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Eventual Periodicity of Linearized BBM Equation using RBFs Meshless Method

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Abstract.: Among the class of dispersive wave equations the BBM equation (Benjamin-Bona-Mahony) is a notable model for long surface gravity waves inside shallow water with small amplitude propagating unidirectionally and has been broadly utilized in research facility and in field investigation of water waves. One more specific subjective part of arrangement for a number of wave equations demonstrated by investigations, which connect along with their large-time behavior named as eventual time periodicity uncovered across solutions to IBVPs (initial-boundary-value-problems). In the present study eventual periodicity of solutions for the linearized BBM equation (IBVP) on a half-line coupled with periodic boundary condition will be explored numerically utilizing meshless technique dependent on RBFs.

AMS (MOS) Subject Classification Codes: 65D12; 65J08; 65L06; 65M20
Key Words: RBFs Meshless Methods, Eventual Periodicity, Linear BBM Equation.

1. Introduction

Water wave propagation phenomena still attract the interest of researchers from many areas and with various objective. As an example, the Benjamin-Bona-Mahony equation denoted by BBM (Benjamin T. B. Bona J. L. and Mahony J. J. (1972)), more familiar like regularized long-wave equation (RLWE) as an illustrative representation of propagation on one directional space of long, low amplitude waves [4, 3]. Prior to 1966 Peregrine introduced this equation in his analysis of bore propagation [30]. In [43] is given a generalized n-dimensional version. In physical applications the BBM equation is renowned, is utilized in investigation of long-wavelength surface waves in liquid, harmonic-crystal acoustic waves, compressible fluid waves with acoustic-gravity, and hydro-magnetic waves cold plasma [22, 23]. In addition it is valid to the study of plasma rossby waves or drift waves in moving fluids [24, 6]. This equation is substitution model to the KdV (Korteweg and de Vries (1895)) equation [27, 13, 25] because of the supposition of large wave length and small wave-amplitude [28, 44, 35, 20, 7, 2, 31]. One more specific qualitative aspects revealed by solutions to (IBVPs) of some wave equations demonstrated through investigations, related through their large-time action called as eventual time periodicity. This attractive and desirable event is demonstrated by a flap-type wave maker installed at one ending of a channel in research tests. As the wave generator oscillates regularly with the period T, it seem that wave amplitude turn out to be periodic after a certain amount of time at each point down the channel [8, 9]. This important and interesting eventual periodic phenomenon was previously discussed in different works, like generalized equations viz KdV, BBM and also its dissipating equivalents, like Burger-type equations [10, 32, 41, 40, 42, 1]. In the present study we numerically investigate whether the corresponding solution u of the following model problem for linearized BBM equation along with specified initial and boundary condition either on half line or on a finite interval is eventually periodic by using meshless method based on RBFs. The boundary-value problem is found to be ill posed for specific cases of robin boundary conditions.

$$\begin{cases} v_t - \gamma v_{xxt} + \mu v_x = f(x,t),\ x \geq 0\ , t \in (0,T] \\ v(x,0) = v_0(x),\ x \geq 0,\ v(0,t) = g(t),\ t \geq 0. \end{cases} \tag{1. 1}$$

Where the source term f and the boundary data g presumed to be periodic of period $\mathrm{T}_0 > 0$.

Meshless methods are emerging and interesting numerical techniques which can solve with no meshing or with a minimum of meshing those engineering and physical problems for which the commonly named mesh-based methods acting on computational nodes applied commercially worldwide for instance finite differences, finite elements and finite volumes are not suited. Element-free galerkin (EFG) method, point interpolation method (PIM), moving least squares (MLS) approximation, boundary element-free method (BEFM) and reproducing kernel particle (RKP) method etc., are some meshless methods. Among the family of meshless methods one of the most prominent meshless method which appears to be really consistent and well ordered system while looking at the interpolation of multidimensional scattered data is RBF method, which recently received considerable and tremendous attention in scientific community due to high flexibility, efficiency and its capacity to gain spectral accuracy to solve complex PDEs, fractional equations and integral equations opposed to other advanced methods [5, 12, 15]. The multi-quadric is well known to be one of the most used kernels in meshless schemes. Hardy had suggested multi-quadric kernels [21] to solve collocation scheme for PDEs employing radial basis function.

2. DESCRIPTION OF RBF PSEUDO-SPECTRAL METHOD

Fasshauer [14] connected the RBF collocation approach to the Pseudo-spectral (PS) scheme which is called as RBF-PS scheme and implemented this scheme for the approximation of 2D Helmholtz and Laplace models also for the Allen-Cahn model together with the piecewise boundary conditions [15]. The authors [17, 18, 26, 38, 39, 29] exploited, utilized and applied this approach to analyze and solve different model PDEs. Here we develop this approach for solution of (1. 1). Assume ψ_j, where j varies from 1 to N, is a linearly independent arbitrary smooth function set that will be used as the basis for our space investigation and $\Xi = \{x_1, x_2, ..., x_N\}$ be a distinct points set in $\mathbb{R}^s$, $s \geq 1$. The approximated solution is given as

$$u^h(x,t) = \sum_{j=1}^{N} \lambda_j(t)\psi_j(x), x \in \Xi, \tag{2. 2}$$

where $h = h_{x,\Xi} = sup_{x\in\Xi} min_{1\leq j\leq N}\|x - x_j\|_2$. Often used well-known radial basis function (RBF) are listed in the following table.

RBF Name	$\psi(r), (r \geq 0), r = \|x - x_j\|_2$
Radial function RBF-(LI)	r
Gaussian function RBF-(GA)	$e^{-(\varepsilon r)^2}$
Laguerre-Gaussian	$(2 - r^2)e^{-r^2}$
Monomial RBF-(MN)	r^{2k-1}
Gneiting	$(1 - r)_+^5(1 + 5r - 27r^2)$
Wendland's	$(1 - \varepsilon r)_+^6(35(\varepsilon r)^2 + 18\varepsilon r + 3)$
Inverse Quadratic function RBF-(IQ)	$\frac{1}{1+(\varepsilon r)^2}$

Where the constant ε is famous as the shape parameter of the RBF which is used to control the shape of functions and is found experimentally to any RBF. Now collocating equation (2. 2) at the grid points x_i, it appears that,

$$u^h(x_i,t) = \sum_{j=1}^{N} \lambda_j(t)\psi(x_i, x_j), 1 \leq i \leq N. \tag{2. 3}$$

The above system in a matrix structure is indicated as

$$\mathbf{u} = \mathbf{B}\lambda, \tag{2. 4}$$

where the interpolation matrix $\mathbf{B}$ launches entries form $\psi(x_i, x_j)$, $1 \leq i, j \leq N$, and the appearance of expansion coefficients vector is stated as $\lambda = [\lambda_1, \lambda_2, ..., \lambda_N]^T$. Derivative of u i.e, u_x utilizing equation (2. 4) can be derived by differentiating the RBF function and again valuating at each point $x_i, 1 \leq i \leq N$, we got in matrix-vector notation

$$\mathbf{u}_x = \mathbf{B}_x\lambda, \tag{2. 5}$$

where matrix $\mathbf{B}_x$ entries are $\frac{d}{dx}\psi(x, x_j)_{x=x_i}, 1 \leq i, j \leq N$. Solving equations (2. 4)-(2. 5) for the unknown values of λ we obtain the differentiation matrix in the form below

$$\mathbf{u}_x = \mathbf{B}_x\mathbf{B^{-1}}\mathbf{u} = \mathbf{D}_x\mathbf{u}. \tag{2. 6}$$

Where $\mathbf{D}_x = \mathbf{B}_x\mathbf{B}^{-\mathbf{1}}$ known as the differentiation matrix. Also it is worth noting that this matrix relies on matrix $\mathbf{B}$ invertibility and notice that for distinct collocation points set the matrix $\mathbf{B}$ is always invertible. We can write the same way

$$\mathbf{u}_{xx} = \mathbf{B}_{xx}\mathbf{B}^{-\mathbf{1}}\mathbf{u} = \mathbf{D}_{xx}\mathbf{u}, \tag{2. 7}$$

where $\mathbf{D}_{xx} = \mathbf{B}_{xx}\mathbf{B}^{-\mathbf{1}}$ having entries of shape $\mathbf{B}_{xx}$ are $\frac{d^2}{dx^2}\psi(x, x_j)_{x=x_i}$, $1 \leq i, j \leq N$. The higher-order differentiation matrices may be obtained in the same manner. The numerical scheme corresponding to the equation (1. 1) with the above differentiation matrices is designated as below

$$\mathbf{v}' - \gamma\mathbf{D}_{xx}\mathbf{v}' + \mu\mathbf{D}_x\mathbf{v} = f(x, t). \tag{2. 8}$$

This equation can be drawn up as

$$(\mathbf{I} - \gamma\mathbf{D}_{xx})\mathbf{v}' = f(x, t) - \mu\mathbf{D}_x\mathbf{v}, \tag{2. 9}$$

where matrix $\mathbf{A} = (\mathbf{I} - \gamma\mathbf{D}_{xx})$ is time free. The pseudo-inverse $\mathbf{A}^\dagger$ of $\mathbf{A}$ can be computed, thus we have

$$\mathbf{v}' = \mathbf{D}\mathbf{v}, \tag{2. 10}$$

where $\mathbf{D} = \mathbf{A}^\dagger(f(x, t) - \mu\mathbf{D}_x)$. Equation (2. 10) is in the form of

$$\mathbf{v}^{'} = \mathbf{F}(\mathbf{v}). \tag{2. 11}$$

Now some solver's ODE such as ode45, ode113, ode23 can be used to discretize this ODE system in time. v_0 is initial vector solution. Any successful ODE solver will pick a suitable time phase δt to resolve ODE system stiffness.

3. Stability Analysis

Since our numerical plan RBF-PS technique converted time-dependent PDEs to an ODEs framework in time. This sort of strategy is known as method of line (MOL) strategy whereby the scheme of coupling ODEs can be solved employing the finite time difference approach e.g, Runge-Kutta methods etc. The stability of line method can be calculated by the famous thumb rule as shown in the work [37]. Line method should be reliable and stable when the spatial discretization operator eigenvalues, scaled and linearized by step size δt, exist in a region of stability for the relevant time-discretization operator. The area of stability is part of a multifaceted plane (complex plane) including of those eigenvalues for which a bounded solution is built by the schemes. Equation's (2. 11) stability relies on the coefficient matrix's eigenvalues [36]. Therefore, to demonstrate the stability of numerical solution of (1. 1) it is necessary to demonstrate that the real term, $Re(\lambda_i)$ of every eigenvalue $Re(\lambda_i)$ of the matrix $\mathbf{F}$ is non-positive i.e., $Re(\lambda_i) \leq 0$ for all $i = 1, 2, ...n$, for more details, see [36]. After measuring the eigenvalues for $\mathbf{D}$ matrices scaled by δt, we analyzed the stable eigenvalue spectrum for the linearized BBM model in Figure.2 and Figure.4.

4. NUMERICAL RESULTS

4.1. **Usage and application of the numerical scheme suggested.** We seek the RBF-PS numerical scheme first for linear and nonlinear BBM equation (Benjamin-Bona-Mahony) with known exact solution [33, 19].

$$\begin{cases} v_t(x,t) - 2v_{xxt}(x,t) + v_x(x,t) = 0, \\ v(x,0) = \exp(-x),\ x \in [0,1] \\ v(0,t) = \exp(-t), \\ v(1,t) = \exp(-1-t), \forall t \geq 0, \\ v(x,t) = \exp(-x-t),\ x \in [0,1], t \geq 0. \end{cases} \tag{4. 12}$$

$$\begin{cases} v_t(x,t) - v_{xxt}(x,t) + vv_x(x,t) = 0, \\ v(x,0) = x,\ x \in [0,1] \\ v(0,t) = 0,\ v(1,t) = \frac{1}{1+t}, \forall t \geq 0, \\ v(x,t) = \frac{x}{1+t},\ x \in [0,1], t \geq 0. \end{cases} \tag{4. 13}$$

The accuracy, efficiency and the success of this scheme for problem (4. 12) and (4. 13) shall be tested in terms of L_∞ error norm given below and is presented in Table.1, Figure.1 and Table.2, Figure.3 respectively for $\delta t = 0.0001$ and $N = 32$, and also for higher values of N.

$$L_\infty = \|u^{ex} - u^{ap}\|_\infty = \max|u^{ex} - u^{ap}|. \tag{4. 14}$$

N	L_∞[34]	L_∞[RBF-PS]
8	5.7806E-03	8.2157E-04
16	5.4363E-03	8.2142E-04
32	3.4051E-03	8.0653E-04
64	1.8811E-03	7.5993E-04
128	9.8606E-03	3.7976E-04
256	5.0452E-04	3.6382E-04

TABLE 1. Comparison table for problem-(4. 12).

4.2. **Linearized BBM equation eventual periodicity.** Now we present the results of our method investigating the eventual periodicity generated with periodic external forcing and also with periodic forcing at the boundary of the BBM equation. We are testing underneath scaled problem solution considered by Shen et al., in [32].

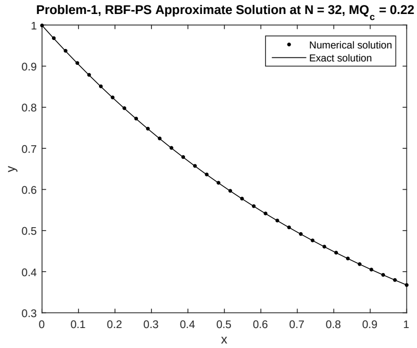

FIGURE 1. Exact and approximate solution for problem-(4. 12).

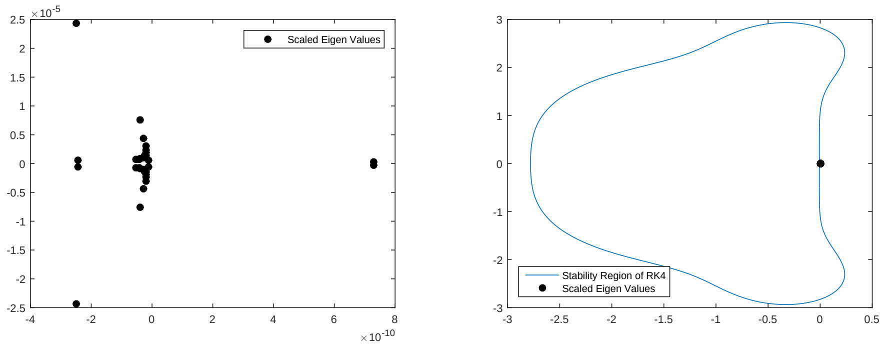

FIGURE 2. Scaled Eigenvalues and Stability region for problem-(4. 12).

N	L_∞[34]	L_∞[RBF-PS]
8	5.7451E-02	1.8980E-02
16	2.9979E-02	9.7860E-04
32	1.5306E-02	9.6062E-05
64	7.7325E-03	3.1285E-05
128	3.8862E-03	2.0602E-05
256	1.9481E-03	1.7595E-05

TABLE 2. Comparison table for problem-(4. 13).

$$\begin{cases} v_t + \alpha v_x + \beta v v_x - \delta v_{xxt} = h_1(t), & x \in [-1,1], \quad t \in [0,T], \\ v(0,t) = 0, & \\ v(1,t) = v_x(1,t) = 0, & t \in [0,T], \\ v(x,0) = v_0(x), & x \in [-1,1], \end{cases} \tag{4. 15}$$

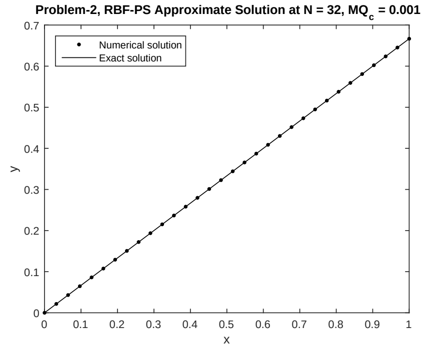

FIGURE 3. Exact and approximate solution for problem-(4. 13).

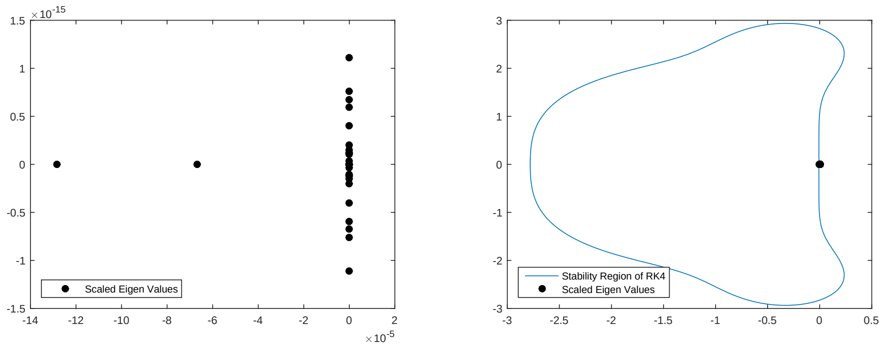

FIGURE 4. Scaled Eigenvalues and Stability region for problem-(4. 13).

and

$$\begin{cases} v_t + \alpha v_x + \beta v v_x - \delta v_{xxt} = 0, & x \in [-1,1], \quad t \in [0,T], \\ v(0,t) = h_2(t), & \\ v(1,t) = v_x(1,t) = 0, & t \in [0,T], \\ v(x,0) = v_0(x), & x \in [-1,1]. \end{cases} \tag{4. 16}$$

Where parameters are the α, β, δ and $h_1(t) = h_2(t) = \sin(20\pi t)\tanh(5t)$. For $v_0 \equiv 0$, equations (4. 15)-(4. 16) are all valid and true approaches of equation (1. 1) accordingly. Plotted here are the solutions having parameters $\alpha = 1.0$, $\delta = 10^{-6}$ and $\beta = 0$. The amplitudes $v(x,t)$ taken into six graphs from Figure.5 and Figure.6 on different points $x = -0.950670$, -0.808460, -0.587280, -0.308720, 0.0 and $x = 0.999650$ in the given domain, and $0 < t < 1.8$ is the time interval. The x and y-axis signify time t and amplitude v in these graphs respectively. Last plot at $x = 0.999650$ indicates the amplitude remaining zero in each problem. The intent of the last plot is to exhibit that at right boundary the wave front has not arrived for $t = 1.8$ and so the amplitudes validity at $x = -0.950670$, -0.808460, -0.587280, -0.308720 and 0.0.

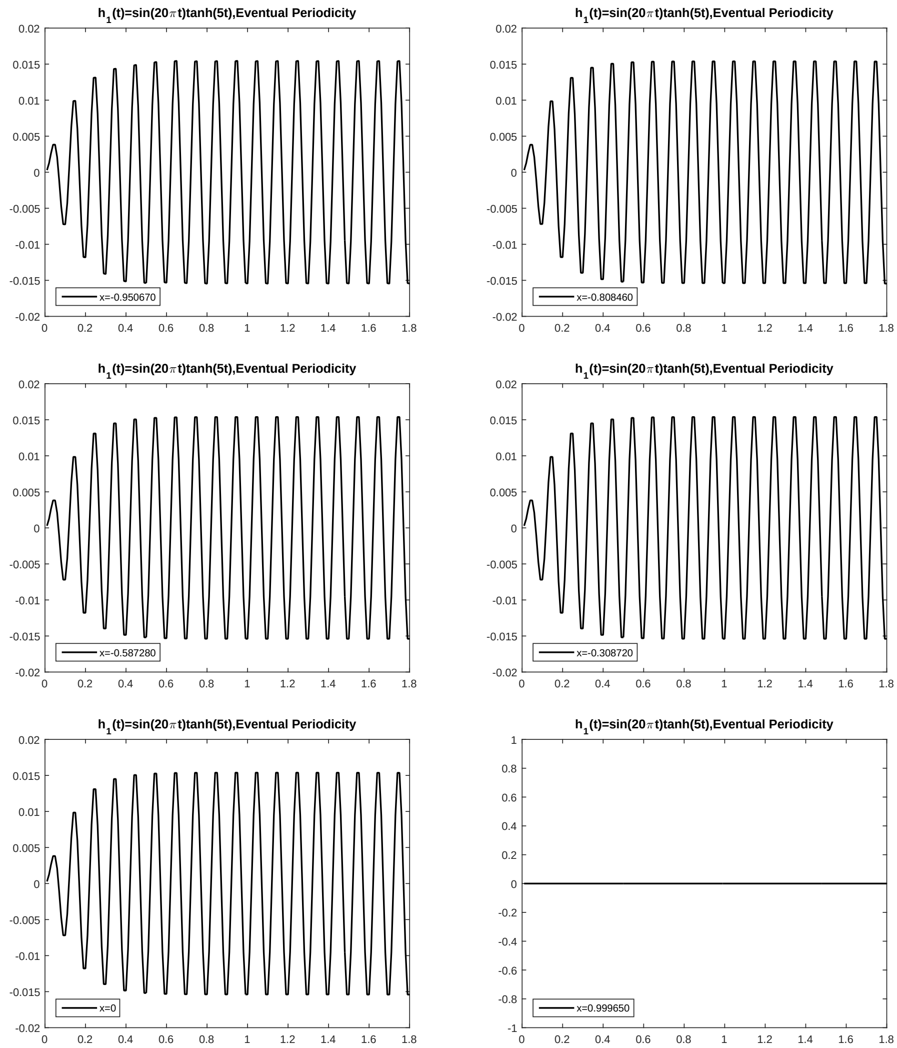

FIGURE 5. Linearized BBM equation eventual periodicity corresponding to (4. 15): at positions -0.950670 ,-0.808460 , -0.587280 , -0.308720 , 0 and 0.999650 seen respectively through six graphs above. Using parameters $\alpha = 1$, $\delta = 10^{-6}$, $\beta = 0$, $x \in [-1, 1]$, $C_{MQ} = 0.0208$, $N = 200$, $\delta t = 0.01$, $tmax = 1.8$, $h_1(t) = \sin(20\pi t) \tanh(5t)$.

5. CONCLUSION

The RBF-PS method in the present research described in detail and implemented for examine the eventual periodicity of IBVPs solutions like linear BBM (Benjamin-Bona-Mahony) equation on a bounded domain. For time integration we combine our method

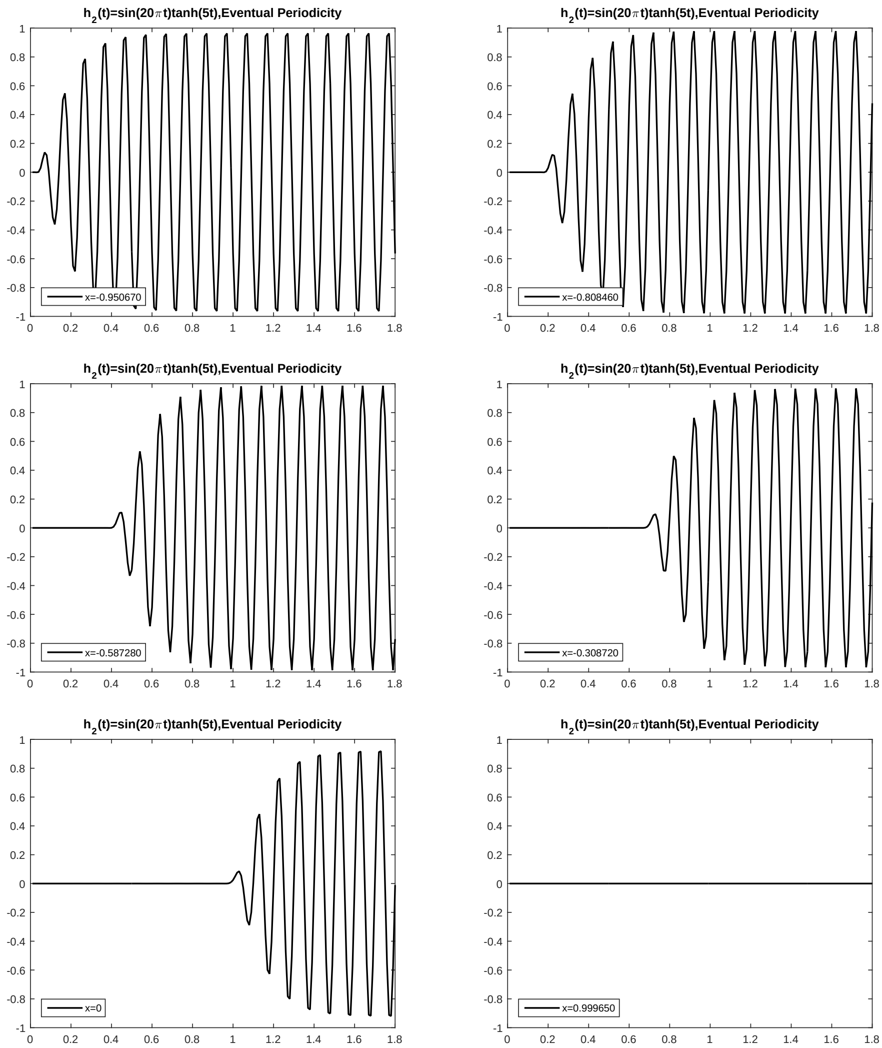

FIGURE 6. Linearized BBM equation eventual periodicity corresponding to (4. 16): at positions -0.950670 ,-0.808460 , -0.587280 , -0.308720 , 0 and 0.999650 seen respectively through six graphs above. Using parameters $\alpha = 1$, $\delta = 10^{-6}$, $\beta = 0$, $x \in [-1, 1]$, $C_{MQ} = 0.3000$, $N = 200$, $\delta t = 0.01$, $tmax = 1.8$, $h_2(t) = \sin(20\pi t)\tanh(5t)$.

with RK4 scheme. Our approache is simpler to construct the numerical scheme for high order nonlinear PDEs. Examples and results are exhibited to expose the efficiency, capacity and the high order accuracy of our proposed methods than other existing methods.

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