establish the value of $h(\mathcal{G}_X, X)$. In this section we determine $\frac{X}{\mathcal{G}_X}$ where $X = \mathbb{U}, \mathbb{O}, \mathbb{A}$.

Lemma 2.1. *For homeomorphism $\mathfrak{a} : \mathbb{A} \to \mathbb{A}$ we have:*
1. $\mathfrak{a}(\{\mathsf{P}_1, \mathsf{P}_3\}) = \{\mathsf{P}_1, \mathsf{P}_3\}$ and $\mathfrak{a}(\Delta) = \Delta$;
2. $\mathfrak{a}(\mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}) = \mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}$,
3. for all $s \in [0, 1]$ there exists $t \in [0, 1]$ with $\mathfrak{a}(\{s\} \times [0, 1]) = \{t\} \times [0, 1]$ and $\mathfrak{a}\{< s, 0 >, < s, 1 >\} = \{< t, 0 >, < t, 1 >\}$;
4. One of the following cases holds:

- a. $\mathfrak{a}(\mathsf{P}_i) = \mathsf{P}_i$ *for* $i = 1, 2, 3, 4$, $\mathfrak{a}(\mathsf{L}_1) = \mathsf{L}_1$ *and* $\mathfrak{a}(\mathsf{L}_3) = \mathsf{L}_3$;
- b. $\mathfrak{a}(\mathsf{P}_1) = \mathsf{P}_3, \mathfrak{a}(\mathsf{P}_2) = \mathsf{P}_4, \mathfrak{a}(\mathsf{P}_3) = \mathsf{P}_1, \mathfrak{a}(\mathsf{P}_4) = \mathsf{P}_2$, $\mathfrak{a}(\mathsf{L}_1) = \mathsf{L}_3$ *and* $\mathfrak{a}(\mathsf{L}_3) = \mathsf{L}_1$.

Proof. **1.** Using the fact that $\mathbb{A}$ has a local countable topological basis on $\mathfrak{x} \in \mathbb{A}$ if and only if $\mathfrak{x} \in \mathbb{A} \setminus \Delta$, we have $\mathfrak{a}(\Delta) = \Delta$. Note that subspace topology on Δ induced by $\mathbb{A}$ coincides with subspace topology on Δ induced by $\mathbb{U}$ hence $\mathfrak{a}(\{\mathsf{P}_1, \mathsf{P}_3\}) = \{\mathsf{P}_1, \mathsf{P}_3\}$.
2. $\mathbb{A}$ has a countable basis $\{B_n : n \geq 1\}$ at $\mathfrak{x} \in \mathbb{A}$ such that all elements of $\{B_n \setminus \{\mathfrak{x}\} : n \geq 1\}$ are connected if and only if $\mathfrak{x} \in \mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}$.
3. Consider $s \in [0, 1]$, using (1) and (2) we have $< a, b >:= \mathfrak{a} < s, 0 >, < c, d >:= \mathfrak{a} < s, 1 >\in \mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_i : 1 \leq i \leq 4\}$. So $b, d \in \{0, 1\}$. Choose $x \in [0, 1]$ and suppose $< u, v >:= \mathfrak{a} < s, x >$. Let $S := \mathfrak{a}(\{s\} \times [0, 1])$. By item (1), $S \cap \Delta = \mathfrak{a} < s, s >=:< t, t >$. Assume that $u \neq t$. Then $< u, u >\notin S$, and the sets:

$$\begin{aligned} U &:= (\{u\} \times ([0, 1] \setminus \{u\})) \cap S = (\{u\} \times [0, 1]) \cap S \,, \\ V &:= (\mathbb{A} \setminus (\{u\} \times [0, 1])) \cap S \,, \end{aligned}$$

form a separation of S ($\mathfrak{a} < s, x > \in U$ when $x \neq s$ and $\mathfrak{a} < s, s > \in V$) which contradicts the connectedness of S. Thus $u = t$ and $\mathfrak{a} < s, x > \in \{t\} \times [0,1]$ for all $x \in [0,1]$., so $\mathfrak{a}(\{s\} \times [0,1]) \subseteq \{t\} \times [0,1]$. In particular $a = c = t$, so $< t, b > = \mathfrak{a} < s, 0 >, < t, d > = \mathfrak{a} < s, 1 >$ with $b, d \in \{0,1\}$ (since $\mathfrak{a} < s, 0 > \neq \mathfrak{a} < s, 1 >$ we have $b \neq d$). Thus $\mathfrak{a} \restriction_{\{s\}\times[0,1]}$: $\{s\} \times [0,1] \to \{t\} \times [0,1]$ is a continuous map with $< t, 0 >, < t, 1 > \in \mathfrak{a}(\{s\} \times [0,1])$ which completes the proof.
4. First suppose $\mathfrak{a}(\mathsf{P}_1) = \mathsf{P}_1$, then by (1), $\mathfrak{a}(\mathsf{P}_3) = \mathsf{P}_3$, so by (3) we have $\mathfrak{a}(\mathsf{L}_1) = \mathsf{L}_1$, $\mathfrak{a}(\mathsf{L}_3) = \mathsf{L}_3$, $\mathfrak{a}(\mathsf{P}_2) = \mathsf{P}_2$ and $\mathfrak{a}(\mathsf{P}_4) = \mathsf{P}_4$.
Now suppose $\mathfrak{a}(\mathsf{P}_1) \neq \mathsf{P}_1$, then by (1), $\mathfrak{a}(\mathsf{P}_1) = \mathsf{P}_3$ and $\mathfrak{a}(\mathsf{P}_3) = \mathsf{P}_1$ so by (3) we have $\mathfrak{a}(\mathsf{L}_1) = \mathsf{L}_3$, $\mathfrak{a}(\mathsf{L}_3) = \mathsf{L}_1$, $\mathfrak{a}(\mathsf{P}_2) = \mathsf{P}_4$ and $\mathfrak{a}(\mathsf{P}_4) = \mathsf{P}_2$. □

Lemma 2.2. *For homeomorphism* $\mathfrak{o} : \mathbb{O} \to \mathbb{O}$ *we have:*
1. $\mathfrak{o} : \mathbb{O} \to \mathbb{O}$ *is order preserving or anti–order preserving;*
2. $\mathfrak{o}(\{\mathsf{P}_1, \mathsf{P}_3\}) = \{\mathsf{P}_1, \mathsf{P}_3\}$*;*
3. $\mathfrak{o}(\mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}) = \mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}$*,*
4. for all $s \in [0,1]$ *there exists* $t \in [0,1]$ *with* $\mathfrak{o}(\{s\} \times [0,1]) = \{t\} \times [0,1]$ *and* $\mathfrak{o}\{< s, 0 >, < s, 1 >\} = \{< t, 0 >, < t, 1 >\}$*;*
5. One of the following cases holds:

a. $\mathfrak{o}(\mathsf{P}_i) = \mathsf{P}_i$, $\mathfrak{o}(\mathsf{L}_i) = \mathsf{L}_i$ *for* $i = 1, 2, 3, 4$ *and* $\mathfrak{o} : \mathbb{O} \to \mathbb{O}$ *is order preserving;*
b. $\mathfrak{o}(\mathsf{P}_1) = \mathsf{P}_3, \mathfrak{o}(\mathsf{P}_2) = \mathsf{P}_4, \mathfrak{o}(\mathsf{P}_3) = \mathsf{P}_1, \mathfrak{o}(\mathsf{P}_4) = \mathsf{P}_2$, $\mathfrak{o}(\mathsf{L}_1) = \mathsf{L}_3$, $\mathfrak{o}(\mathsf{L}_2) = \mathsf{L}_4$, $\mathfrak{o}(\mathsf{L}_3) = \mathsf{L}_1$, $\mathfrak{o}(\mathsf{L}_4) = \mathsf{L}_2$ *and* $\mathfrak{o} : \mathbb{O} \to \mathbb{O}$ *is anti–order preserving.*

Proof. **2.** Use (1) and $\mathsf{P}_1 = \max \mathbb{O}$, $\mathsf{P}_3 = \min \mathbb{O}$.
3. Use the fact that all open neighbourhoods of $\mathfrak{x} \in \mathbb{O}$ are non–metrizable if and only if $\mathfrak{x} \in \mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}$.
4. Consider $s \in [0,1]$, using (2) and (3) we have $< a, b > := \mathfrak{o} < s, 0 >, < c, d > := \mathfrak{o} < s, 1 > \in \mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_i : 1 \leq i \leq 4\}$. So $b, d \in \{0,1\}$. Choose $x \in [0,1]$ and suppose $< t, v > := \mathfrak{o} < s, x >$. If $t \neq a$ then we may choose $r \in \{\frac{a+t}{2}, \frac{a+2t}{3}, \frac{a+3t}{4}, \frac{a+4t}{5}\} \setminus \{a, c, t\}$. Then $< r, 0 > \notin \mathfrak{o}(\{s\} \times [0,1])$ and for:

$$\begin{aligned} U &:= \{< z, w > \in \mathbb{O} : < z, w > \prec_\ell < r, 0 >\} \cap \mathfrak{o}(\{s\} \times [0,1]) , \\ V &:= \{< z, w > \in \mathbb{O} : < r, 0 > \prec_\ell < z, w >\} \cap \mathfrak{o}(\{s\} \times [0,1]) , \end{aligned}$$

U, V is a separation of $\mathfrak{o}(\{s\} \times [0,1])$ which is in contradiction with connectedness of $\mathfrak{o}(\{s\} \times [0,1])$. Thus $t = a$ and $\mathfrak{o}(\{s\} \times [0,1]) \subseteq \{t\} \times [0,1]$. In particular $a = c = t$, so $< t, b > = \mathfrak{o} < s, 0 >, < t, d > = \mathfrak{o} < s, 1 >$ with $b, d \in \{0,1\}$. So $\mathfrak{o} \restriction_{\{s\}\times[0,1]}$: $\{s\} \times [0,1] \to \{t\} \times [0,1]$ is a continuous map with $< t, 0 >, < t, 1 > \in \mathfrak{o}(\{s\} \times [0,1])$ which completes the proof.
5. (a) Suppose $\mathfrak{o} : \mathbb{O} \to \mathbb{O}$ is order preserving. So $\mathfrak{o}(\mathsf{P}_1) = \mathfrak{o}(\min \mathbb{O}) = \min \mathbb{O} = \mathsf{P}_1$ and $\mathfrak{o}(\mathsf{P}_3) = \mathfrak{o}(\max \mathbb{O}) = \max \mathbb{O} = \mathsf{P}_3$, also by (3) we have

$$\mathfrak{o}(\mathsf{P}_2) = \mathfrak{o}(\min(\mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\})) = \min(\mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}) = \mathsf{P}_2$$

and

$$\mathfrak{o}(\mathsf{P}_4) = \mathfrak{o}(\max(\mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\})) = \max(\mathsf{L}_2 \cup \mathsf{L}_4 \cup \{\mathsf{P}_2, \mathsf{P}_4\}) = \mathsf{P}_4 \, .$$

Hence by (4) we have $\mathfrak{o}(\mathsf{L}_1)=\mathsf{L}_1$ and $\mathfrak{o}(\mathsf{L}_4)=\mathsf{L}_4$. Consider $s\in[0,1]$, by (4) there exists $t\in[0,1]$ with $\mathfrak{o}(\{s\}\times[0,1])=\{t\}\times[0,1]$ so

$$\mathfrak{o}<s,0>=\mathfrak{o}(\min(\{s\}\times[0,1]))=\min\mathfrak{o}(\{s\}\times[0,1])=\min(\{t\}\times[0,1])=<t,0>,$$

which shows $\mathfrak{o}(\mathsf{L}_1\cup\{\mathsf{P}_1,\mathsf{P}_2\})\subseteq\mathsf{L}_1\cup\{\mathsf{P}_1,\mathsf{P}_2\}$ and $\mathfrak{o}(\mathsf{L}_4)\subseteq\mathsf{L}_4$; also by a similar method we have $\mathfrak{o}<s,1>=<t,1>$ which leads to $\mathfrak{o}(\mathsf{L}_2)\subseteq\mathsf{L}_2$. Use (2) to obtain $\mathfrak{o}(\mathsf{L}_2)=\mathsf{L}_2$ and $\mathfrak{o}(\mathsf{L}_4)=\mathsf{L}_4$.
(b) Use a similar method described in the proof of (a). □

Theorem 2.3. $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ *is an order preserving homeomorphism if and only if there exist order preserving homeomorphism* $\theta:[0,1]\to[0,1]$ *and* $\mu:\underset{t\mapsto\mu_t}{[0,1]\to[0,1]^{[0,1]}}$ *such that for all* $t\in[0,1]$, $\mu_t:[0,1]\to[0,1]$ *is an order preserving homeomorphism and* $\mathfrak{o}<s,t>=<\theta(s),\mu_s(t)>$.
Also $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ *is an anti–order preserving homeomorphism if and only if there exist anti–order preserving homeomorphism* $\theta:[0,1]\to[0,1]$ *and* $\mu:\underset{t\mapsto\mu_t}{[0,1]\to[0,1]^{[0,1]}}$ *such that for all* $t\in[0,1]$, $\mu_t:[0,1]\to[0,1]$ *is an anti–order preserving homeomorphism and* $\mathfrak{o}<s,t>=<\theta(s),\mu_s(t)>$.

Proof. First suppose $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ is an order preserving homeomorphism, by Lemma 2.2 for each $s\in[0,1]$ there exists $t\in[0,1]$ with $\mathfrak{o}(\{s\}\times[0,1])=\{t\}\times[0,1]$, let $\theta(s):=t$. Also by Lemma 2.2 (since $\mathfrak{o}\restriction_{\mathsf{L}_2\cup\{\mathsf{P}_2,\mathsf{P}_3\}}:\mathsf{L}_2\cup\{\mathsf{P}_2,\mathsf{P}_3\}\to\mathsf{L}_2\cup\{\mathsf{P}_2,\mathsf{P}_3\}$ is order preserving and bijection), $\theta:[0,1]\to[0,1]$ is order preserving and bijection, thus it is an order preserving homeomorphism on $[0,1]$. Now for $s\in[0,1]$, considering homeomorphism $\mathfrak{o}\restriction_{\{s\}\times[0,1]}:\{s\}\times[0,1]\to\{\theta(s)\}\times[0,1]$, we may define homeomorphism $\mu_s:[0,1]\to[0,1]$ with $\mathfrak{o}<s,t>=<\theta(s),\mu_s(t)>$. For $x,y\in[0,1]$ with $x\leq y$ since $<s,x>\preceq_\ell<s,y>$ we have

$$<\theta(s),\mu_s(x)>=\mathfrak{o}<s,x>\preceq_\ell\mathfrak{o}<s,y>=<\theta(s),\mu_s(y)>$$

which leads to $\mu_s(x)\leq\mu_s(y)$ and $\mu_s:[0,1]\to[0,1]$ is order preserving too.
Conversely, consider order preserving homeomorphism $\theta:[0,1]\to[0,1]$ and $\mu:\underset{t\mapsto\mu_t}{[0,1]\to[0,1]^{[0,1]}}$ such that for all $t\in[0,1]$, $\mu_t:[0,1]\to[0,1]$ is an order preserving homeomorphism and define $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ with $\mathfrak{o}<s,t>=<\theta(s),\mu_s(t)>$. It's clear that $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ is order preserving and bijective which leads to continuity of $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ under order topology.
In order to complete the proof consider homeomorphism $\underset{<s,t>\mapsto<1-s,1-t>}{\varphi:\mathbb{O}\to\mathbb{O}}$ and note that $\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ is an anti–order preserving homeomorphism if and only if $\varphi\circ\mathfrak{o}:\mathbb{O}\to\mathbb{O}$ is an order preserving homeomorphism. □

Note. If $\mathfrak{a}:\mathbb{A}\to\mathbb{A}$ is a homeomorphism,then there exist a homeomorphism $\theta:[0,1]\to[0,1]$ with $\theta(\{0,1\})=\{0,1\}$ and $\mu:\underset{t\mapsto\mu_t}{[0,1]\to[0,1]^{[0,1]}}$ such that for all $t\in[0,1]$, $\mu_t:[0,1]\to[0,1]$ is a homeomorphism with $\mu_t(t)=\theta(t)$ and $\mathfrak{a}<s,t>=<\theta(s),\mu_s(t)>$ (note that $\mathfrak{a}\restriction_{\Delta\cup\{\mathsf{P}_1,\mathsf{P}_3\}}:\Delta\cup\{\mathsf{P}_1,\mathsf{P}_3\}\to\Delta\cup\{\mathsf{P}_1,\mathsf{P}_3\}$ is a homeomorphism).

Corollary 2.4. *For homeomorphisms $p, q : [0,1] \to [0,1]$, consider*

$$p \times q : \underset{<s,t> \mapsto <p(s),q(t)>}{[0,1] \times [0,1] \to [0,1] \times [0,1]} ,$$

then we have:

1. $p \times q : \mathbb{A} \to \mathbb{A}$ *is a homeomorphism if and only if* $p = q$*;*
2. $p \times q : \mathbb{O} \to \mathbb{O}$ *is a homeomorphism if and only if* $p \circ q : [0,1] \to [0,1]$ *is order preserving;*
3. $p \times q : \mathbb{U} \to \mathbb{U}$ *is a homeomorphism.*

Proof. **1.** If $p \times q : \mathbb{A} \to \mathbb{A}$ is a homeomorphism, then by Lemma 2.1 we have $p \times q(\Delta) = \Delta$, thus for all $t \in [0,1]$ we have $< p(t), q(t) >= p \times q(t,t) \in \Delta$ which shows $p(t) = q(t)$ and leads to $p = q$.
2. Suppose $p \times q : \mathbb{O} \to \mathbb{O}$ is a homeomorphism, by Lemma 2.2 one of the following cases holds:

- $p \times q : \mathbb{O} \to \mathbb{O}$ is order preserving: in this case $p, q : [0,1] \to [0,1]$ are order preserving too, thus $p \circ q : [0,1] \to [0,1]$ is order preserving;
- $p \times q : \mathbb{O} \to \mathbb{O}$ is anti–order preserving: in this case $p, q : [0,1] \to [0,1]$ are anti–order preserving too, thus $p \circ q : [0,1] \to [0,1]$ is order preserving.

Using two cases above $p \circ q : [0,1] \to [0,1]$ is order preserving.
Conversely suppose $p \circ q : [0,1] \to [0,1]$ is order preserving, thus either "$p, q : [0,1] \to [0,1]$ are order preserving" or "$p, q : [0,1] \to [0,1]$ are anti–order preserving". Use Theorem 2.3 to complete the proof of this item. □

Theorem 2.5. *We have:*

$$\begin{aligned}
\frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}} &= \{\{\mathsf{P}_1, \mathsf{P}_3\}, \{\mathsf{P}_2, \mathsf{P}_4\}, \mathsf{L}_1 \cup \mathsf{L}_3, \mathsf{L}_2 \cup \mathsf{L}_4, \Delta \setminus \{\mathsf{P}_1, \mathsf{P}_3\}, ((0,1) \times (0,1)) \setminus \Delta\} , \\
\frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}} &= \{\{\mathsf{P}_1, \mathsf{P}_3\}, \{\mathsf{P}_2, \mathsf{P}_4\}, \mathsf{L}_1 \cup \mathsf{L}_3, \mathsf{L}_2 \cup \mathsf{L}_4, (0,1) \times (0,1)\} , \\
\frac{\mathbb{U}}{\mathcal{G}_{\mathbb{U}}} &= \{(0,1) \times (0,1), \mathbb{U} \setminus ((0,1) \times (0,1))\} .
\end{aligned}$$

Proof. We prove case by case. Note that $\varphi : X \to X$ with $\varphi < s,t >=< 1-s, 1-t >$ (for $(s,t) \in X$) for $X = \mathbb{A}, \mathbb{O}, \mathbb{U}$ is homeomorphism. Also for $x, y \in (0,1)$ consider hommeoorphism $f_{x,y} : [0,1] \to [0,1]$ with:

$$f_{x,y}(t) = \begin{cases} \dfrac{y}{x} t & 0 \le t \le x , \\ \\ \dfrac{(1-y)t + (y-x)}{1-x} & x \le t \le 1 . \end{cases}$$

Now we have:

A1. $\{\mathsf{P}_1, \mathsf{P}_3\} \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$: Use Lemma 2.1 and note that $\varphi(\mathsf{P}_1) = \mathsf{P}_3$.
A2. $\{\mathsf{P}_2, \mathsf{P}_4\} \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$: Use Lemma 2.1 and note that $\varphi(\mathsf{P}_2) = \mathsf{P}_4$.
A3. $\mathsf{L}_1 \cup \mathsf{L}_3 \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$: Using Lemma 2.1 we have $\mathcal{G}_{\mathbb{A}} < 0, \frac{1}{2} > \subseteq \mathcal{G}_{\mathbb{A}} \mathsf{L}_1 \subseteq \mathsf{L}_1 \cup \mathsf{L}_3$. For $x \in (0,1)$ consider homeomorphism $h : \mathbb{A} \to \mathbb{A}$ with $h < 0, t >=< 0, f_{\frac{1}{2},x}(t) >$

and $h < s,t > = < s,t >$ for $s \neq 0$. Then $< 0,x > = h < 0,\frac{1}{2} > \in \mathcal{G}_{\mathbb{A}} < 0,\frac{1}{2} >$, thus $\mathsf{L}_1 \subseteq \mathcal{G}_{\mathbb{A}} < 0,\frac{1}{2} >$, so $\mathsf{L}_1 \cup \mathsf{L}_3 = \varphi(\mathsf{L}_1) \cup \mathsf{L}_1 \subseteq \mathcal{G}_{\mathbb{A}} < 0,\frac{1}{2} >$ which leads to $\mathsf{L}_1 \cup \mathsf{L}_3 = \mathcal{G}_{\mathbb{A}} < 0,\frac{1}{2} > \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$.

A4. $\mathsf{L}_2 \cup \mathsf{L}_4 \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$: Using Lemma 2.1 we have $\mathcal{G}_{\mathbb{A}} < \frac{1}{2},0 > \subseteq \mathcal{G}_{\mathbb{A}}\mathsf{L}_4 \subseteq \mathsf{L}_2 \cup \mathsf{L}_4$. For $x \in (0,1)$ consider homeomorphism $h : \mathbb{A} \to \mathbb{A}$ with $h < s,t > = < f_{\frac{1}{2},x}(s), f_{\frac{1}{2},x}(t) >$, thus $< x,0 > = h < \frac{1}{2},0 > \in \mathcal{G}_{\mathbb{A}} < \frac{1}{2},0 >$ which leads to $\mathsf{L}_4 \subseteq \mathcal{G}_{\mathbb{A}} < \frac{1}{2},0 >$. Thus $\mathsf{L}_2 = \varphi(\mathsf{L}_4) \subseteq \varphi(\mathcal{G}_{\mathbb{A}} < \frac{1}{2},0 >) = \mathcal{G}_{\mathbb{A}} < \frac{1}{2},0 >$ which leads to $\mathsf{L}_2 \cup \mathsf{L}_4 = \mathcal{G}_{\mathbb{A}} < \frac{1}{2},0 > \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$.

A5. $\Delta \setminus \{\mathsf{P}_1,\mathsf{P}_3\} \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$: Using Lemma 2.1 we have $\mathcal{G}_{\mathbb{A}} < \frac{1}{2},\frac{1}{2} > \subseteq \Delta \setminus \{\mathsf{P}_1,\mathsf{P}_3\}$. For $x \in (0,1)$ consider homeomorphism $h : \mathbb{A} \to \mathbb{A}$ with $h < s,t > = < f_{\frac{1}{2},x}(s), f_{\frac{1}{2},x}(t) >$ so $< x,x > = h < \frac{1}{2},\frac{1}{2} > \in \mathcal{G}_{\mathbb{A}} < \frac{1}{2},\frac{1}{2} >$ which shows $\Delta \setminus \{\mathsf{P}_1,\mathsf{P}_3\} \subseteq \mathcal{G}_{\mathbb{A}} < \frac{1}{2},\frac{1}{2} >$.

A6. $((0,1)\times(0,1)) \setminus \Delta \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$: Consider $< a,b >, < c,d > \in ((0,1)\times(0,1)) \setminus \Delta$, using (A1), ..., (A5) we have $\mathcal{G}_{\mathbb{A}} < a,b > \subseteq ((0,1)\times(0,1)) \setminus \Delta$. Consider the following cases:

<u>I.</u> $b < a$, $d < c$ and $a \leq c$. In this case consider homeomorphism $h : \mathbb{A} \to \mathbb{A}$ with $h < s,t > = < f_{a,c}(s), f_{a,c}(t) >$, thus $h < a,b > = < c, f_{a,c}(b) >$ (note that $b < a$ thus $f_{a,c}(b) < f_{a,c}(a) = c$). Define $p : \mathbb{A} \to \mathbb{A}$ with:

$$p < s,t > := \begin{cases} < s, \dfrac{d}{f_{a,c}(b)} t > & s = c, 0 \leq t \leq f_{a,c}(b)\,, \\[2ex] < s, \dfrac{(d-c)t + (f_{a,c}(b) - d)c}{f_{a,c}(b) - c} > & s = c, f_{a,c}(b) \leq t \leq c\,, \\[2ex] < s,t > & \text{otherwise}\,, \end{cases}$$

then $h,p \in \mathcal{G}_{\mathbb{A}}$ and

$$< c,d > = p < c, f_{a,c}(b) > = p(h < a,b >) \in \mathcal{G}_{\mathbb{A}} < a,b > \ .$$

<u>II.</u> $b < a$, $d < c$ and $c \leq a$. By case (I) we have $< a,b > \in \mathcal{G}_{\mathbb{A}} < c,d >$ thus there exists $j \in \mathcal{G}_{\mathbb{A}}$ with $< a,b > = j < c,d >$ so $< c,d > = j^{-1} < a,b > \in \mathcal{G}_{\mathbb{A}} < a,b >$.

<u>III.</u> $b < a$ and $d > c$. Choose $e \in (0,c)$ by cases (I) and (II) we have $< c,e > \in \mathcal{G}_{\mathbb{A}} < a,b >$. Define $q : \mathbb{A} \to \mathbb{A}$ with:

$$q < s,t > := \begin{cases} < c, \dfrac{d-1}{e} t + 1 > & 0 \leq t \leq e, s = c\,, \\[2ex] < c, \dfrac{(d-c)t + (e-d)c}{e-c} > & e \leq t \leq c, s = c\,, \\[2ex] < c, \dfrac{c(1-t)}{1-c} > & c \leq t \leq 1, s = c\,, \\[2ex] < s,t > & t \neq d\,, \end{cases}$$

then $q \in \mathcal{G}_{\mathbb{A}}$ and $< c,d >= q < c,e > \in q\mathcal{G}_{\mathbb{A}} < a,b >= \mathcal{G}_{\mathbb{A}} < a,b >$.
Using cases (I,II, III) we have $((0,1)\times(0,1))\setminus\Delta \subseteq \mathcal{G}_{\mathbb{A}} < a,b >$ which leads to $((0,1)\times(0,1))\setminus\Delta = \mathcal{G}_{\mathbb{A}} < a,b > \in \frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}}$.

O1. $\{\mathsf{P}_1,\mathsf{P}_3\} \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$: Use Lemma 2.2 and note that $\varphi(\mathsf{P}_1)=\mathsf{P}_3$.

O2. $\{\mathsf{P}_2,\mathsf{P}_4\} \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$: Use Lemma 2.2 and note that $\varphi(\mathsf{P}_2)=\mathsf{P}_4$.

O3. $\mathsf{L}_2\cup\mathsf{L}_4 \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$: By Lemma 2.2, $\mathcal{G}_{\mathbb{O}} < \frac{1}{2},0 > \subseteq \mathsf{L}_2\cup\mathsf{L}_4$. For $x\in(0,1)$ consider $h:\mathbb{O}\to\mathbb{O}$ with $h < s,t >=< f_{\frac{1}{2},x}(s),t >$ so $< x,0 >= h < \frac{1}{2},0 > \in \mathcal{G}_{\mathbb{O}} < \frac{1}{2},0 >$ and $< x,1 >= h(\varphi < \frac{1}{2},0 >) \in \mathcal{G}_{\mathbb{O}} < \frac{1}{2},0 >$, thus $\mathsf{L}_2\cup\mathsf{L}_4 \subseteq \mathcal{G}_{\mathbb{O}} < \frac{1}{2},0 >$ which leads to $\mathsf{L}_2\cup\mathsf{L}_4 = \mathcal{G}_{\mathbb{O}} < \frac{1}{2},0 > \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$.

O4. $\mathsf{L}_1\cup\mathsf{L}_3 \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$: By Lemma 2.2, $\mathcal{G}_{\mathbb{O}} < 0,\frac{1}{2} > \subseteq \mathsf{L}_1\cup\mathsf{L}_3$. For $x\in(0,1)$ consider $h:\mathbb{O}\to\mathbb{O}$ with $h < s,t >=< s,f_{\frac{1}{2},x}(t) >$ so $< 0,x >= h < 0,\frac{1}{2} > \in \mathcal{G}_{\mathbb{O}} < 0,\frac{1}{2} >$ and $< 1,x >= h(\varphi < 0,\frac{1}{2} >) \in \mathcal{G}_{\mathbb{O}} < 0,\frac{1}{2} >$, thus $\mathsf{L}_2\cup\mathsf{L}_4 \subseteq \mathcal{G}_{\mathbb{O}} < 0,\frac{1}{2} >$ which leads to $\mathsf{L}_2\cup\mathsf{L}_4 = \mathcal{G}_{\mathbb{O}} < 0,\frac{1}{2} > \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$.

O5. $(0,1)\times(0,1) \in \frac{\mathbb{O}}{\mathcal{G}_{\mathbb{O}}}$: Using (O1), (O2), (O3) and (O4) we have $\mathcal{G}_{\mathbb{O}} < \frac{1}{2},\frac{1}{2} > \subseteq (0,1)\times(0,1)$. Choose $< x,y > \in (0,1)\times(0,1)$ and define $h:\mathbb{O}\to\mathbb{O}$ with $h < s,t >=< f_{\frac{1}{2},x}(s), f_{\frac{1}{2},y}(t) >$, then $< x,y >= h < \frac{1}{2},\frac{1}{2} >$ which shows $(0,1)\times(0,1)\subseteq \mathcal{G}_{\mathbb{O}} < \frac{1}{2},\frac{1}{2} >$ and completes the proof.

□

Devaney chaos. We say transformation group (G,X) is topological transitive if for all nonempty and open subsets U,V of X we have $U\cap GV\neq\varnothing$. We say that $x\in X$ is a periodic point of transformation group (G,X) if $st(x):=\{g\in G: gx=x\}$ is a subgroup of finite index of G. Transformation group (G,X) is Devaney chaotic if it is topological transitive and the collection of its periodic points is dense in X [3]. We say that $x\in X$ is an almost periodic point of (G,X) if $\overline{Gx}$ is a minimal subset of X (i.e., it is a closed invariant subset of (G,X) without any proper subset which is a closed invariant subset of (G,X)) [4]. All periodic points of (G,X) are almost periodic. Using the following theorem we show that transformation groups $(\mathcal{G}_{\mathbb{A}},\mathbb{A})$, $(\mathcal{G}_{\mathbb{O}},\mathbb{O})$ and $(\mathcal{G}_{\mathbb{U}},\mathbb{U})$ are not Devaney chaotic.

Theorem 2.6. *For $X=\mathbb{A},\mathbb{O}$, the transformation group (G,X) is not topological transitive, in particular it is not Devaney chaotic. However $(\mathcal{G}_{\mathbb{U}},\mathbb{U})$ is topological transitive.*

Proof. For $X=\mathbb{A},\mathbb{O}$ the sets $U:=(0,1)\times(0,1)$ and $V:=\mathsf{L}_1\cup\mathsf{L}_3$ are open subsets of X and by Theorem 2.5 we have $GU\cap V\subseteq\mathcal{G}_XU\cap V=U\cap V=\varnothing$ thus (G,X) is not topological transitive. □

Note. $\mathfrak{x}$ is an almost periodic point of $(\mathcal{G}_{\mathbb{A}},\mathbb{A})$ (resp. $(\mathcal{G}_{\mathbb{O}},\mathbb{O})$) if and only if $\mathfrak{x}$ is a periodic point. Also $\{\mathsf{P}_i: 1\leq i\leq 4\}$ is the collection of all its periodic points. Moreover $(\mathcal{G}_{\mathbb{U}},\mathbb{U})$ does not have any periodic point, but $\{< s,t > \in\mathbb{U}: \{s,t\}\cap\{0,1\}\neq\varnothing\}$ is the collection of its almost periodic points.

3. COMPUTING $P_h(\mathbb{A})$, $P_h(\mathbb{O})$ AND $P_h(\mathbb{U})$

Now we are ready to find out $P_h(\mathbb{A})$, $P_h(\mathbb{O})$ and $P_h(\mathbb{U})$. We show $P_h(\mathbb{A})=\{n:n\geq 5\}\cup\{+\infty\}$, $P_h(\mathbb{O})=\{n:n\geq 4\}\cup\{+\infty\}$ and $P_h(\mathbb{U})=\{n:n\geq 1\}\cup\{+\infty\}$.

Theorem 3.1. $h(\mathcal{G}_{\mathbb{A}}, \mathbb{A}) = 5$, $h(\mathcal{G}_{\mathbb{O}}, \mathbb{O}) = 4$, $h(\mathcal{G}_{\mathbb{U}}, \mathbb{U}) = 1$.

Proof. Use Theorem 2.5. □

Theorem 3.2. $P_h(\mathbb{A}) = \{n : n \geq 5\} \cup \{+\infty\}$, $P_h(\mathbb{O}) = \{n : n \geq 4\} \cup \{+\infty\}$, $P_h(\mathbb{U}) = \{n : n \geq 1\} \cup \{+\infty\}$.

Proof. Computing $P_h(\mathbb{A})$. By Theorem 3.1, it's evident that $5 \in P_h(\mathbb{A}) \subseteq \{n : n \geq 5\} \cup \{+\infty\}$. For $n \geq 1$ choose $t_1, \ldots, t_n \in (0,1)$ with $\frac{1}{2} = t_1 < \cdots, t_n$ and let

$$\begin{aligned}
\mathcal{H}_{\mathbb{A}} &:= \{f \in \mathcal{G}_{\mathbb{A}} : f(\mathsf{P}_1) = \mathsf{P}_1\} \\
\mathcal{K}_0 &:= \{f \in \mathcal{G}_{\mathbb{A}} : f < 0, t_1 >=< 0, t_1 >, \ldots, f < 0, t_n >=< 0, t_n >\} (\subseteq \mathcal{H}_{\mathbb{A}}) \\
\mathcal{K}_1 &:= \{f \in \mathcal{K}_0 : f(\{< 0, \frac{1}{j} >: j \geq 2\} \cup \{< 0, \frac{1}{2} - \frac{1}{j} >: j \geq 3\}) = \\
&\qquad \{< 0, \frac{1}{j} >: j \geq 2\} \cup \{< 0, \frac{1}{2} - \frac{1}{j} >: j \geq 3\}\} \\
\mathcal{K}_2 &:= \{f \in \mathcal{G}_{\mathbb{A}} : f(\{< 0, \frac{1}{2} >, < 1, \frac{1}{2} >\}) = \{< 0, \frac{1}{2} >, < 1, \frac{1}{2} >\}\} \\
\mathcal{K}_3 &:= \{f \in \mathcal{G}_{\mathbb{A}} : f(\{< i, \frac{1}{j} >: j \geq 2, i = 0, 1\} \cup \{< i, 1 - \frac{1}{j} >: j \geq 2, i = 0, 1\}) = \\
&\qquad \{< i, \frac{1}{j} >: j \geq 2, i = 0, 1\} \cup \{< i, 1 - \frac{1}{j} >: j \geq 2, i = 0, 1\}\}
\end{aligned}$$

Then $\mathcal{H}_{\mathbb{A}}$ is a proper normal subgroup of $\mathcal{G}_{\mathbb{A}}$ with index 2 and $\mathcal{G}_{\mathbb{A}} = \mathcal{H}_{\mathbb{A}} \cup \varphi \mathcal{H}_{\mathbb{A}}$ (where $\varphi < s, t >=< 1 - s, 1 - t >$). Moreover using a similar method described in Theorem 2.5 we have:

$$\begin{aligned}
\frac{\mathbb{A}}{\mathcal{H}_{\mathbb{A}}} &= \{\{\mathsf{P}_1\}, \{\mathsf{P}_2\}, \{\mathsf{P}_3\}, \{\mathsf{P}_4\}, \mathsf{L}_1, \mathsf{L}_3, \mathsf{L}_2 \cup \mathsf{L}_4, \Delta \setminus \{\mathsf{P}_1, \mathsf{P}_3\}, ((0,1) \times (0,1)) \setminus \Delta\} \\
\frac{\mathbb{A}}{\mathcal{K}_0} &= (\frac{\mathbb{A}}{\mathcal{H}_{\mathbb{A}}} \setminus \{\mathsf{L}_1\}) \cup \{\{< 0, t_1 >\}, \ldots . \{< 0, t_n >\}, \\
&\qquad \{0\} \times (0, t_1), \{0\} \times (t_1, t_2), \ldots, \{0\} \times (t_{n-1}, t_n), \{0\} \times (t_n, 1)\} \\
\frac{\mathbb{A}}{\mathcal{K}_1} &= (\frac{\mathbb{A}}{\mathcal{K}_0} \setminus \{\{0\} \times (0, t_1)\}) \cup \{\{< 0, \frac{1}{j} >: j \geq 2\} \cup \{< 0, \frac{1}{2} - \frac{1}{j} >: j \geq 3\}, \\
&\qquad \{0\} \times ((0, t_1) \setminus \{\frac{1}{j} : j \geq 2\} \cup \{\frac{1}{2} - \frac{1}{j} : j \geq 3\})\} \\
\frac{\mathbb{A}}{\mathcal{K}_2} &= (\frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}} \setminus \{\mathsf{L}_1 \cup \mathsf{L}_3\}) \cup \{\{< 0, \frac{1}{2} >, < 1, \frac{1}{2} >\}, \\
&\qquad (\{0\} \times (0, \frac{1}{2})) \cup (\{1\} \times (\frac{1}{2}, 1)), (\{0\} \times (\frac{1}{2}, 1)) \cup (\{1\} \times (0, \frac{1}{2}))\} \\
\frac{\mathbb{A}}{\mathcal{K}_3} &= (\frac{\mathbb{A}}{\mathcal{G}_{\mathbb{A}}} \setminus \{\mathsf{L}_1 \cup \mathsf{L}_3\}) \cup \\
&\qquad \{\{< i, \frac{1}{j} >: j \geq 2, i = 0, 1\} \cup \{< i, 1 - \frac{1}{j} >: j \geq 2, i = 0, 1\}, \\
&\qquad (\mathsf{L}_1 \cup \mathsf{L}_3) \setminus (\{< i, \frac{1}{j} >: j \geq 2, i = 0, 1\} \cup \{< i, 1 - \frac{1}{j} >: j \geq 2, i = 0, 1\})\}
\end{aligned}$$

which leads to $h(\mathcal{H}_{\mathbb{A}},\mathbb{A})=8$, $h(\mathcal{K}_0,\mathbb{A})=8+2n$, $h(\mathcal{K}_1,\mathbb{A})=8+2n+1$, $h(\mathcal{K}_2,\mathbb{A})=7$, $h(\mathcal{K}_3,\mathbb{A})=6$, $h(\{id_{\mathbb{A}}\},\mathbb{A})=+\infty$. Hence $P_h(\mathbb{A})=\{n:n\geq 5\}\cup\{+\infty\}$.
Computing $P_h(\mathbb{O})$. By Theorem 3.1, it's evident that $4\in P_h(\mathbb{O})\subseteq\{n:n\geq 4\}\cup\{+\infty\}$. For $n\geq 1$ choose $t_1,\ldots,t_n\in(0,1)$ with $\frac{1}{2}=t_1<\cdots,t_n$ and let

$$\begin{aligned}
\mathcal{H}_{\mathbb{O}} &:= \{f\in\mathcal{G}_{\mathbb{O}}: f(\mathsf{P}_1)=\mathsf{P}_1\}\\
\mathcal{J}_0 &:= \{f\in\mathcal{G}_{\mathbb{O}}: f<0,t_1>=<0,t_1>,\ldots,f<0,t_n>=<0,t_n>\}(\subseteq\mathcal{H}_{\mathbb{O}})\\
\mathcal{J}_1 &:= \{f\in\mathcal{J}_0: f(\{<0,\frac{1}{j}>:j\geq 2\}\cup\{<0,\frac{1}{2}-\frac{1}{j}>:j\geq 3\})=\\
&\qquad \{<0,\frac{1}{j}>:j\geq 2\}\cup\{<0,\frac{1}{2}-\frac{1}{j}>:j\geq 3\}\}\\
\mathcal{J}_2 &:= \{f\in\mathcal{G}_{\mathbb{O}}: f(\{<0,\frac{1}{2}>,<1,\frac{1}{2}>\})=\{<0,\frac{1}{2}>,<1,\frac{1}{2}>\}\}\\
\mathcal{J}_3 &:= \{f\in\mathcal{G}_{\mathbb{O}}: f(\{<i,\frac{1}{j}>:j\geq 2,i=0,1\}\cup\{<i,1-\frac{1}{j}>:j\geq 2,i=0,1\})=\\
&\qquad \{<i,\frac{1}{j}>:j\geq 2,i=0,1\}\cup\{<i,1-\frac{1}{j}>:j\geq 2,i=0,1\}\}\\
\mathcal{J}_4 &:= \{f\in\mathcal{G}_{\mathbb{O}}: f(\{\frac{1}{2}\}\times(0,1))=\{\frac{1}{2}\}\times(0,1)\}
\end{aligned}$$

One can verify $h(\mathcal{H}_{\mathbb{O}},\mathbb{O})=8$, $h(\mathcal{J}_0,\mathbb{O})=8+2n$, $h(\mathcal{J}_1,\mathbb{O})=8+2n+1$, $h(\mathcal{J}_2,\mathbb{O})=6$, $h(\mathcal{J}_3,\mathbb{A})=5$, $h(\mathcal{J}_4,\mathbb{O})=7$, $h(\{id_{\mathbb{O}}\},\mathbb{O})=+\infty$. Hence $P_h(\mathbb{A})=\{n:n\geq 4\}\cup\{+\infty\}$.
Computing $P_h(\mathbb{U})$. By Theorem 3.1, it's evident that $1\in P_h(\mathbb{U})\subseteq\{n:n\geq 1\}\cup\{+\infty\}$. For $n\geq 1$ choose distinct $\mathfrak{x}_1,\ldots,\mathfrak{x}_n\in(0,1)\times(0,1)$, so $h(\{f\in\mathcal{G}_{\mathbb{U}}:f(\mathfrak{x}_1)=\mathfrak{x}_1,\ldots,f(\mathfrak{x}_n)=\mathfrak{x}_n\},\mathbb{U})=n+1$ and $h(\{id_{\mathbb{U}}\},\mathbb{U})=+\infty$ which completes the proof. □

4. Acknowledgments

The authors are grateful to the research division of the University of Tehran for the grant which supported this research.

References

[1] F. Ayatollah Zadeh Shirazi, N. Golestani, *On classification of transformation semigroups: indicator sequences and indicator topological spaces*, Filomat, **26,** No.2 (2012) 313–329.
[2] F. Ayatollah Zadeh Shirazi, M. A. Mahmoodi, M. Raeisi, *On distaity of a transformation semigroup with one point compactification of a discrete space as phase space*, Iranian Journal of Science and Technology, Transaction A: Science, **40,** No.4 (2016) 209–217.
[3] T. Ceccherini, M. Coornaert, *Sensitivity and Devaney's chaos in uniform spaces*, Journal of dynamical and control systems, **19,** No.3 (2013) 349–357.
[4] R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin Inc., 1969.
[5] M. Garg, R. Das, *Exploring stronger forms of transitivity on $G-$spaces*, Mat. Vesnik, **69,** No.3 (2017) 164–175.
[6] K. Kuwakubo, *The theory of transformation groups*, Oxford University Press, 1991.
[7] J.–H. Mai, S. Shao, *The structure of graph maps without periodic points*, Topology and its Applications, **154,** No.14 (2007), 2714–2728.
[8] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Classics in Mathematics, Reprint of 1980, Springer–Verlag, 2008.
[9] L. A. Steen, J. A. Seebach, *Counterexamples in topology*, Holt, Rinehart and Winston, Inc., 1970.

[10] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate texts in Mathematics, 226, Springer–Verlag 2005.