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On Some New Grüss Inequalities Concerning to Caputo k-fractional Derivative

Sajid Iqbal
Department of Mathematics,
University of Management and Technology,
Sialkot Campus,Sialkot, Punjab, 51310, Pakistan
Email: sajidiqbal@skt.umt.edu.pk;

Shahzad Ahmad
Department of Mathematics,
University of Management and Technology,
Sialkot Campus,Sialkot, Punjab, 51310, Pakistan
Email: shahzad.ahmed@skt.umt.edu.pk

Muhammad Adil Khan
Department of Mathematics,
University of Peshawar, Peshawar, Pakistan
Email: madilkhan@uop.edu.pk

Muhammad Samraiz
Department of Mathematics,
University of Sargodha, Sargodha, Pakistan
Email: muhammad.samraiz@uos.edu.pk

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Abstract. In this article, we establish some new integral inequalities on fractional calculus operator i.e. k-Caputo fractional derivative operator. As a consequence, we obtain new variety of fractional integral inequalities. Also we apply the Young's inequality to find new versions of inequalities for the generalized fractional derivative. Such results for this new and generalized fractional derivative are very useful and worthwhile in the fields of differential equations and fractional differential calculus which has very deep connection with the real world problems. These results may motivate further research in different areas of pure and applied sciences.

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1. INTRODUCTION

Fractional calculus refers to the study of integral and derivative operators of fractional order. This subject is as significant as calculus itself and have been of huge importance in the last few decades (see for example [1], [2], [3], [7], [8], [9], [10], [11], [12]). Fractional calculus has been applied in different areas of engineering, science, finance, applied mathematics, bio engineering etc. Mathematical inequalities are important to the study of mathematics as well as many related fields and their uses are broad in scope.

Grüss inequality is stated in the next theorem.

Theorem 1.1. [6] *Let $\Re$ be a set of real numbers, $m, M, n, N \in \Re$, and $\Omega, \Upsilon : [\tau_1, \tau_2] \to \Re$ be two positive functions such that $m \leq \Omega(\mu) \leq M$, $n \leq \Upsilon(\mu) \leq N$, for $\mu \in [\tau_1, \tau_2]$. Then*

$$\left| \frac{1}{\tau_2 - \tau_1} \int\limits_{\tau_1}^{\tau_2} \Omega(\mu)\Upsilon(\mu)d\mu - \frac{1}{(\tau_2 - \tau_1)^2} \int\limits_{\tau_1}^{\tau_2} \Omega(\mu)d\mu \int\limits_{\tau_1}^{\tau_2} \Upsilon(\mu)d\mu \right| \leq \frac{1}{4}(M - m)(N - n), \tag{1. 1}$$

where the constant $\frac{1}{4}$ cannot be improved.

Following definition of Caputo k-fractional derivatives is given in [5].

Definition 1.2. *Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $\Omega \in AC^n[a, b]$. The left and right sided Caputo k-fractional derivatives of order α are defined as follows:*

$${}^C D^{\alpha,k}_{a+}\Omega(\varrho) = \frac{1}{k\Gamma_k(n - \frac{\alpha}{k})} \int_a^\varrho \frac{\Omega^{(n)}(\chi)}{(\varrho - \chi)^{\frac{\alpha}{k} - n + 1}} d\chi, \varrho > a \tag{1. 2}$$

and

$${}^C D^{\alpha,k}_{b-}\Omega(\varrho) = \frac{(-1)^n}{k\Gamma_k(n - \frac{\alpha}{k})} \int_\varrho^b \frac{\Omega^{(n)}(\chi)}{(\chi - \varrho)^{\frac{\alpha}{k} - n + 1}} d\chi, \varrho < b \tag{1. 3}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function (Diaz et al. in [4]) defined as

$$\Gamma_k(\alpha) = \int_0^\infty \chi^{\alpha - 1} e^{\frac{-\chi^k}{k}} d\chi,$$

also

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative of order n exists, then Caputo k-fractional derivative $({}^C D^{\alpha,1}_{a+}\Omega)(x)$ coincides with $\Omega^{(n)}(x)$.

2. MAIN RESULTS

The first main result is given in next theorem.

Theorem 2.1. *Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha]+1$, $\Omega, \Psi_1, \Psi_2 \in AC^n[a, b]$. and let $\left({}^C D_{a+}^{\alpha,k} f\right)$ denotes the Caputo k-fractional derivative of order $\alpha > 0$. Suppose that the exist ${\Psi_1}^{(n)}, {\Psi_2}^{(n)}$ such that*

$$ {\Psi_1}^{(n)}(\xi) \leq \Omega^{(n)}(\xi) \leq {\Psi_2}^{(n)}(\xi), \tag{2. 4}$$

for all $\xi \in [0, \infty)$. Then

$$\begin{aligned} &\left({}^C D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^C D_{a+}^{\alpha,k}\Omega\right)(\xi) + \left({}^C D_{a+}^{\alpha,k}\Psi_2\right)(\xi)\left({}^C D_{a+}^{\alpha,k}\Omega\right)(\xi) \\ \geq\ &\left({}^C D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^C D_{a+}^{\alpha,k}\Psi_2\right)(\xi) + \left({}^C D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^C D_{a+}^{\alpha,k}\Omega\right)(\xi). \end{aligned} \tag{2. 5}$$

Proof. Using (2. 4) for all $\gamma \geq 0$, $\delta \geq 0$, we have

$$[{\Psi_2}^{(n)}(\gamma) - \Omega^{(n)}(\gamma)][\Omega^{(n)}(\delta) - {\Psi_1}^{(n)}(\delta)] \geq 0,$$

then

$${\Psi_2}^{(n)}(\gamma)\Omega^{(n)}(\delta) + {\Psi_1}^{(n)}(\delta)\Omega^{(n)}(\gamma) \geq {\Psi_1}^{(n)}(\delta){\Psi_2}^{(n)}(\gamma) + \Omega^{(n)}(\gamma)\Omega^{(n)}(\delta). \tag{2. 6}$$

If we multiplying by $\frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k(\nu(n-\mu))}$ on both sides of (2. 6) and integrating the resulting identity for the variable γ over the interval (a, ξ) we get

$$\begin{aligned} &\Omega^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}{\Psi_2}^{(n)}(\gamma)d\gamma \\ +\ &{\Psi_1}^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}\Omega^{(n)}(\gamma)d\gamma \\ \geq\ &{\Psi_1}^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}{\Psi_2}^{(n)}(\gamma)d\gamma \\ +\ &\Omega^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}\Omega^{(n)}(\gamma)d\gamma, \end{aligned}$$

which can be written as

$$\begin{aligned} &\Omega^{(n)}(\delta)\left({}^C D_{a+}^{\alpha,k}\Psi_2\right)(\xi) + \Psi_1^{(n)}(\delta)\left({}^C D_{a+}^{\alpha,k}\Omega\right)(\xi) \\ \geq\ &{\Psi_1}^{(n)}(\delta)\left({}^C D_{a+}^{\alpha,k}\Psi_2\right)(\xi) + \Omega^{(n)}(\delta)\left({}^C D_{a+}^{\alpha,k}\Omega\right)(\xi). \end{aligned} \tag{2. 7}$$

Now multiplying by $\frac{(\xi-\delta)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ on both sides of (2. 7) and integrating the resulting identity for the variable δ on the interval (a, ξ) we get

$$\begin{aligned}&\left({}^{C}D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)+\left({}^{k}D_{a+}^{\mu,\nu}\Psi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\\ \geq\ &\left({}^{C}D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Psi_2\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi).\end{aligned}$$

This complete the proof. □

Corollary 2.2. *Let $m, M \in \Re$, with $m < M$, and $k, \xi > 0$. Let $\Omega^{(n)}$ be a positive function such that $m \leq \Omega^{(n)}(\xi) \leq M$. Then*

$$\begin{aligned}&mR(\xi)({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi)+MR(\xi)^{C}D_{a+}^{\alpha,k}\Omega(\xi)\\ \geq\ &mMR(\xi)R(\xi)+({}^{C}D_{a+}^{\alpha,k}\Omega(\xi))({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi),\end{aligned}$$

where

$$R(\xi)=\frac{(\xi-a)^{n-\frac{\alpha}{k}}}{\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}. \tag{2. 8}$$

Remark 2.3. *Take $k = 1$ in Theorem 2.1 and Corollary 2.2 we get the results for the Caputo fractional derivative.*

Theorem 2.4. *Let $k > 0$, and let $\left({}^{C}D_{a+}^{\alpha,k}f\right)$ denotes the Caputo k-fractional derivative of order $\alpha > 0$, and let $\Omega^{(n)}$ and $\Upsilon^{(n)}$ be two positive functions on $[a, \xi)$. Suppose that (2. 4) holds and there exist integrable functions $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$ on $[a, \xi)$ such that*

$$\varphi_1^{(n)}(\xi)\leq\Upsilon^{(n)}(\xi)\leq\varphi_2^{(n)}(\xi). \tag{2. 9}$$

Then the following inequalities holds:

$$\begin{aligned}a.\quad&\left({}^{C}D_{a+}^{\alpha,k}\varphi_1\right)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\Psi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi)\\ &\geq\left({}^{C}D_{a+}^{\alpha,k}\varphi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Psi_2\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi),\end{aligned}$$

$$\begin{aligned}b.\quad&\left({}^{C}D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\varphi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\\ &\geq\left({}^{C}D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\varphi_2\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi),\end{aligned}$$

$$\begin{aligned}c.\quad&\left({}^{C}D_{a+}^{\alpha,k}\Psi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\varphi_2\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi)\\ &\geq\left({}^{C}D_{a+}^{\alpha,k}\Psi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon_2\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\varphi_2\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi),\end{aligned}$$

$$\begin{aligned}d.\quad&\left({}^{C}D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\varphi_1\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi)\\ &\geq\left({}^{C}D_{a+}^{\alpha,k}\Psi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi)+\left({}^{C}D_{a+}^{\alpha,k}\varphi_1\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi).\end{aligned}$$

Proof. For all $\xi \in [0, \infty)$, it follows from (2. 4) and (2. 9) we get

$$[\Psi_2{}^{(n)}(\gamma)-\Omega^{(n)}(\gamma))][\Upsilon^{(n)}(\delta)-\varphi_1^{(n)}(\delta))]\geq 0.$$

Then

$$\Psi_2{}^{(n)}(\gamma)\Upsilon^{(n)}(\delta)+\varphi_1^{(n)}(\delta)\Omega^{(n)}(\gamma)\geq\varphi_1^{(n)}(\delta)\Psi_2{}^{(n)}(\gamma)+\Omega^{(n)}(\gamma)\Upsilon^{(n)}(\delta).$$

Multiplying by $\frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k(\nu(n-\mu))}$ on both side and integrating the resulting identity for the variable γ over the interval $[a,\xi)$ we have that

$$\begin{aligned}
&\Upsilon^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}\Psi_2{}^{(n)}(\gamma)d\gamma\\
+&\ \varphi_1^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}\Omega^{(n)}(\gamma)d\gamma\\
\geq&\ \varphi_1^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}\Psi_2{}^{(n)}(\gamma)d\gamma\\
+&\ \Upsilon^{(n)}(\delta)\frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^\xi(\xi-\gamma)^{n-\frac{\alpha}{k}-1}\Omega^{(n)}(\gamma)d\gamma,
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\Upsilon^{(n)}(\delta)\left({}^CD_{a+}^{\alpha,k}\Psi_2\right)(\xi)+\varphi_1^{(n)}(\delta)\left({}^CD_{a+}^{\alpha,k}\Omega\right)(\xi)\\
\geq&\ \varphi_1^{(n)}(\delta)\left({}^CD_{a+}^{\alpha,k}\Psi_2\right)(\xi)+\Upsilon^{(n)}(\delta)\left({}^CD_{a+}^{\alpha,k}\Omega\right)(\xi).
\end{aligned}$$

Again multiplying by $\frac{(\xi-\delta)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ on both side and integrating the resulting identity for the variable δ over the interval $[a,\xi)$ we have that

$$\begin{aligned}
&\left({}^CD_{a+}^{\alpha,k}\varphi_1\right)(\xi)\left({}^CD_{a+}^{\alpha,k}\Omega\right)(\xi)+\left({}^CD_{a+}^{\alpha,k}\Psi_2\right)(\xi)\left({}^CD_{a+}^{\alpha,k}\Upsilon\right)(\xi)\\
\geq&\ \left({}^CD_{a+}^{\alpha,k}\varphi_1\right)(\xi)\left({}^CD_{a+}^{\alpha,k}\Psi_2\right)(\xi)+\left({}^CD_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^CD_{a+}^{\alpha,k}\Upsilon\right)(\xi)
\end{aligned}$$

This complete the part (a).
To prove the part (b)-(d) following inequalities shall be used.

$$(b).\qquad(\varphi_2^{(n)}(\gamma)-\Upsilon^{(n)}(\gamma))(\Omega^{(n)}(\delta)-\Psi_1{}^{(n)}(\delta)))\geq 0.$$

$$(c).\qquad(\Psi_2{}^{(n)}(\gamma)-\Omega^{(n)}(\gamma))(\Upsilon^{(n)}(\delta)-\varphi_2^{(n)}(\delta))\geq 0.$$

$$(d).\qquad(\Psi_1^{(n)}(\gamma)-\Omega^{(n)}(\gamma))(\Upsilon^{(n)}(\delta)-\varphi_1^{(n)}(\delta)))\geq 0.$$

□

Corollary 2.5. *Let $\Omega^{(n)}$ and $\Upsilon^{(n)}$ be two positive functions on $[a,\xi)$. Suppose that there exist real constant m,M,n,N such that $m\leq\Omega^{(n)}(\xi)\leq M$, $n\leq\Upsilon^{(n)}(\xi)\leq N$ for all*

$\xi \in [0, \infty)$. *Then*

$$\begin{aligned} a. \quad & nR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi) + MR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) \\ & \geq nMR(\xi)R(\xi) + \left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) \end{aligned}$$

$$\begin{aligned} b. \quad & mR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) + NR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi) \\ & \geq mNR(\xi)mR(\xi) + \left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) \\ c. \quad & NMR(\xi)R(\xi) + \left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) \\ & \geq MR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) + NR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi) \end{aligned}$$

$$\begin{aligned} d. \quad & nmR(\xi)R(\xi)\left({}^{C}D_{a+}^{\alpha,k}\varphi_1\right)(\xi) + \left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) \\ & \geq mR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Upsilon\right)(\xi) + nmR(\xi)\left({}^{C}D_{a+}^{\alpha,k}\Omega\right)(\xi), \end{aligned}$$

$R(\xi)$ *is defined by (2. 8).*

Lemma 2.6. *Let* $k > 0$, *and let* $\left({}^{C}D_{a+}^{\alpha,k}f\right)$ *denotes the Caputo k-fractional derivative of order* $\alpha > 0$, *and let* ${\Psi_1}^{(n)}$ *and* ${\Psi_2}^{(n)}$ *be two integrable functions on* $[0, \infty)$. *Then*

$$\begin{aligned} & R(\xi)(({}^{C}D_{a+}^{\alpha,k}\Omega^2)(\xi)) - (({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi))^2 \\ = \;& (({}^{C}D_{a+}^{\alpha,k}\Psi_2)(\xi) - ({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi))({}^{C}D_{a+}^{\alpha,k}\Omega(\xi) - {}^{C}D_{a+}^{\alpha,k}\Psi_1(\xi)) \\ - \;& R(\xi)(({}^{C}D_{a+}^{\alpha,k}\Psi_2)(\xi) - ({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi))(({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi) - ({}^{C}D_{a+}^{\alpha,k}\Psi_1)(\xi)) \\ + \;& R(\xi){}^{C}D_{a+}^{\alpha,k}(\Psi_1(\xi)\Omega(\xi)) - (({}^{C}D_{a+}^{\alpha,k}\Psi_1)(\xi)({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi)) \\ + \;& R(\xi){}^{C}D_{a+}^{\alpha,k}(\Psi_2(\xi)\Omega(\xi)) - ({}^{C}D_{a+}^{\alpha,k}\Psi_2)(\xi)({}^{C}D_{a+}^{\alpha,k}\Omega)(\xi) \\ - \;& R(\xi){}^{C}D_{a+}^{\alpha,k}(\Psi_1(\xi)\Psi_2) + ({}^{C}D_{a+}^{\alpha,k}\Psi_1)(\xi)({}^{C}D_{a+}^{\alpha,k}\Psi_2)(\xi), \end{aligned}$$

$R(\xi)$ *is defined by (2. 8).*

Proof. Since $\gamma, \delta > 0$, we have

$$\begin{aligned} & ({\Psi_2}^{(n)}(\delta) - \Omega^{(n)}(\delta))(\Omega^{(n)}(\gamma) - {\Psi_1}^{(n)}(\gamma)) \\ + \;& ({\Psi_2}^{(n)}(\gamma) - \Omega^{(n)}(\gamma))(\Omega^{(n)}(\delta) - {\Psi_1}^{(n)}(\delta)) \\ - \;& ({\Psi_2}^{(n)}(\gamma) - \Omega^{(n)}(\gamma))(\Omega^{(n)}(\delta) - {\Psi_1}^{(n)}(\delta)) \\ - \;& ({\Psi_2}^{(n)}(\delta) - \Omega^{(n)}(\delta))(\Omega^{(n)}(\delta) - {\Psi_1}^{(n)}(\delta)) \\ = \;& (\Omega^{(n)}(\gamma))^2 + (\Omega^{(n)}(\delta))^2 - 2\Omega^{(n)}(\gamma)\Omega^{(n)}(\delta) \\ + \;& \Psi_2^{(n)}(\delta)\Omega^{(n)}(\gamma) + \Psi_1^{(n)}(\gamma)\Omega^{(n)}(\delta) - \Psi_1^{(n)}(\gamma)\Psi_2^{(n)}(\delta) + \Psi_2^{(n)}(\gamma)\Omega^{(n)}(\delta) \\ + \;& \Psi_1^{(n)}(\delta)\Omega^{(n)}(\gamma) - \Psi_1^{(n)}(\delta)\Psi_2^{(n)}(\gamma) - \Psi_2^{(n)}(\gamma)\Omega^{(n)}(\gamma) + \Psi_1^{(n)}(\gamma)\Psi_1^{(n)}(\gamma) \\ - \;& \Psi_1^{(n)}(\gamma)\Omega^{(n)}(\gamma) + \Psi_2^{(n)}(\delta)\Omega^{(n)}(\delta) + \Psi_1^{(n)}(\delta)\Psi_2^{(n)}(\delta) - \Psi_1^{(n)}(\delta)\Omega^{(n)}(\delta). \end{aligned}$$

Multiplying both sides by $\frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ and integrating for the variable γ over the interval $[a,\xi]$ we get

$$\begin{aligned}
& (\Psi_2^{(n)}(\delta)-\Omega^{(n)}(\delta))({}^CD_{a+}^{\alpha,k}\Omega(\xi)-{}^CD_{a+}^{\alpha,k}\Psi_1(\xi)) \\
+ \;& ({}^CD_{a+}^{\alpha,k}\Psi_2(\xi)-{}^CD_{a+}^{\alpha,k}\Omega(\xi))(\Omega^{(n)}(\delta)-\Psi_1^{(n)}(\delta))) \\
- \;& {}^CD_{a+}^{\alpha,k}((\Psi_2(\xi)-\Omega(\xi))(\Omega(\delta)-\Psi_1(\delta)))) \\
- \;& (\Psi_2^{(n)}(\delta)-\Omega^{(n)}(\delta))(\Omega^{(n)}(\delta)-\Psi_1^{(n)}(\delta)))R(\xi) \\
= \;& ({}^CD_{a+}^{\alpha,k}\Omega^2(\xi))+(R(\xi)\Omega^2(\delta))-2{}^CD_{a+}^{\alpha,k}\Omega(\xi)\Omega^{(n)}(\delta) \\
+ \;& \Psi_2^{(n)}(\delta){}^CD_{a+}^{\alpha,k}\Omega(\xi)+{}^CD_{a+}^{\alpha,k}\Psi_1(\xi)\Omega^{(n)}(\delta) \\
+ \;& {}^CD_{a+}^{\alpha,k}\Psi_1(\xi)\Psi_2^{(n)}(\delta)+\Psi_2^{(n)}(\delta)\Omega^{(n)}(\delta) \\
+ \;& \Psi_1^{(n)}(\delta){}^CD_{a+}^{\alpha,k}\Omega(\xi)-\Psi_1^{(n)}(\delta){}^CD_{a+}^{\alpha,k}\Psi_2(\xi) \\
- \;& {}^CD_{a+}^{\alpha,k}(\Psi_2(\delta)\Omega(\xi))+{}^CD_{a+}^{\alpha,k}(\Psi_1(\xi)\Psi_2(\xi))R(\xi) \\
- \;& {}^CD_{a+}^{\alpha,k}(\Psi_1(\xi)\Omega(\xi))-\Psi_2^{(n)}(\delta)\Omega^{(n)}(\delta)R(\xi) \\
+ \;& \Psi_1^{(n)}(\delta)\Psi_2^{(n)}(\delta)R(\xi)-\Psi_1^{(n)}(\delta)\Omega^{(n)}(\delta)R(\xi).
\end{aligned}$$

Multiplying both sides by $\frac{(\xi-\delta)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ and integrating for the variable δ over the interval $[a,\xi]$ we get

$$\begin{aligned}
& ({}^CD_{a+}^{\alpha,k}\Psi_2(\xi)-{}^CD_{a+}^{\alpha,k}\Omega(\xi))({}^CD_{a+}^{\alpha,k}\Omega(\xi)-{}^CD_{a+}^{\alpha,k}\Psi_1(\xi)) \\
+ \;& ({}^CD_{a+}^{\alpha,k}\Psi_2(\xi)-{}^CD_{a+}^{\alpha,k}\Omega(\xi))({}^CD_{a+}^{\alpha,k}\Omega(\xi)-{}^CD_{a+}^{\alpha,k}\Psi_1(\xi))) \\
- \;& {}^CD_{a+}^{\alpha,k}((\Psi_2(\xi)-\Omega(\xi))(\Omega(\xi)-\Psi_1(\xi)))R(\xi) \\
- \;& ({}^CD_{a+}^{\alpha,k}\Psi_2(\xi)-{}^CD_{a+}^{\alpha,k}\Omega(\xi))({}^CD_{a+}^{\alpha,k}\Omega(\xi)-{}^CD_{a+}^{\alpha,k}\Psi_1(\xi)))R(\xi) \\
= \;& ({}^CD_{a+}^{\alpha,k}\Omega^2(\xi))R(\xi)+R(\xi)(R(\xi)\Omega^2(\xi))-2{}^CD_{a+}^{\alpha,k}\Omega(\xi){}^CD_{a+}^{\alpha,k}\Omega(\xi) \\
+ \;& {}^CD_{a+}^{\alpha,k}\Psi_2(\xi){}^CD_{a+}^{\alpha,k}\Omega(\xi)+{}^CD_{a+}^{\alpha,k}\Psi_1(\xi){}^CD_{a+}^{\alpha,k}\Omega(\xi) \\
+ \;& {}^CD_{a+}^{\alpha,k}\Psi_1(\xi){}^CD_{a+}^{\alpha,k}\Psi_2(\xi)+{}^CD_{a+}^{\alpha,k}\Psi_2(\xi){}^CD_{a+}^{\alpha,k}\Omega(\xi) \\
+ \;& {}^CD_{a+}^{\alpha,k}\Psi_1(\xi){}^CD_{a+}^{\alpha,k}\Omega(\xi)-{}^CD_{a+}^{\alpha,k}\Psi_1(\xi){}^CD_{a+}^{\alpha,k}\Psi_2(\xi) \\
- \;& R(\xi){}^CD_{a+}^{\alpha,k}(\Psi_2(\xi)\Omega(\xi))+{}^CD_{a+}^{\alpha,k}(\Psi_1(\xi)\Psi_1(\xi))R(\xi) \\
- \;& R(\xi){}^CD_{a+}^{\alpha,k}(\Psi_1(\xi)\Omega(\xi))-{}^CD_{a+}^{\alpha,k}(\Psi_2(\xi)\Omega(\xi))R(\xi) \\
+ \;& R(\xi){}^CD_{a+}^{\alpha,k}(\Psi_1(\xi)\Psi_2(\xi))-{}^CD_{a+}^{\alpha,k}(\Psi_1(\xi)\Omega(\xi))R(\xi).
\end{aligned}$$

This complete the proof of lemma. □

Corollary 2.7. *Let $m<M$, $k>0$, and let $\Omega^{(n)}$ be a positive function on $[a,\xi)$ such that $m\le\Omega^{(n)}(\xi)\le M$. Then*

$$\begin{aligned}
& R(\xi)({}^CD_{a+}^{\alpha,k}\Omega^2(\xi))-({}^CD_{a+}^{\alpha,k}\Omega(\xi))^2 \\
= \;& \left(MR(\xi)-{}^CD_{a+}^{\alpha,k}\Omega(\xi)\right)\left({}^CD_{a+}^{\alpha,k}\Omega(\xi)-mR(\xi)\right)-{}^CD_{a+}^{\alpha,k}((M-\Omega(\rho))(\Omega(\rho)-m)).
\end{aligned}$$

$R(\xi)$ is defined by (2. 8).

Theorem 2.8. *Let $k>0$, and let $\left({}^C D_{a+}^{\alpha,k} f\right)$ denotes the Caputo k-fractional derivative of order $\alpha>0$, and let $\Omega^{(n)}$ $\Psi_1^{(n)}$, ${\Psi_2}^{(n)}$, $\varphi_1^{(n)}$, and $\varphi_2^{(n)}$, be integrable functions on $[a,\xi)$. If conditions (2. 4) and (2. 9) are satisfied then*

$$\begin{aligned} &\left|R(\xi)^C D_{a+}^{\alpha,k}(\Omega(\xi)\Upsilon(\xi)) - ({}^C D_{a+}^{\alpha,k}\Omega)(\xi)({}^C D_{a+}^{\alpha,k}\Upsilon)(\xi)\right| \\ &\leq \sqrt{T(\Omega,\Psi_1,\Psi_2)T(\Upsilon,\varphi_1,\varphi_2)}, \end{aligned} \tag{2. 10}$$

where

$$\begin{aligned} T(\Omega,\Psi_1,\Psi_2) &= (({}^C D_{a+}^{\alpha,k}\Psi_2)(\xi) - ({}^C D_{a+}^{\alpha,k}\Omega)(\xi))(({}^C D_{a+}^{\alpha,k}\Omega)(\xi) - ({}^C D_{a+}^{\alpha,k}\Psi_1)(\xi)) \\ &+ R(\xi)({}^C D_{a+}^{\alpha,k}\Psi_1(\xi)\Omega)(\xi) - ({}^C D_{a+}^{\alpha,k}\Psi_1)(\xi)({}^C D_{a+}^{\alpha,k}\Omega)(\xi) \\ &+ R(\xi)({}^C D_{a+}^{\alpha,k})(\Psi_2(\xi)\Omega(\xi)) - ({}^C D_{a+}^{\alpha,k}\Psi_2)(\xi)({}^C D_{a+}^{\alpha,k}\Omega)(\xi) \\ &+ ({}^C D_{a+}^{\alpha,k}\Psi_1)(\xi)({}^C D_{a+}^{\alpha,k}\Psi_2)(\xi) - R(\xi)({}^C D_{a+}^{\alpha,k})(\Psi_1\Psi_2(\xi)), \end{aligned}$$

and

$$\begin{aligned} T(\Upsilon,\varphi_1,\varphi_2) &= (({}^C D_{a+}^{\alpha,k}\varphi_2)(\xi) - ({}^C D_{a+}^{\alpha,k}\Upsilon)(\xi))(({}^C D_{a+}^{\alpha,k}\Upsilon)(\xi) - ({}^C D_{a+}^{\alpha,k}\varphi_1)(\xi)) \\ &+ R(\xi)({}^C D_{a+}^{\alpha,k})(\varphi_1(\xi)\Upsilon(\xi)) - ({}^C D_{a+}^{\alpha,k}\varphi_1)(\xi)({}^C D_{a+}^{\alpha,k}\Upsilon)(\xi) \\ &+ R(\xi)({}^C D_{a+}^{\alpha,k})(\varphi_2(\xi)\Upsilon(\xi)) - ({}^C D_{a+}^{\alpha,k}\varphi_2)(\xi)({}^C D_{a+}^{\alpha,k}\Upsilon)(\xi) \\ &+ ({}^C D_{a+}^{\alpha,k}\varphi_1)((\xi)^C D_{a+}^{\alpha,k}\varphi_2)(\xi) - R(\xi)({}^C D_{a+}^{\alpha,k})(\varphi_1(\xi)\varphi_2(\xi)), \end{aligned}$$

$R(\xi)$ is defined by (2. 8).

Proof. Let $\xi>0$, $\gamma,\delta\in[a,\xi]$, $\Omega^{(n)}$, $\Upsilon^{(n)}$ be two positive function on $[0,\infty]$ such that conditions (2. 4) and (2. 9) are satisfied and $T(\gamma,\delta)$ defined by

$$T(\gamma,\delta) = (\Omega^{(n)}(\gamma) - \Omega^{(n)}(\delta))(\Upsilon^{(n)}(\gamma) - \Upsilon^{(n)}(\delta)). \tag{2. 11}$$

Taking the product on both side (2. 11) by $\frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}(\xi-\delta)^{n-\frac{\alpha}{k}-1}}{2(k\Gamma_k\left(n-\frac{\alpha}{k}\right))^2}$ integrating for the variables γ and δ over the interval $[a,\xi]$ we get

$$\begin{aligned} &\int_a^\xi\int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}}(\xi-\delta)^{n-\frac{\alpha}{k}}}{2(k\Gamma_k\left(n-\frac{\alpha}{k}\right))^2}T(\gamma,\delta)d\gamma d\delta \\ =\ &\frac{\xi^{\frac{\nu(n-\mu)}{k}}}{\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}({}^C D_{a+}^{\alpha,k})(\Omega(\xi)\Upsilon(\xi)) - ({}^C D_{a+}^{\alpha,k}\Omega)(\xi)({}^C D_{a+}^{\alpha,k}\Upsilon)(\xi). \end{aligned} \tag{2. 12}$$

Applying Cauchy Schwarz Inequality we get

$$\begin{aligned}
&\left(\int_a^\xi\int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}(\xi-\delta)^{n-\frac{\alpha}{k}-1}}{2(k\Gamma_k\left(n-\frac{\alpha}{k}\right))^2}\right.\\
&\left.(\Omega^{(n)}(\gamma)-\Omega^{(n)}(\delta))(\Upsilon^{(n)}\gamma)-\Upsilon^{(n)}(\delta)d\gamma d\delta\right)^2\\
\leq\ & \int_a^\xi\int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}}(\xi-\delta)^{n-\frac{\alpha}{k}-1}}{2(k\Gamma_k\left(n-\frac{\alpha}{k}\right))^2}(\Omega^{(n)}(\gamma)-\Omega^{(n)}(\delta))^2 d\gamma d\delta\\
\times\ & \int_a^\xi\int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}}(\xi-\delta)^{n-\frac{\alpha}{k}}}{2(k\Gamma_k\left(n-\frac{\alpha}{k}\right))^2}(\Upsilon^{(n)}(\gamma)-\Upsilon^{(n)}(\delta))^2 d\gamma d\delta
\end{aligned} \tag{2. 13}$$

From (2. 12) and (2. 13) we get

$$\begin{aligned}
&(R(\xi)(({}^CD^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon(\xi))-({}^CD^{\alpha,k}_{a+}\Omega)(\xi)({}^CD^{\alpha,k}_{a+}\Upsilon)(\xi))^2\\
\leq\ & (R(\xi)({}^CD^{\alpha,k}_{a+}\Omega^2)(\xi)-(({}^CD^{\alpha,k}_{a+}\Omega)(\xi))^2)\\
\times\ & (R(\xi)({}^CD^{\alpha,k}_{a+}\Upsilon^2)(\xi)-(({}^CD^{\alpha,k}_{a+}\Upsilon)(\xi))^2)
\end{aligned}$$

Since

$$(\Psi_2{}^{(n)}(\xi)-\Omega^{(n)}(\xi))(\Omega^{(n)}(\xi)-\Psi_1{}^{(n)}(\xi)))\geq 0,$$

and

$$(\varphi_2^{(n)}(\xi)-\Upsilon^{(n)}(\xi))(\Upsilon^{(n)}(\xi)-\varphi_1^{(n)}(\xi))\geq 0,$$

we have

$$R(\xi)(\Psi_2{}^{(n)}(\xi)-\Omega^{(n)}(\xi))(\Omega^{(n)}(\xi)-\Psi_1{}^{(n)}(\xi)))\geq 0,$$

and

$$R(\xi)(\varphi_2^{(n)}(\xi)-\Upsilon^{(n)}(\xi))(\Upsilon^{(n)}(\xi)-\varphi_1^{(n)}(\xi))\geq 0.$$

Thus from the Lemma 2.6 we have

$$\begin{aligned}
&R(\xi)(({}^CD^{\alpha,k}_{a+}\Omega^2)(\xi))-(({}^CD^{\alpha,k}_{a+}\Omega)(\xi))^2\\
\leq\ & (({}^CD^{\alpha,k}_{a+}\Psi_2)(\xi)-({}^CD^{\alpha,k}_{a+}\Omega)(\xi))(({}^CD^{\alpha,k}_{a+}\Omega)(\xi)-({}^CD^{\alpha,k}_{a+}\Psi_1)(\xi))\\
+\ & R(\xi)({}^CD^{\alpha,k}_{a+})(\Psi_1(\xi)\Omega(\xi))-({}^CD^{\alpha,k}_{a+}\Psi_1)(\xi)({}^CD^{\alpha,k}_{a+}\Omega)(\xi)\\
+\ & R(\xi)({}^CD^{\alpha,k}_{a+})(\Psi_2(\xi)\Omega(\xi))-({}^CD^{\alpha,k}_{a+}\Psi_2)(\xi)({}^CD^{\alpha,k}_{a+}\Omega)(\xi)\\
+\ & ({}^CD^{\alpha,k}_{a+}\Psi_1)(\xi)({}^CD^{\alpha,k}_{a+}\Psi_2)(\xi)-R(\xi)({}^CD^{\alpha,k}_{a+})(\Psi_1(\xi)\Psi_2(\xi))\\
=\ & T(\Omega,\Psi_1,\Psi_2)
\end{aligned} \tag{2. 14}$$

and

$$
\begin{aligned}
& R(\xi)(({}^{C}D^{\alpha,k}_{a+}\Upsilon^{2})(\xi))-(({}^{C}D^{\alpha,k}_{a+}\Upsilon)(\xi))^{2}\\
\leq\ & (({}^{C}D^{\alpha,k}_{a+}\varphi_{2})(\xi)-({}^{C}D^{\alpha,k}_{a+}\Upsilon)(\xi))(({}^{C}D^{\alpha,k}_{a+}\Upsilon)(\xi)-({}^{C}D^{\alpha,k}_{a+}\varphi_{1})(\xi))\\
+\ & R(\xi)({}^{C}D^{\alpha,k}_{a+})(\varphi_{1}(\xi)\Upsilon(\xi))-({}^{C}D^{\alpha,k}_{a+}\varphi_{1})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon)(\xi)\\
+\ & R(\xi)({}^{C}D^{\alpha,k}_{a+})(\varphi_{2}(\xi)\Upsilon(\xi))-({}^{C}D^{\alpha,k}_{a+}\varphi_{2})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon)(\xi)\\
+\ & ({}^{C}D^{\alpha,k}_{a+}\varphi_{1})(\xi)({}^{C}D^{\alpha,k}_{a+}\varphi_{2})(\xi)-R(\xi)({}^{C}D^{\alpha,k}_{a+})(\varphi_{1}(\xi)\varphi_{2}(\xi))\\
=\ & T(\Upsilon,\varphi_{1},\varphi_{2}) \qquad (2.\ 15)
\end{aligned}
$$

Therefore, the inequality (2. 10) follows from (2. 14) and (2. 15). This complete the proof. □

Corollary 2.9. *Let $m, M, n, N \in \Re$, $T(\Omega,\Psi_1,\Psi_2) = T(\Omega, m, M)$ and $T(\Upsilon,\varphi_1,\varphi_2) = T(\Upsilon, n, N)$. Then the inequality (2. 10) reduces to*

$$
\left|R(\xi)({}^{C}D^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon(\xi))-({}^{C}D^{\alpha,k}_{a+}\Omega)(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon)(\xi)\right|\leq (R(\xi))^{2}\,(M-m)(N-n).
$$

Theorem 2.10. *Let $k>0$ and $\Omega^{(n)}$ and $\Upsilon^{(n)}$ be two positive functions defined on $[0,\infty)$. Then the following inequalities holds:*

1. $q\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)+p\ ({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)\geq pq\dfrac{1}{R(\xi)}({}^{k}D^{\mu,\nu}_{a+}\Upsilon)(\rho)({}^{C}D^{\alpha,k}_{a+}\Omega)(\xi),$

2. $q\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{p})(\xi)+p\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{q})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)$
   $\geq pq(({}^{C}D^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon(\xi)))^{2},$

3. $q\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)+p\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{q})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{p})(\xi)$
   $\geq pq(({}^{C}D^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon^{p-1}(\xi))(({}^{C}D^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon^{q-1}(\xi)),$

4. $q\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)+p\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)$
   $\geq pq({}^{C}D^{\alpha,k}_{a+})(\Omega^{p-1}(\xi)\Upsilon^{q-1}(\xi))({}^{C}D^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon(\xi)),$

5. $q\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{2})(\xi)+p\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{2})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)$
   $\geq pq({}^{C}D^{\alpha,k}_{a+})(\Omega(\xi)\Upsilon(\xi))({}^{k}D^{\mu,\nu}_{a+})(\Omega^{\frac{2}{q}}(\xi)\Upsilon^{\frac{2}{p}}(\xi)),$

6. $q\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{2})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{q})(\xi)+p\ ({}^{C}D^{\alpha,k}_{a+}\Omega^{p})(\xi)({}^{C}D^{\alpha,k}_{a+}\Upsilon^{2})(\xi)$
   $\geq pq({}^{C}D^{\alpha,k}_{a+})(\Omega^{\frac{2}{p}}(\xi)\Upsilon^{\frac{2}{q}}(\xi))({}^{C}D^{\alpha,k}_{a+})(\Omega^{p-1}(\xi)\Upsilon^{q-1}(\xi)),$

7. $q\ ({}^{C}D^{\alpha,k}_{a+})(\Omega^{2}(\xi)\Upsilon^{q}(\xi))+p\ ({}^{C}D^{\alpha,k}_{a+})(\Omega^{2}(\xi)\Upsilon^{p})(\xi)$
   $\geq pq\dfrac{1}{R(\xi)}({}^{C}D^{\alpha,k}_{a+})(\Omega^{\frac{2}{p}}(\xi)\Upsilon^{q-1}(\xi))({}^{C}D^{\alpha,k}_{a+})(\Omega^{\frac{2}{q}}(\xi)\Upsilon^{p-1}(\xi)),$

$R(\xi)$ is defined by (2. 8).

Proof. By Young's Inequality we have

$$\frac{a^p}{p}+\frac{a^q}{q}\geq ab,(a,b\geq 0,p,q>1,\frac{1}{p}+\frac{1}{q}=1),$$

put $a=\Omega^{(n)}(\gamma)$, and $b=\Upsilon^{(n)}(\delta)$, we have

$$\frac{(\Omega^{(n)}(\gamma))^p}{p}+\frac{(\Upsilon^{(n)}(\delta))^q}{q}\geq \Omega^{(n)}(\gamma)\Upsilon^{(n)}(\delta),$$

for all $\Omega^{(n)}(\gamma),\Upsilon^{(n)}(\delta)\geq 0$.
Multiplying by $\frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ and integrating for the variable γ over the interval $[a,\xi]$ we get

$$\int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\frac{(\Omega^{(n)}(\gamma))^p}{p}d\gamma+\int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\frac{(\Upsilon^{(n)}(\delta))^q}{q}d\gamma$$
$$\geq \int_a^\xi \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\Omega^{(n)}(\gamma)\Upsilon^{(n)}(\delta)d\gamma,$$

and it becomes

$$\frac{1}{p}{}^CD^{\alpha,k}_{a+}\Omega^p(\xi)+\frac{1}{q}\Upsilon^q(\delta)R(\xi)\geq \Upsilon(\delta){}^CD^{\alpha,k}_{a+}\Omega(\xi).$$

Again Multiplying by $\frac{(\xi-\delta)^{n-\frac{\alpha}{k}}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ and integrating for the variable δ over the interval $[a,\xi]$ we get

$$\frac{1}{p}({}^CD^{\alpha,k}_{a+}\Omega)^p(\xi)R(\xi)+\frac{1}{q}{}^CD^{\alpha,k}_{a+}\Upsilon^q(\xi)R(\xi)\geq {}^kD^{\mu,\nu}_{a+}\Upsilon(\rho){}^CD^{\alpha,k}_{a+}\Omega(\xi),$$

which implies that

$$\frac{1}{p}({}^CD^{\alpha,k}_{a+}\Omega)^p(\xi)+\frac{1}{q}({}^CD^{\alpha,k}_{a+}S)^q(\delta)\geq \frac{1}{R(\xi)}{}^kD^{\mu,\nu}_{a+}\Upsilon(\rho){}^CD^{\alpha,k}_{a+}\Omega(\xi).$$

This complete the proof of part (1).
The reaming inequalities can be proved using Young's inequality in the similar manner by

taking:

2. $a = \Omega^{(n)}(\gamma)\Upsilon^{(n)}(\delta),$ and $b = \Omega^{(n)}(\delta)\Upsilon^{(n)}(\gamma).$

3. $a = \dfrac{\Omega^{(n)}(\gamma)}{\Upsilon^{(n)}(\gamma)},$ and $b = \dfrac{\Omega^{(n)}(\delta)}{\Upsilon^{(n)}(\delta)},$
$\Upsilon^{(n)}(\gamma)\Upsilon^{(n)}(\delta) \neq 0.$

4. $a = \dfrac{\Omega^{(n)}(\delta)}{\Omega^{(n)}(\gamma)},$ and $b = \dfrac{\Upsilon^{(n)}(\delta)}{\Upsilon^{(n)}(\gamma)},$
$\Omega^{(n)}(\gamma)\Upsilon^{(n)}(\delta) \neq 0.$

5. $a = \Omega^{(n)}(\gamma){\Upsilon^{(n)}}^{\frac{2}{p}}(\delta),$
and $b = {\Omega^{(n)}}^{\frac{2}{q}}(\delta)\Upsilon^{(n)}(\gamma).$

6. $a = \dfrac{{\Omega^{(n)}}^{\frac{2}{p}}(\gamma)}{(\Omega^{(n)}(\delta))},$ and $b = \dfrac{({\Upsilon^{(n)}}^{\frac{2}{q}})(\gamma)}{\Upsilon^{(n)}(\gamma)},$
$\Omega^{(n)}(\delta)\Upsilon^{(n)}(\delta) \neq 0.$

7. $a = \dfrac{({\Omega^{(n)}}^{\frac{2}{p}})(\gamma)}{\Upsilon^{(n)}(\delta)},$ and $b = \dfrac{({\Omega^{(n)}}^{\frac{2}{q}})(\delta)}{\Upsilon^{(n)}(\gamma)},$
$\Upsilon^{(n)}(\gamma)\Upsilon^{(n)}(\delta) \neq 0.$

□

Example 2.11. *Let* $k > 0$ *and* $\varphi_2^{(n)}(\gamma)$ *be a positive function defined on* $[0, \infty)$ *and let* $m = \min\limits_{0 \leq \gamma \leq \xi} \frac{\Omega^{(n)}(\gamma)}{\Upsilon^{(n)}(\gamma)}$ *and* $M = \max\limits_{0 \leq \gamma \leq \xi} \frac{\Omega^{(n)}(\gamma)}{\Upsilon^{(n)}(\gamma)}$*. Then one can have*

$$0 \leq ({}^{C}D_{a+}^{\alpha,k}\Omega^2)(\xi)({}^{C}D_{a+}^{\alpha,k}\Upsilon^2)(\xi) \leq \frac{(m+M)^2}{4mM}({}^{C}D_{a+}^{\alpha,k})(\Omega\Upsilon(\xi)) \tag{2. 16}$$

Proof. It follows from (2. 16)

$$\left(\frac{(\Omega^{(n)})(\gamma)}{(\Upsilon^{(n)})(\gamma)} - m\right)\left(M - \frac{(\Omega^{(n)})(\gamma)}{(\Upsilon^{(n)})(\gamma)}\right)({\Upsilon^{(n)}}^2)(\gamma) \geq 0,$$

and

$${\Omega^{(n)}}^2(\gamma) + mM{\Upsilon^{(n)}}^2(\gamma) \leq (m+M)\Omega^{(n)}(\gamma)\Upsilon^{(n)}(\gamma).$$

Multiplying by $\frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}$ and integrating for the variable γ over the interval $[a,\xi]$ we get

$$\int_a^{\xi} \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \Omega^{(n)^2})(\xi)d\gamma + mM\int_a^{\xi} \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \Upsilon^{(n)^2}(\xi)d\gamma$$

$$\leq (m+M)\int_a^{\xi} \frac{(\xi-\gamma)^{n-\frac{\alpha}{k}-1}}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \Omega^{(n)}(\gamma)\Upsilon^{(n)}(\gamma)d\gamma$$

This implies that

$$({}^C D_{a+}^{\alpha,k}\Omega^2)(\xi) + mM({}^C D_{a+}^{\alpha,k}\Upsilon^2)(\xi) \leq (m+M)({}^C D_{a+}^{\alpha,k}\Omega)(\gamma)\Upsilon^{(n)}(\gamma). \tag{2. 17}$$

Alternatively it follows from

$$\left(\sqrt{{}^C D_{a+}^{\alpha,k}\Omega^2(\xi)} - \sqrt{mM{}^C D_{a+}^{\alpha,k}\Upsilon^2(\xi)}\right)^2 \geq 0$$

that

$$2\sqrt{({}^C D_{a+}^{\alpha,k}\Omega^2)(\xi)}\sqrt{mM({}^C D_{a+}^{\alpha,k}\Upsilon^2)(\xi)} \leq (m+M)({}^C D_{a+}^{\alpha,k}\Omega)(\gamma)\Upsilon(\gamma). \tag{2. 18}$$

Therefore,

$$4mM({}^C D_{a+}^{\alpha,k}\Omega^2)(\xi)({}^C D_{a+}^{\alpha,k}\Upsilon^2)(\xi) \leq (m+M)^2(({}^C D_{a+}^{\alpha,k}\Omega)(\gamma)\Upsilon(\gamma))^2$$

follows from (2. 17) and (2. 18), and proof is complete. □

REFERENCES

[1] T. Akman, B. Yildiz, D. Baleanu, *New discretization of CaputoFabrizio derivative,* Comput. Appl. Math. **37,** No.3 (2018) 3307–3333.

[2] G. A. Anastassiou, *Fractional Differentiation inequalities,* Springer science+Businness Media, LLC, Dordrecht the Netherlands, (2009).

[3] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods,* World Scientific, Hackensack (2012).

[4] R. Diaz, E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol,* Divulg. Mat., **15**, (2007), 179-192.

[5] G. Farid, A. Javed, A. U. Rehman, *On Hadamard inequalities for n-times differentiable functions which are relative convex via Caputo k-fractional derivatives,* Nonlinear Anal. Forum., (to appear).

[6] G. Grüss, *Uber das Maximum des absoluten Betrages* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx,$ Math. Z. **39,** No.1(1935) 215–226.

[7] F. Jarad, T. Abdeljawad, D. Baleanu, *On the generalized fractional derivatives and their Caputo modification,* J. Nonlinear Sci. Appl. **10**, No.5(2017) 2607–2619

[8] U.N.Katugampola, *New approach to generalized fractional integral,* Appl. Math. Comput. 218, (2011) 860–865.

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, 204 , Elsevier. New York-London, (2006).

[10] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London (2010).

[11] D.S. Oliveira, E. Capelas de Oliveira, *HilferKatugampola fractional derivative* (2017, submitted). arXiv:1705.07733

[12] T. Z. Tomovski, R. Hilfer, and H. M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions,* Integral Transforms Spec. Funct. 21 (2010).