Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(7)(2020) pp. 11-24

Weighted Simpson's Type Inequalities for HA-convex Functions

Muhammad Amer Latif
Department of Basic Science,
Deanship of Preparatory Year, King Faisal University, Al Hasa, Kingdom of Saudi Arabia,
Email: m_amer_latif@hotmail.com

Sabir Hussain
Department of Mathematics,
University of Engineering and Technology Lahore, Pakistan,
Email: sabirhus@gmail.com

Madeeha Baloch
Department of Mathematics,
University of Sialkot, Pakistan,
Email: madeehabaloch@yahoo.com

Received: 31 March, 2019 / Accepted: 17 July, 2020 / Published online: 22 July, 2020

Abstract.: In this paper, some new weighted Simpson type integral inequalities are presented for the class of HA-convex functions.

AMS (MOS) Subject Classification Codes: 26A33; 41A55; 26D15; 26E60
Key Words: Simpson's inequality; Convex Functions; HA-Convex Function.

1. INTRODUCTION

It is known that a function $\varphi : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is convex if

$$\varphi\left(\theta u_1 + (1-\theta)\, u_2\right) \leq \theta \varphi\left(u_1\right) + (1-\theta)\, \varphi\left(u_2\right)$$

holds for all $u_1, u_2 \in \mathbb{I}$, $\theta \in [0,1]$.

One of the important inequalities for convex functions is the Hermite-Hadamard inequality stated as follows.

Let $\varphi : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $\alpha, \beta \in \mathbb{I}$ with $\alpha < \beta$. Then

$$(\beta - \alpha)\, \varphi(A) \leq \int_\alpha^\beta \varphi(u)\, du \leq \frac{\varphi(\alpha) + \varphi(\beta)}{2} (\beta - \alpha)\,, \tag{1. 1}$$

where $A = \frac{\alpha+\beta}{2}$.

A number of variants of the inequality (1. 1) have been obtained by researchers from all over the world in the past three decades.

In [12], İşcan introduced the concept of harmonically convex functions as given in the definition below.

Definition 1.1. [12] *Let* $\mathbb{I} \subset \mathbb{R} \backslash \{0\}$*, a function* $\varphi : \mathbb{I} \to \mathbb{R}$ *is harmonically convex on* $\mathbb{I}$*, if*

$$\varphi\left(\frac{u_1 u_2}{\theta u_1 + (1-\theta) u_2}\right) \leq \theta \varphi(u_2) + (1-\theta) \varphi(u_1) \tag{1. 2}$$

for all $u_1, u_2 \in \mathbb{I}$ *and* $\theta \in [0,1]$*. If (1. 2) is reversed, then* φ *is harmonically concave.*

In [12], İşcan also proved the following Hermite-Hadamard type inequality for this class of functions and established some new results connected with its right-hand side.

Theorem 1.2. [12] *Let* $\varphi : \mathbb{I} \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ *be a harmonically convex function on* $\mathbb{I}^\circ$ *and* $\alpha, \beta \in \mathbb{I}^\circ$ *with* $\alpha < \beta$ *such that* $\varphi^{'} \in L([\alpha, \beta])$*. Then*

$$(\beta - \alpha) \varphi(H) \leq \alpha\beta \int_\alpha^\beta \frac{\varphi(u)}{u^2} du \leq \frac{\varphi(\alpha) + \varphi(\beta)}{2} (\beta - \alpha), \tag{1. 3}$$

where $H = \frac{2\alpha\beta}{\alpha+\beta}$*. The above inequalities are sharp.*

One of the important quadrature formulae for functions of one real variable is the Simpson rule.

The Simpson inequality states that if $\varphi : [\alpha, \beta] \to \mathbb{R}$ is a four times continuously differentiable mapping on (α, β) and $\left\|\varphi^{(4)}\right\|_\infty = \sup\limits_{q \in (\alpha,\beta)} \left|\varphi^{(4)}(q)\right| < \infty$, then

$$\left| \int_\alpha^\beta \varphi(u) du - \frac{\beta - \alpha}{3} \left[\frac{\varphi(\alpha) + \varphi(\beta)}{2} + 2\varphi(A) \right] \right| \leq \frac{1}{2880} \left\|\varphi^{(4)}\right\|_\infty \cdot (\beta - \alpha)^5, \tag{1. 4}$$

where $A = \frac{\alpha+\beta}{2}$.

It attracted huge interest in the last two decades with many results concerning its error estimate. There is a substantial literature on the generalizations of Simpson's inequality, Simpson type integral inequalities, Hermite-Hadamard type and Fejér type integral inequalities using a variety of convexity conditions, see for example [1]-[39] and the references cited therein.

2. Weighted Simpson's type inequalities for harmonically-convex functions

Throughout the paper, we will use the notations $U_1(q)$ and $U_2(q)$ respectively for $\frac{2\alpha\beta}{(1-q)\alpha + (1+q)\beta}$ and $\frac{2\alpha\beta}{(1+q)\alpha + (1-q)\beta}$.

The following definition will be used in the sequel of the paper to establish the weighted Simpson type inequalities for harmonically-convex functions.

Definition 2.1. [20] *A function* $\phi : [\alpha, \beta] \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ *is said to be harmonically symmetric with respect to* H *if*

$$\phi(u) = \phi\left(\frac{1}{\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{u}}\right)$$

holds for all $u \in [\alpha, \beta]$.

Here we point out some assumptions to prove our results.

Assumption 1: Let $\varphi : \mathbb{I} \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ or $\varphi : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^{\circ}$ and α, $\beta \in \mathbb{I}^{\circ}$ with $\alpha < \beta$ and let $\phi : [\alpha, \beta] \to [0, \infty)$ be a continuous positive mapping and harmonically symmetric to H.

Lemma 2.2. *According to assumption 1 with* $\varphi : \mathbb{I} \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ *and* $\varphi^{'} \in L_1 [\alpha, \beta]$. *Then*

$$\frac{1}{8}\left(\frac{\alpha\beta}{\beta-\alpha}\right)[\varphi(\alpha)+6\varphi(H)+\varphi(\beta)]\int\limits_{\alpha}^{\beta}\frac{\phi(u)}{u^2}du-\frac{\alpha\beta}{\beta-\alpha}\int\limits_{\alpha}^{\beta}\frac{\phi(u)\varphi(u)}{u^2}du=\frac{\beta-\alpha}{4\alpha\beta}$$
$$\times\left\{\int\limits_0^1 p_1(q)(U_1(q))^2\varphi^{'}(U_1(q))\,dq+\int\limits_0^1 p_2(q)(U_2(q))^2\varphi^{'}(U_2(q))\,dq\right\}, \qquad (2.\ 5)$$

where

$$p_1(q)=\frac{3}{4}\int\limits_0^1\phi(U_1(s))\,ds-\int\limits_0^q\phi(U_1(s))\,ds,$$

$$p_2(q)=-\frac{3}{4}\int\limits_0^1\phi(U_2(s))\,ds+\int\limits_0^q\phi(U_2(s))\,ds$$

and $H = \frac{2\alpha\beta}{\alpha+\beta}$.

Proof. By integration by parts, we have

$$\begin{aligned}
\mathbb{I}_1 &= \int\limits_0^1 p_1(q)(U_1(q))^2\varphi^{'}(U_1(q))\,dq \\
&= -\left(\frac{2\alpha\beta}{\beta-\alpha}\right)\int\limits_0^1 p_1(q)\left[-\left(\frac{\beta-\alpha}{2\alpha\beta}\right)\right](U_1(q))^2\varphi^{'}(U_1(q))\,dq \\
&= -\left(\frac{2\alpha\beta}{\beta-\alpha}\right)\int\limits_0^1\left[\frac{3}{4}\int\limits_0^1\phi(U_1(s))\,ds-\int\limits_0^q\phi(U_1(s))\,ds\right]d\left[\varphi(U_1(q))\right] \\
&= -\left(\frac{2\alpha\beta}{\beta-\alpha}\right)\left[\frac{3}{4}\int\limits_0^1\phi(U_1(s))\,ds-\int\limits_0^q\phi(U_1(s))\,ds\right]\varphi(U_1(q))\Bigg|_0^1 \\
&\quad -\left(\frac{2\alpha\beta}{\beta-\alpha}\right)\int\limits_0^1\phi(U_1(q))\varphi(U_1(q))\,dq \\
&= \left(\frac{2\alpha\beta}{\beta-\alpha}\right)\left[\frac{\varphi(\alpha)}{4}\int\limits_0^1\phi(U_1(q))\,dq+\frac{3}{4}\varphi(H)\int\limits_0^1\phi(U_1(q))\,dq\right]
\end{aligned}$$

$$-\left(\frac{2\alpha\beta}{\beta-\alpha}\right)\int_{0}^{1}\phi\left(U_{1}\left(q\right)\right)\varphi\left(U_{1}\left(q\right)\right)dq.$$

By making the substitution $u=U_{1}\left(q\right)$, we get

$$\begin{aligned}\mathbb{I}_{1}=&\left(\frac{\alpha\beta}{\beta-\alpha}\right)^{2}\left[\varphi\left(\alpha\right)+3\varphi\left(H\right)\right]\int_{\alpha}^{H}\frac{\phi\left(u\right)}{u^{2}}du\\&-\left(\frac{2\alpha\beta}{\beta-\alpha}\right)^{2}\int_{\alpha}^{H}\frac{\phi\left(u\right)\varphi\left(u\right)}{u^{2}}du.\end{aligned}$$

Similarly, we can have

$$\begin{aligned}\mathbb{I}_{2}=&\int_{0}^{1}p_{2}\left(q\right)\left(U_{2}\left(q\right)\right)^{2}\varphi^{'}\left(U_{2}\left(q\right)\right)dq\\=&\left(\frac{\alpha\beta}{\beta-\alpha}\right)^{2}\left[3\varphi\left(H\right)+\varphi\left(\beta\right)\right]\int_{H}^{\beta}\frac{\phi\left(u\right)}{u^{2}}du\\&-\left(\frac{2\alpha\beta}{\beta-\alpha}\right)^{2}\int_{\frac{2\alpha\beta}{\alpha+\beta}}^{\beta}\frac{\phi\left(u\right)\varphi\left(u\right)}{u^{2}}du.\end{aligned}$$

Since $\phi(u)$ is harmonically symmetric with respect to H, we have

$$\int_{\alpha}^{H}\frac{\phi\left(u\right)}{u^{2}}du=\int_{H}^{\beta}\frac{\phi\left(u\right)}{u^{2}}du=\frac{1}{2}\int_{\alpha}^{\beta}\frac{\phi\left(u\right)}{u^{2}}du.$$

Thus, we have

$$\begin{aligned}&\frac{\beta-\alpha}{4\alpha\beta}\left(\mathbb{I}_{1}+\mathbb{I}_{2}\right)\\&=\frac{1}{8}\left(\frac{\alpha\beta}{\beta-\alpha}\right)\left[\varphi\left(\alpha\right)+6\varphi\left(H\right)+\varphi\left(\beta\right)\right]\int_{\alpha}^{\beta}\frac{\phi\left(u\right)}{u^{2}}du\\&-\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\phi\left(u\right)\varphi\left(u\right)}{u^{2}}du.\end{aligned}$$

Hence the proof of the theorem is done. □

Remark 2.3. *Throughout this manuscript we will use the following notation for the sake of convenience*

$$\mu(\alpha,\beta;\varphi,\phi)=\frac{1}{8}\left(\frac{\alpha\beta}{\beta-\alpha}\right)[\varphi(\alpha)+6\varphi(H)+\varphi(\beta)]\int_{\alpha}^{\beta}\frac{\phi(u)}{u^2}du - \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\phi(u)\varphi(u)}{u^2}du$$

Corollary 2.4. *Under the assumptions of Lemma 2.2, the following inequality holds*

$$\mu(\alpha,\beta;\varphi,\phi)\leq\frac{\beta-\alpha}{4\alpha\beta}\|\phi\|_{[\alpha,\beta],\infty}\left\{\int_0^1\left(\frac{3}{4}-q\right)(U_1(q))^2\varphi'(U_1(q))\,dq + \int_0^1\left(q-\frac{3}{4}\right)(U_2(q))^2\varphi'(U_2(q))\,dq\right\}. \quad (2.\ 6)$$

Proof. Proof follows from the fact that

$$\|\phi\|_{[\alpha,H],\infty}\leq\|\phi\|_{[\alpha,\beta],\infty}$$

and

$$\|\phi\|_{[H,\beta],\infty}\leq\|\phi\|_{[\alpha,\beta],\infty}.$$

□

Theorem 2.5. *According to assumption 1 with $\varphi:\mathbb{I}\subseteq(0,\infty)\to\mathbb{R}$ and $\varphi'\in L_1[\alpha,\beta]$. If $\left|\varphi'\right|^{\mathbf{v}}$ is HA-convex on $[\alpha,\beta]$ for $\mathbf{v}\geq 1$, then*

$$|\mu(\alpha,\beta;\varphi,\phi)|\leq\frac{(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}\left(\frac{5}{16}\right)^{1-\frac{1}{\mathbf{v}}}H^2 \times\left\{\left[\alpha_1(\mathbf{v},y)\left|\varphi'(\alpha)\right|^{\mathbf{v}}+\alpha_2(\mathbf{v},y)\left|\varphi'(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}} + \left[\alpha_2(\mathbf{v},-y)\left|\varphi'(\alpha)\right|^{\mathbf{v}}+\alpha_1(\mathbf{v},-y)\left|\varphi'(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right\}, \quad (2.\ 7)$$

where

$$\alpha_1(\mathbf{v},y)=-\frac{64-8y(6y(\mathbf{v}-1)-1)(2\mathbf{v}-3)-2^{4\mathbf{v}}(4+3y)^{2-2\mathbf{v}}(8+y(14\mathbf{v}-15))}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)} + \frac{8(1+y)^{1-2\mathbf{v}}\left(8+y(18\mathbf{v}-11)+y^2\left(8\mathbf{v}^2-2\mathbf{v}-7\right)\right)}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)},$$

$$\alpha_2(\mathbf{v},y)=\frac{64-2^{4\mathbf{v}}(4+3y)^{2-2\mathbf{v}}(8+y(9-2\mathbf{v}))+8y(6y(\mathbf{v}-\mathbf{1})-7)(2\mathbf{v}-\mathbf{3})}{128y^3(\mathbf{v}-\mathbf{1})(2\mathbf{v}-\mathbf{3})(2\mathbf{v}-\mathbf{1})}$$

$$+\frac{8(1+y)^{2-2q}(8+y(5+2\mathbf{v}))}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)},$$

$y=\frac{\beta-\alpha}{\alpha+\beta}$ *and* $H=\frac{2\alpha\beta}{\alpha+\beta}$.

Proof. From (2. 6) and using the power-mean inequality, we have

$$\begin{aligned}|\mu(\alpha,\beta;\varphi,\phi)| &\le \frac{(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}\\ &\times\left\{\left(\int_0^1\left|\frac{3}{4}-q\right|dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1\left|\frac{3}{4}-q\right|(U_1(q))^{2\mathbf{v}}\left|\varphi'(U_1(q))\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right.\\ &\left.+\left(\int_0^1\left|q-\frac{3}{4}\right|dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1\left|q-\frac{3}{4}\right|(U_2(q))^{2\mathbf{v}}\left|\varphi'(U_2(q))\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right\}. \quad (2.\ 8)\end{aligned}$$

By using the HA-convexity of $\left|\varphi'\right|^{\mathbf{v}}$ on $[\alpha,\beta]$ for $\mathbf{v}\ge 1$, we get

$$\begin{aligned}&\int_0^1\left|\frac{3}{4}-q\right|(U_1(q))^{2\mathbf{v}}\left|\varphi'(U_1(q))\right|^{\mathbf{v}}dq\\ &\le\left|\varphi'(\alpha)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)\left(\frac{1+q}{2}\right)(U_1(q))^{2\mathbf{v}}dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)\left(\frac{1+q}{2}\right)(U_1(q))^{2\mathbf{v}}dq\right]\\ &+\left|\varphi'(\beta)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)\left(\frac{1-q}{2}\right)(U_1(q))^{2\mathbf{v}}dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)\left(\frac{1-q}{2}\right)(U_1(q))^{2\mathbf{v}}dq\right]\\ &=H^{2\mathbf{v}}\left[-\frac{64-8y(6y(\mathbf{v}-1)-1)(2\mathbf{v}-3)-2^{4\mathbf{v}}(4+3y)^{2-2\mathbf{v}}(8+y(14\mathbf{v}-15))}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right.\\ &\left.+\frac{8(1+y)^{1-2\mathbf{v}}\left(8+y(18\mathbf{v}-11)+y^2\left(8\mathbf{v}^2-2\mathbf{v}-7\right)\right)}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right]\left|\varphi'(\alpha)\right|^{\mathbf{v}}\\ &+H^{2\mathbf{v}}\left[\frac{64-2^{4\mathbf{v}}(4+3y)^{2-2\mathbf{v}}(8+y(9-2\mathbf{v}))+8y(6y(\mathbf{v}-1)-7)(2\mathbf{v}-3)}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right.\\ &\left.+\frac{8(1+y)^{2-2q}(8+y(5+2\mathbf{v}))}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right]\left|\varphi'(\beta)\right|^{\mathbf{v}} \quad (2.\ 9)\end{aligned}$$

and

$$\int_0^1\left|q-\frac{3}{4}\right|(U_2(q))^{2\mathbf{v}}\left|\varphi'(U_2(q))\right|^{\mathbf{v}}dq$$

$$\leq \left|\varphi^{'}(\alpha)\right|^{\mathbf{v}}\left[\int_{0}^{\frac{3}{4}}\left(\frac{3}{4}-q\right)\left(\frac{1-q}{2}\right)(U_2(q))^{2\mathbf{v}}\,dq.+\int_{\frac{3}{4}}^{1}\left(q-\frac{3}{4}\right)\left(\frac{1-q}{2}\right)(U_2(q))^{2\mathbf{v}}\,dq\right]$$

$$+\left|\varphi^{'}(\beta)\right|^{\mathbf{v}}\left[\int_{0}^{\frac{3}{4}}\left(\frac{3}{4}-q\right)\left(\frac{1+q}{2}\right)(U_2(q))^{2\mathbf{v}}\,dq+\int_{\frac{3}{4}}^{1}\left(q-\frac{3}{4}\right)\left(\frac{1+q}{2}\right)(U_2(q))^{2\mathbf{v}}\,dq\right]$$

$$= H^{2\mathbf{v}}\left[\frac{64+8y(6y(1-\mathbf{v})-1)(2\mathbf{v}-3)-2^{4\mathbf{v}}(4-3y)^{2-2\mathbf{v}}(8-y(14\mathbf{v}-15))}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right.$$

$$\left.-\frac{8(1-y)^{1-2\mathbf{v}}\left(8-y(18\mathbf{v}-11)+y^2\left(8\mathbf{v}^2-2\mathbf{v}-7\right)\right)}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right]\left|\varphi^{'}(\beta)\right|^{\mathbf{v}}$$

$$+H^{2\mathbf{v}}\left[-\frac{64-2^{4\mathbf{v}}(4-3y)^{2-2\mathbf{v}}(8-y(9-2\mathbf{v}))-8y(6y(1-\mathbf{v})-7)(2\mathbf{v}-3)}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right.$$

$$\left.+\frac{8(1-y)^{2-2q}(y(5+2\mathbf{v})-8)}{128y^3(\mathbf{v}-1)(2\mathbf{v}-3)(2\mathbf{v}-1)}\right]\left|\varphi^{'}(\alpha)\right|^{\mathbf{v}}, \quad (2.\ 10)$$

where $y=\frac{\beta-\alpha}{\alpha+\beta}$ and $H=\frac{2\alpha\beta}{\alpha+\beta}$.
Using (2. 9) and (2. 10) in (2. 8) we get (2. 7). □

Theorem 2.6. *According to assumption 1 with* $\varphi:\mathbb{I}\subseteq(0,\infty)\to\mathbb{R}$ *and* $\varphi^{'}\in L_1[\alpha,\beta]$*. If* $\left|\varphi^{'}\right|^{\mathbf{v}}$ *is HA-convex on* $[\alpha,\beta]$ *for* $\mathbf{v}>1$*, then*

$$|\mu(\alpha,\beta;\varphi,\phi)|\leq\frac{H^2(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}\left(\frac{1-\mathbf{v}}{1+\mathbf{v}}\right)^{1-\frac{1}{\mathbf{v}}}$$

$$\times\left\{[\beta_1(y)]^{1-\frac{1}{\mathbf{v}}}\left[\beta_2(\mathbf{v})\left|\varphi^{'}(\alpha)\right|^{\mathbf{v}}+\beta_3(\mathbf{v})\left|\varphi^{'}(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right.$$

$$\left.+[\beta_1(-y)]^{1-\frac{1}{\mathbf{v}}}\left[\beta_3(\mathbf{v})\left|\varphi^{'}(\alpha)\right|^{\mathbf{v}}+\beta_2(\mathbf{v})\left|\varphi^{'}(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right\}, \quad (2.\ 11)$$

where

$$\beta_1(y)=y^{-1}\left((1+y)^{\frac{1+\mathbf{v}}{1-\mathbf{v}}}-1\right),\beta_2(\mathbf{v})=\frac{2^{-2\mathbf{v}-5}(8\mathbf{v}+15)+2^{-2\mathbf{v}-5}\times3^{\mathbf{v}+1}(4\mathbf{v}+11)}{(\mathbf{v}+1)(\mathbf{v}+2)},$$

$$\beta_3(\mathbf{v})=\frac{2^{-2\mathbf{v}-5}+2^{-2\mathbf{v}-5}\times3^{\mathbf{v}+1}(4\mathbf{v}+5)}{(\mathbf{v}+1)(\mathbf{v}+2)},$$

$y=\frac{\beta-\alpha}{\alpha+\beta}$ *and* $H=\frac{2\alpha\beta}{\alpha+\beta}$.

Proof. From (2. 6) and using the Hölder integral inequality, we have

$$|\mu(\alpha,\beta;\varphi,\phi)|\leq\frac{(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}$$

$$\times\left\{\left(\int_0^1 (U_1(q))^{\frac{2\mathbf{v}}{\mathbf{v}-1}}\,dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1\left|\frac{3}{4}-q\right|^{\mathbf{v}}\left|\varphi'\left(U_1(q)\right)\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right.$$
$$\left.+\left(\int_0^1 (U_2(q))^{\frac{2\mathbf{v}}{\mathbf{v}-1}}\,dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1\left|q-\frac{3}{4}\right|^{\mathbf{v}}\left|\varphi'\left(U_2(q)\right)\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right\}. \quad (2.\ 12)$$

By using the HA-convexity of $\left|\varphi'\right|^{\mathbf{v}}$ on $[\alpha,\beta]$ for $\mathbf{v}>1$, we get

$$\int_0^1\left|\frac{3}{4}-q\right|^{\mathbf{v}}\left|\varphi'\left(U_1(q)\right)\right|^{\mathbf{v}}dq$$
$$\leq\left|\varphi'(\alpha)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)^{\mathbf{v}}\left(\frac{1+q}{2}\right)dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)^{\mathbf{v}}\left(\frac{1+q}{2}\right)dq\right]$$
$$+\left|\varphi'(\beta)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)^{\mathbf{v}}\left(\frac{1-q}{2}\right)dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)^{\mathbf{v}}\left(\frac{1-q}{2}\right)dq\right]$$
$$=\left(\frac{2^{-2\mathbf{v}-5}\times 3^{\mathbf{v}+1}\left(4\mathbf{v}+11\right)+2^{-2\mathbf{v}-5}\left(8\mathbf{v}+15\right)}{\left(\mathbf{v}+1\right)\left(\mathbf{v}+2\right)}\right)\left|\varphi'(\alpha)\right|^{\mathbf{v}}$$
$$+\left(\frac{2^{-2\mathbf{v}-5}+2^{-2\mathbf{v}-5}\times 3^{\mathbf{v}+1}\left(4\mathbf{v}+5\right)}{\left(\mathbf{v}+1\right)\left(\mathbf{v}+2\right)}\right)\left|\varphi'(\beta)\right|^{\mathbf{v}} \quad (2.\ 13)$$

and

$$\int_0^1\left|q-\frac{3}{4}\right|^{\mathbf{v}}\left|\varphi'\left(U_2(q)\right)\right|^{\mathbf{v}}dq$$
$$\leq\left|\varphi'(\alpha)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)^{\mathbf{v}}\left(\frac{1-q}{2}\right)dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)^{\mathbf{v}}\left(\frac{1-q}{2}\right)dq\right]$$
$$+\left|\varphi'(\beta)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)^{\mathbf{v}}\left(\frac{1+q}{2}\right)dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)^{\mathbf{v}}\left(\frac{1+q}{2}\right)dq\right]$$
$$=\left(\frac{2^{-2\mathbf{v}-5}\times 3^{\mathbf{v}+1}\left(4\mathbf{v}+5\right)+2^{-2\mathbf{v}-5}}{\left(\mathbf{v}+1\right)\left(\mathbf{v}+2\right)}\right)\left|\varphi'(\alpha)\right|^{\mathbf{v}}$$
$$+\left(\frac{2^{-2\mathbf{v}-5}\left(8\mathbf{v}+15\right)+2^{-2\mathbf{v}-5}\times 3^{\mathbf{v}+1}\left(4\mathbf{v}+11\right)}{\left(\mathbf{v}+1\right)\left(\mathbf{v}+2\right)}\right)\left|\varphi'(\beta)\right|^{\mathbf{v}}. \quad (2.\ 14)$$

We also observe that

$$\int_0^1 (U_1(q))^{\frac{2\mathbf{v}}{\mathbf{v}-1}}\,dq = \left(\frac{1-\mathbf{v}}{1+\mathbf{v}}\right) H^{\frac{2\mathbf{v}}{\mathbf{v}-1}} y^{-1}\left((1+y)^{\frac{1+\mathbf{v}}{1-\mathbf{v}}}-1\right) \tag{2. 15}$$

and

$$\int_0^1 (U_2(q))^{\frac{2\mathbf{v}}{\mathbf{v}-1}}\,dq = \left(\frac{1-\mathbf{v}}{1+\mathbf{v}}\right) H^{\frac{2\mathbf{v}}{\mathbf{v}-1}} y^{-1}\left(1-(1-y)^{\frac{1+\mathbf{v}}{1-\mathbf{v}}}\right), \tag{2. 16}$$

where $y=\frac{\beta-\alpha}{\alpha+\beta}$ and $H=\frac{2\alpha\beta}{\alpha+\beta}$.
Applying (2. 13)-(2. 16) in (2. 12), we get (2. 11). □

Theorem 2.7. *According to the assumptions of Theorem 2.6, we have*

$$\begin{aligned}
|\mu(\alpha,\beta;\varphi,\phi)| \le{}& \frac{H^2(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}\left(\frac{\mathbf{v}-1}{2\mathbf{v}-1}\right)^{1-\frac{1}{\mathbf{v}}} \\
&\times\left[4^{-\frac{2\mathbf{v}-1}{\mathbf{v}-1}}\left(3^{\frac{2\mathbf{v}-1}{\mathbf{v}-1}}+1\right)\right]^{1-\frac{1}{\mathbf{v}}}\left(\frac{1}{4(2\mathbf{v}-1)(\mathbf{v}-1)y^2}\right)^{\frac{1}{\mathbf{v}}} \\
&\times\left\{\left[\gamma_1(\mathbf{v},y)\left|\varphi'(\alpha)\right|^{\mathbf{v}}+\gamma_2(\mathbf{v},y)\left|\varphi'(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right. \\
&\left.+\left[\gamma_2(\mathbf{v},-y)\left|\varphi'(\alpha)\right|^{\mathbf{v}}+\gamma_1(\mathbf{v},-y)\left|\varphi'(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right\},
\end{aligned} \tag{2. 17}$$

where

$$\gamma_1(\mathbf{v},y) = 1+2(\mathbf{v}-1)y-(1+y)^{1-2\mathbf{v}}(1+(4\mathbf{v}-3)y),$$
$$\gamma_2(\mathbf{v},y) = (1+y)^{2-2\mathbf{v}}+2(\mathbf{v}-1)y-1$$

$y=\frac{\beta-\alpha}{\alpha+\beta}$ *and* $H=\frac{2\alpha\beta}{\alpha+\beta}$.

Proof. From (2. 6) and using the Hölder integral inequality, we have

$$\begin{aligned}
|\mu(\alpha,\beta;\varphi,\phi)| \le{}& \frac{(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta} \\
&\times\left\{\left(\int_0^1\left|\frac{3}{4}-q\right|^{\frac{\mathbf{v}}{\mathbf{v}-1}}dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1 (U_1(q))^{2\mathbf{v}}\left|\varphi'(U_1(q))\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right. \\
&\left.+\left(\int_0^1\left|q-\frac{3}{4}\right|^{\frac{\mathbf{v}}{\mathbf{v}-1}}dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1 (U_2(q))^{2\mathbf{v}}\left|\varphi'(U_2(q))\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right\}.
\end{aligned} \tag{2. 18}$$

Since $\left|\varphi'\right|^{\mathbf{v}}$ is HA-convex on $[\alpha,\beta]$ for $\mathbf{v}>1$, we get

$$\int_0^1 (U_1(q))^{2\mathbf{v}} \left|\varphi'(U_1(q))\right|^{\mathbf{v}} dq$$
$$\leq \left|\varphi'(\alpha)\right|^{\mathbf{v}} \int_0^1 \left(\frac{1+q}{2}\right)(U_1(q))^{2\mathbf{v}} dq + \left|\varphi'(\beta)\right|^{\mathbf{v}} \int_0^1 \left(\frac{1-q}{2}\right)(U_1(q))^{2\mathbf{v}} dq$$
$$= \frac{H^{2\mathbf{v}}\left[1+2(\mathbf{v}-1)y-(1+y)^{1-2\mathbf{v}}(1+(4\mathbf{v}-3)y)\right]}{4(2\mathbf{v}-1)(\mathbf{v}-1)y^2}\left|\varphi'(\alpha)\right|^{\mathbf{v}}$$
$$+\frac{H^{2\mathbf{v}}\left[(1+y)^{2-2\mathbf{v}}+2(\mathbf{v}-1)y-1\right]}{4(2\mathbf{v}-1)(\mathbf{v}-1)y^2}\left|\varphi'(\beta)\right|^{\mathbf{v}} \quad (2.\ 19)$$

and

$$\int_0^1 (U_2(q))^{2\mathbf{v}} \left|\varphi'(U_2(q))\right|^{\mathbf{v}} dq$$
$$\leq \left|\varphi'(\alpha)\right|^{\mathbf{v}} \int_0^1 \left(\frac{1-q}{2}\right)(U_2(q))^{2\mathbf{v}} dq + \left|\varphi'(\beta)\right|^{\mathbf{v}} \int_0^1 \left(\frac{1+q}{2}\right)[U_2(q)]^{2\mathbf{v}} dq$$
$$= \frac{H^{2\mathbf{v}}\left[(1-y)^{2-2\mathbf{v}}+2(1-\mathbf{v})y-1\right]}{4(2\mathbf{v}-1)(\mathbf{v}-1)y^2}\left|\varphi'(\alpha)\right|^{\mathbf{v}}$$
$$+\frac{H^{2\mathbf{v}}\left[1+2(1-\mathbf{v})y-(1-y)^{1-2\mathbf{v}}(1+(3-4\mathbf{v})y)\right]}{4(2\mathbf{v}-1)(\mathbf{v}-1)y^2}\left|\varphi'(\beta)\right|^{\mathbf{v}}. \quad (2.\ 20)$$

We notice that

$$\int_0^1 \left|\frac{3}{4}-q\right|^{\frac{\mathbf{v}}{\mathbf{v}-1}} dq = \int_0^1 \left|q-\frac{3}{4}\right|^{\frac{\mathbf{v}}{\mathbf{v}-1}} dq = 4^{-\frac{2\mathbf{v}-1}{\mathbf{v}-1}}\left(\frac{\mathbf{v}-1}{2\mathbf{v}-1}\right)\left(3^{\frac{2\mathbf{v}-1}{\mathbf{v}-1}}+1\right). \quad (2.\ 21)$$

Applying (2. 19)-(2. 21) in (2. 18), we get (2. 17). □

Theorem 2.8. *According to the assumptions of Theorem 2.5, we have*

$$|\mu(\alpha,\beta;\varphi,\phi)| \leq \frac{H^2(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}$$
$$\times\left\{[q_1(y)]^{1-\frac{1}{\mathbf{v}}}\left[\alpha_1(y)\left|\varphi'(\alpha)\right|^{\mathbf{v}}+\alpha_2(y)\left|\varphi'(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right.$$
$$\left.+[q_1(-y)]^{1-\frac{1}{\mathbf{v}}}\left[\alpha_2(-y)\left|\varphi'(\alpha)\right|^{\mathbf{v}}+\alpha_1(-y)\left|\varphi'(\beta)\right|^{\mathbf{v}}\right]^{\frac{1}{\mathbf{v}}}\right\}, \quad (2.\ 22)$$

where

$$q_1(y) = \frac{3y+2}{4y(1+y)} + \frac{1}{y^2}\ln\left(\frac{16(1+y)}{(4+3y)^2}\right),$$

$$\alpha_1(y) = \frac{3y^2-3y-4}{8y^2(1+y)} + \frac{(y-8)}{8y^3}\ln\left(\frac{16(1+y)}{(4+3y)^2}\right),$$

$$\alpha_2(y) = \frac{4+3y}{8y^2} + \frac{(8+7y)}{8y^3}\ln\left(\frac{16(1+y)}{(4+3y)^2}\right),$$

$y = \frac{\beta-\alpha}{\alpha+\beta}$ *and* $H = \frac{2\alpha\beta}{\alpha+\beta}$.

Proof. From (2. 6) and using the power-mean inequality, we have

$$|\mu(\alpha,\beta;\varphi,\phi)| \le \frac{(\beta-\alpha)\|\phi\|_{[\alpha,\beta],\infty}}{4\alpha\beta}$$
$$\times\left\{\left(\int_0^1\left|\frac{3}{4}-q\right|(U_1(q))^2dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1\left|\frac{3}{4}-q\right|(U_1(q))^2\left|\varphi'(U_1(q))\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right.$$
$$\left.+\left(\int_0^1\left|q-\frac{3}{4}\right|(U_2(q))^2dq\right)^{1-\frac{1}{\mathbf{v}}}\left(\int_0^1\left|\frac{3}{4}-q\right|(U_2(q))^2\left|\varphi'(U_2(q))\right|^{\mathbf{v}}dq\right)^{\frac{1}{\mathbf{v}}}\right\}. \tag{2. 23}$$

By using the HA-convexity of $\left|\varphi'\right|^{\mathbf{v}}$ on $[\alpha,\beta]$ for $\mathbf{v}\ge 1$, we get

$$\int_0^1\left|\frac{3}{4}-q\right|(U_1(q))^2\left|\varphi'(U_1(q))\right|^{\mathbf{v}}dq$$
$$\le \left|\varphi'(\alpha)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)\left(\frac{1+q}{2}\right)(U_1(q)))^2dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)\left(\frac{1+q}{2}\right)(U_1(q))^2dq\right]$$
$$+\left|\varphi'(\beta)\right|^{\mathbf{v}}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)\left(\frac{1-q}{2}\right)(U_1(q))^2dq+\int_{\frac{3}{4}}^1\left(q-\frac{3}{4}\right)\left(\frac{1-q}{2}\right)(U_1(q))^2dq\right]$$
$$= H^2\left[\frac{3y^2-3y-4}{8y^2(1+y)}+\frac{(y-8)}{8y^3}\ln\left(\frac{16(1+y)}{(4+3y)^2}\right)\right]\left|\varphi'(\alpha)\right|^{\mathbf{v}}$$
$$+H^2\left[\frac{4+3y}{8y^2}+\frac{(8+7y)}{8y^3}\ln\left(\frac{16(1+y)}{(4+3y)^2}\right)\right]\left|\varphi'(\beta)\right|^{\mathbf{v}} \tag{2. 24}$$

and

$$\int_0^1\left|q-\frac{3}{4}\right|(U_2(q))^2\left|\varphi'(U_2(q))\right|^{\mathbf{v}}dq$$

$$\leq \left|\varphi'(\alpha)\right|^{\mathbf{v}} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-q\right)\left(\frac{1-q}{2}\right)(U_2(q))^2\,dq + \int_{\frac{3}{4}}^{1} \left(q-\frac{3}{4}\right)\left(\frac{1-q}{2}\right)(U_2(q))^2\,dq\right]$$

$$+ \left|\varphi'(\beta)\right|^{\mathbf{v}} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-q\right)\left(\frac{1+q}{2}\right)(U_2(q))^2\,dq + \int_{\frac{3}{4}}^{1} \left(q-\frac{3}{4}\right)\left(\frac{1+q}{2}\right)(U_2(q))^2\,dq\right]$$

$$= H^2\left[\frac{4-3y}{8y^2} - \frac{(8-7y)}{8y^3}\ln\left(\frac{16(1-y)}{(4-3y)^2}\right)\right]\left|\varphi'(\alpha)\right|^{\mathbf{v}}$$

$$+ H^2\left[\frac{3y^2+3y-4}{8y^2(1-y)} + \frac{(y+8)}{8y^3}\ln\left(\frac{16(1-y)}{(4-3y)^2}\right)\right]\left|\varphi'(\beta)\right|^{\mathbf{v}}. \quad (2.\ 25)$$

We also have

$$\int_0^1 \left|\frac{3}{4}-q\right|(U_1(q))^2dq = \int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)(U_1(q))^2dq + \int_{\frac{3}{4}}^{1}\left(q-\frac{3}{4}\right)(U_1(q))^2dq$$

$$= H^2\left[\frac{3y+2}{4y(1+y)} + \frac{1}{y^2}\ln\left(\frac{16(1+y)}{(4+3y)^2}\right)\right] \quad (2.\ 26)$$

and

$$\int_0^1 \left|\frac{3}{4}-q\right|(U_2(q))^2dq = \int_0^{\frac{3}{4}}\left(\frac{3}{4}-q\right)(U_2(q))^2dq + \int_{\frac{3}{4}}^{1}\left(q-\frac{3}{4}\right)(U_2(q))^2dq$$

$$= H^2\left[\frac{3y-2}{4y(1-y)} + \frac{1}{y^2}\ln\left(\frac{16(1-y)}{(4-3y)^2}\right)\right], \quad (2.\ 27)$$

where $y = \frac{\beta-\alpha}{\alpha+\beta}$ and $H = \frac{2\alpha\beta}{\alpha+\beta}$.
Using (2. 25)-(2. 27) in (2. 23) we get (2. 22). □

3. ACKNOWLEDGMENTS

The authors are deeply indebted to the anonymous reviewer who read the manuscript very carefully and gave very important and useful suggestions to improve the manuscript.

REFERENCES

[1] J. H. Chen, X. J. Huang, *Some new inequalities of Simpson's type for s-convex functions via fractional integrals*, Filomat, **31**, No.15(2017) 4989–4997.

[2] L. Chun, F. Qi, *Inequalities of Simpson type for functions whose third derivatives are extended s-convex functions and applications to means*, J. Comput. Anal. Appl., **19**, No.3 (2015) 555–569.

[3] S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl., **5** (2000) 533-579.

[4] T. S. Du, Y. J. Li, Z. Q. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions*, Appl. Math. Comput., **293** (2017) 358–369.

[5] T. S. Du, J. G. Liao, Y. J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions*, J. Nonlinear Sci. Appl., **9**(2016) 3112-3126.

[6] K. -C. Hsu, S. -R. Hwang, K. -L. Tseng, *Some extended Simpson-type inequalities and applications*, Bull. Iran. Math. Soc., **43**, No.2 (2017) 409-425.

[7] S. Hussain, S. Qaisar, *Generalizations of Simpson's type inequalities through preinvexity and prequasiinvexity*, Punjab Univ. J. Math, **46**, No.2 (2014) 1-9.

[8] S. Hussain, S. Qaisar, *More results on Simpson's type inequality through convexity for twice differentiable continuous mappings*, SpringerPlus **5** (2017) Article ID 77.

[9] V. H. Huy, Q. A. Ngô, *New inequalities of Simpson-like type involving n knots and the mth derivative*, Math. Comput. Model. Dyn. Syst., **52** No.3-4 (2010) 522-528.

[10] İ. İşcan, *Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions*, Journal of Mathematics, **2014** (2014) Article ID 346305, 10 pages.

[11] İ. İşcan, S. Turhan, S. Maden, *Hermite-Hadamard and Simpson-like type inequalities for differentiable p-quasi-convex functions*, Filomat, **31**, No.19(2017) 5945-5953.

[12] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat., **43**,No.6 (2014) 935 .942.

[13] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejér type inequalities for harmonically-convex functions with applications to special means*, International Journal of Analysis and Applications **13**, No. 1 (2017) 1-14.

[14] M. A. Latif, *New Fejer and Hermite-Hadamard type inequalities for differentiable p-convex mappings*, Punjab Univ. j. math. **51**, No. 2 (2019) 39-59.

[15] M. A. Latif and W. Irshad, *Some Fejer and Hermite-Hadamard type inequalities considering ϵ-convex and(σ, ϵ)-convex functions*, Punjab Univ. j. math. **50**, No. 3 (2018) 13-24.

[16] M. A. Latif, *Estimates of Hermite-Hadamard inequality for twice differentiable harmonically-convex functions with applications*, Punjab Univ. j. math. **50**, No. 1 (2018) 1-13.

[17] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some ϕ-analogues of Hermite-Hadamard inequality for s-convex functions in the second sense and related estimates*, Punjab Univ. j. math. **48**, No. 2 (2016) 147-166.

[18] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions*, Punjab Univ. j. math. **47**, No. 1 (2015) 57-72.

[19] M. A. Latif and S. Hussain, *New Hermite-Hadamard type inequalities for harmonically-convex functions*, Punjab Univ. j. math. **51**, No. 6 (2019) 1-17.

[20] M. A. Latif, S. S. Dragomir and E. Momoniat, *Fejer type inequalities for harmonically-convex functions with applications*, Journal of Applied Analysis and Computation, **7**, NO.3 (2017) 795-813.

[21] M. A. Latif and S. Hussain, *Weighted Simpson's type inequalities for GA-convex functions*, Punjab Univ. J. Math, **51** (10) (2019) 93-106.

[22] Y. J. Li, T. S. Du, *A generalization of Simpson type inequality via differentiable functions using extended $(s, m)_{\varphi}$-preinvex functions*, J. Comput. Anal. Appl., **22**, NO.4 (2017) 613-632.

[23] Y. J. Li, T. S. Du, B. Yu, *Some new integral inequalities of Hadamard-Simpson type for extended (s, m)-preinvex functions*, Ital. J. Pure Appl. Math., **36** (2016) 583-600.

[24] W. J. Liu, *Some Simpson type inequalities for h-convex and (α, m)-convex functions*, J. Comput. Anal. Appl., **16**, No.5 (2014) 1005-1012.

[25] C. -Y. Luo, T. S. Du , M. Kunt, Y. Zhang, *Certain new bounds considering the weighted Simpson-like type inequality and applications*, J. Inequal. Appl., **2018** (2018) 2018:332.

[26] M. Matloka, *Weighted Simpson type inequalities for h-convex functions*, J. Nonlinear Sci. Appl., **10** (2017) 5770-5780.

[27] M. A. Noor, K. I. Noor, M. U. Awan, *Simpson-type inequalities for geometrically relative convex functions*, Ukr. Math. J., **70**, No.7 (2018) 992-1000.

[28] J. Pecaric, S. Varovanec, *A note on Simpson's inequality for Lipschitzian functions*, Soochow J. Math., **27**, No.1 (2001) 53-57.

[29] S. Qaisar, C. J. He, S. Hussain, *A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications*, J. Inequal. Appl., **2013** 158 (2013).

[30] F. Qi, B. Y. Xi, *Some integral inequalities of Simpson type for GA-ε-convex functions*, Georgian Math. J., **20** (2013) 775-788.

[31] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., **60** (2010) 2191-2199.

[32] E. Set, A. O. Akdemir, M. E. Özdemir, *Simpson type integral inequalities for convex functions via Riemann-Liouville integrals*, Filomat, **31**, No.14 (2017) 4415-4420.

[33] Y. Shuang, Y. Wang, F. Qi, *Integral inequalities of Simpson's type for (α, m)-convex functions*, J. Nonlinear Sci. Appl., **9**, No.12 (2016) 6364-6370.

[34] M. Tunç, E. Gov, S. Balgecti, *Simpson type quantum integral inequalities for convex functions*, Miskolc Math. Notes, **19**, No.1 (2018) 649-664.

[35] M. Tunç, C. Yildiz, A. Ekinci, *On some inequalities of Simpson's type via h-convex functions*, Hacet. J. Math. Stat., **42**, No.4 (2013) 309-317.

[36] Y. Wang, S. H. Wang, F. Qi, *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex*, Facta Univ., Ser. Math. Inform., **28**, No.2 (2013) 151-159.

[37] B. Y. Xi, F. Qi, *Integral inequalities of Simpson type for logarithmically convex functions*, Adv. Stud. Contemp. Math. (Kyungshang), **23**, No.4 (2013), 559-566.

[38] Z. Q. Yang, Y. J. Li, T. S Du, *A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions*, Ital. J. Pure Appl. Math., **35** (2015) 327-338.

[39] C. Zhou, C. Peng, T. S. Du, *Simpson-like type inequalities for relative semi-(α, m)-logarithmically convex functions*, J. Nonlinear Sci. Appl., **10**, No.8 (2017).