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Linear Algebraic Approach to Formulate A New Recurrence Relation for Bernoulli Numbers from the Power-Sum of Natural Numbers with Experiments on Pedagogy

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Abstract. In this article, a new recurrence relation formula for Bernoulli numbers have been derived, and sum of integer exponents of natural numbers has been revisited from this novel perspective. Some interesting pedagogical experiments on wording and presentation of mathematical derivation have been attempted, and development from first principle have been undertaken in line with this experimental approach.

AMS (MOS) Subject Classification Code: 11B68.
Key Words: Bernoulli Numbers, Power Sum, Global Series, Globalized Local Series.

1. Introduction

Sum of integer exponent of natural numbers is algebraically not very intricate, and sum for specific integer exponents (especially 1,2,3) is easily available on introductory algebra books. Calculating higher exponents as well as generalizing the formula [4, 12] have been already done. This paper aims to present an algebraic approach to this problem to the readers with minimal knowledge of number theory, combinatorics, calculus and linear algebra. Concisely, only Binomial theorem and Cramer's rule of solving system of linear equation [5] (including systems of linear equations, finding determinant) along with elementary or high school algebra is required to understand this article. An understanding of large summation and product operators as well as factorial symbol, permutation-combination symbols are necessary as well. Another significant class of readers expected here are those working on STEM education and pedagogy in general, and mathematics education in particular.

A mathematical sequence called Bernoulli number is intimately related to this sum of integer exponents of natural numbers. This sequence of numbers frequently encountered in number theory, has been extensively known for several centuries if not two millennia [11]. The series have been extensively computed and codified [17, 18] and also have been generalized [3]. Although not within scope of this article, the series of Bernoulli numbers can be written as Taylor series expansion of several transcendental functions (tangent, hyperbolic tangent for example).

2. Methodology and Evaluation of Approach

A great number of recurrences and explicit forms for the Bernoulli numbers are already known [16]. Some other definitions of Bernoulli numbers are:

1. Recursive definition:

$$B_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}, \text{ where } B_0 = 1 \tag{2. 1}$$

2. Explicit definition: In 1893, Louis Saalschutz listed [16] a total of 38 explicit formulas for the Bernoulli numbers. One of them is:

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n \tag{2. 2}$$

Some other explicit formulas developed are also reported in the literature [8].

3. Generating Function:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \tag{2. 3}$$

[7]

L. Euler also presented a generating function [19] for the Bernoulli numbers. Some other various mathematical functions generated from the Bernoulli Numbers are reported in the literature [10]. Many fundamental properties of Bernoulli numbers may be shown simply and directly using those generating functions [2]. Evaluating the zeta function [21], determining asymptotics of Stirling's formula [6], and approximating the harmonic series [20] are only a few of the applications of bernoulli numbers. Numerous applications of bernoulli numbers have been developed by Merca, M. [13, 14, 15] in the recent yeras.

In this article, we have proved a new recursive relation for Bernoulli numbers. This article revolves on two-interrelated topics, one of which is a result leading from calculating sum of (non-negative) integer exponent of natural numbers, and requires not much background knowledge other than those mentioned before. However, the second part contains extensive use of row transformation, and reader needs to know their implications before understanding what is meant by them. Still, the linear algebra rules are preliminary enough in second part.

In order to reduce pre-requisite content for the article but maintain an "air of intricacy", (which may be considered as an experimental pedagogical methodology for the time being and may be cast aside) a first principle development except for binomial theorem and Cramer's rule have been maintained. Instead of standard terminologies, some candid glossaries have been used, and extensive explanatory remarks have also been added, which are hoped by authors to be consistent with first-principle derivation.

The wording to determine preliminaries of the article which have been required for both mathematical discussion at hand (Power-sum of natural number and power-sum to recurrence relation) and terse mathematical derivations of second discussion have been planned to counterbalance each other and complement the two aspects of this paper. The approach of determinants in order to evaluate sum of power of natural numbers is not new [22]; and as it has been mentioned before, it is not claimed that this approach to calculate Bernoulli numbers (having a quadratic asymptotic time complexity) is very efficient. In fact, much more efficient formulae to compute Bernoulli numbers have been proved [9]. Some attention on the two parallel derivations of the paper can potentially indicate redundancy, and this is not entirely accidental. As the authors have tried to use a minimal number of glossaries and high level of prerequisite knowledge is attempted not to be required to understand the contents, citations are treated as auxiliary advanced topics of interest, and significant number of already proved theorems have been redeveloped. While row transformations have been used in Bernoulli-number derivation part, Cramer's rule have been extensively used in power-sum of natural numbers part. This is also an attempt to bring variety in mathematical approach, and choice of elegancy certainly have varied between the two authors in this respect. It is strongly emphasized that, the pedagogical linguistic content is much more prevalent in power-sum of natural numbers part.

Including own experimental approaches in a review paper is not entirely new, which is believed by the authors to be done in various fields of STEM research. Stripping completely a scientific discipline off standard glossaries, although obviously not the goal or demonstration of this article, is not entirely new either [1]. There can be cases where researcher is own object of study (perhaps in psychology, for example), but no such paper have been within notice of present authors. Sample size for such study would be too small to be scientifically meaningful and reliable. All of these aforementioned approaches in this paragraph to pedagogy have been implicitly, albeit partially and often sparsely, included in the paper.

The detailed, but often verbose explanation of steps, which are usually skipped in a typical mathematical article as the authors consider the skipped steps very obvious (and think the audiences would consider the skipped steps to be same) have been avoided here. This approach certainly adds to volume of article without adding any extract of substance, but have been followed by the present authors so that they are not confused by work of themselves while compiling the work and retracing these steps every time they append new perspectives afresh and redact old ones if necessary. Since language and philosophy of this paper related to experimental pedagogy (which is relevant to this article) is inextricably linked to the linguistic style utilized to re-derive already known mathematical formulae, these wordings cannot be discarded; discarding these will significantly hamper linguistics and philosophy of the article. Upon completion of the work, these concise but slightly verbose steps have been retained, only to let researchers on mathematics education to examine the psychological approach of the present authors under circumstances mentioned.

It is recognized that, such mode of self-psychoanalysis is not the standard of educational researchers or psychologists at present, but this paper can be said to be a feeble but preliminary attempt to incorporate this mode of self-analysis within mainstream research, which is believed by the authors to help subject-specific self-analysis of young prospective researchers and academicians alike. Similarly, the authors, instead of finding out proper nomenclature for a given technical term (which they considered diverging from subject-matter), proposed names according to what they considered fitting with context, often including some critical "abuse of notation" which mentally simplified authors' approach. It is to emphasize that, wording-choice of authors can be of some importance as well.

3. Power-Sum of Natural Numbers

It follows from binomial theorem that,

$$x^n - (x-1)^n = nx^{n-1} - \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} - \ldots - (-1)^r\binom{n}{r}x^{n-r} + \ldots - (-1)^n$$

$$(x-1)^n - (x-2)^n = n(x-1)^{n-1} - \binom{n}{2}(x-1)^{n-2} + \binom{n}{3}(x-1)^{n-3} - \ldots - (-1)^r\binom{n}{r}(x-1)^{n-r} + \ldots - (-1)^n$$

$$(x-2)^n - (x-3)^n = n(x-2)^{n-1} - \binom{n}{2}(x-2)^{n-2} + \binom{n}{3}(x-2)^{n-3} - \ldots - (-1)^r\binom{n}{r}(x-2)^{n-r} + \ldots - (-1)^n$$

$$\ldots\ldots\ldots$$

$$1^n - 0^n = n(1)^{n-1} - \binom{n}{2}(1)^{n-2} + \binom{n}{3}(1)^{n-3} - \ldots - (-1)^r\binom{n}{r}(1)^{n-r} + \ldots - (-1)^n$$

It is to be noted that, while binomial expansion of $(x-1)^n$ has $(n+1)$ terms, $x^n - (x-1)^n$ has n terms. For $(x-1)^n$, the $(r+1)^{th}$ term of the expansion is $(-1)^r\binom{n}{r}x^{n-r}$, while for the expression $x^n - (x-1)^n$, the first term of $(x-1)^n$ i.e., x^n cancels out, so $-(-1)^r\binom{n}{r}x^{n-r}$ becomes the r^{th} term. Adding the expressions:

$$x^n = n\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} - \binom{n}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} + \binom{n}{3}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-3} + \ldots - (-1)^r\binom{n}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r} + \ldots - (-1)^n\binom{n}{n}\sum_{\substack{x=1\\x\in N}}^{x=x} x^0$$

$$= \sum_{\substack{r=1\\r\in N}}^{r=n} -(-1)^r\binom{n}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r}$$

This equation can be written into matrix form as

$$\begin{bmatrix} \binom{n}{1} & -\binom{n}{2} & \ldots & -(-1)^r\binom{n}{r} & \ldots & -(-1)^n\binom{n}{n} \end{bmatrix} \begin{bmatrix} \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^0 \end{bmatrix} = \begin{bmatrix} x^n \end{bmatrix}$$

Now the similar equations of the form:

$$x^{n-1} = 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} + \binom{n-1}{1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} - \binom{n-1}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-3} + \ldots - (-1)^r\binom{n-1}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r-1} + \ldots - (-1)^{n-1}\binom{n-1}{n-1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{0}$$

$$x^{n-2} = 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} + 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} + \binom{n-2}{1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-3} - \binom{n-2}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-4} + \ldots - (-1)^r\binom{n-2}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r-2} + \ldots - (-1)^{n-2}\binom{n-2}{n-2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{0}$$

$$x^{n-3} = 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} + 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} + 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-3} + \binom{n-3}{1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-4} - \binom{n-3}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-5} + \ldots - (-1)^r\binom{n-3}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r-3} + \ldots - (-1)^{n-3}\binom{n-3}{n-3}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{0}$$

$$x^{n-k} = 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} + 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} + \cdots + \binom{n-k}{1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-1} - \binom{n-k}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-2} + \ldots - (-1)^r\binom{n-k}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r-k} + \ldots - (-1)^{n-k}\binom{n-k}{n-k}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{0}$$

To explain proceeding arguments concisely, local series is referred as, either writing a single or 'summed over all preceding natural numbers' expression of the form:

$$x^{n-k} - (x-1)^{n-k} = (n-k)x^{n-k-1} - \binom{n-k}{2}x^{n-k-2} + \ldots - (-1)^r\binom{n-k}{r}x^{n-k-r} + \ldots - (-1)^{n-k}$$

Or, upon adding all terms , that is

$$x^{n-k} = \binom{n-k}{1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-1} - \binom{n-k}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-2} + \ldots - (-1)^r\binom{n-k}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r-k} + \ldots - (-1)^{n-k}\binom{n-k}{n-k}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{0}$$

Putting $k = 0$, the series obtained at both cases are referred as global series. Now, in order to make all other local series start from $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1}$ instead of $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-1}$, the coefficients of k number of terms, namely, $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-i}$ $(1 \le i \le k)$ has to be equated with zero, and the expression should be recast into as:

$$x^{n-k} - (x-1)^{n-k} = 0x^{n-1} + 0x^{n-2} + \cdots + (n-k)x^{n-k-1} - \binom{n-k}{2}x^{n-k-2} + \ldots - (-1)^r\binom{n-k}{r}x^{n-k-r} + \ldots - (-1)^{n-k}$$

And

$$x^{n-k} = 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} + 0\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} + \cdots + \binom{n-k}{1}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-1} - \binom{n-k}{2}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-2}$$
$$+ \ldots - (-1)^r\binom{n-k}{r}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r-k} + \ldots - (-1)^{n-k}\binom{n-k}{n-k}\sum_{\substack{x=1\\x\in N}}^{x=x} x^0$$

This expression of local series is referred as globalized local series, and adding these zero terms are referred as "globalization". For $(x-1)^{n-k}$, the $(r+1)^{th}$ term of the expansion is $(-1)^r\binom{n-k}{r}x^{n-k-r}$, while for the expression $x^{n-k}-(x-1)^{n-k}$, the first term of $(x-1)^{n-k}$ cancels out, so $(-1)^r\binom{n-k}{r}x^{n-k-r}$ becomes the r^{th} term counted from beginning of series expansion of $x^{n-k}-(x-1)^{n-k}$. But if term count is started from coefficient of x^n, for sum of terms like $x^{n-1}-(x-1)^{n-1}$ from $x=1$ to $x=x$, (which turns out to be x^{n-1}), coefficient of $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1}$ is zero. First term for local series (i.e. $x^{n-1}-(x-1)^{n-1}$) matches the second exponent of second term for global series (i.e. $x^n-(x-1)^n$). Hence, r^{th} term of local series is $(r+1)^{th}$ term of global series; the first term of "globalized" local series is zero. Globalized local series still contains n terms, $n-1$ terms from local series, and one zero term preceding it. Hence the $(r+1)^{th}$ term of globalized local series, that is, r^{th} term of local series is $-(-1)^r\binom{n-1}{r}x^{n-r-1}$. And similarly, for sum of terms like $x^{n-2}-(x-1)^{n-2}$, from $x=1$ to $x=x$, (which turns out to be x^{n-2}), two zeroes for exponents of $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1}$ and $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2}$ appears, while local term is globalized. And the $(r+2)^{th}$ term of globalized local series, that is, r^{th} term of local series is $-(-1)^r\binom{n-2}{r}x^{n-r-2}$. There are $n-2$ terms from local series. By similar continuation of arguments, $\sum_{\substack{x=1\\x\in N}}^{x=x}(x^{n-k}-(x-1)^{n-k})$ series requires addition of k number of zero terms for globalization, and so, $(r+k)^{th}$ term of globalized local series, that is, r^{th} term of local series is $-(-1)^r\binom{n-k}{r}x^{n-k-r}$. There are $n-k$ terms from local series. So, it must follow that, r^{th} term of globalized local series, that is, $(r-k)^{th}$ term of local series is $-(-1)^{r-k}\binom{n-k}{r-k}x^{n-k-r+k} = -(-1)^{r-k}\binom{n-k}{n-r}x^{n-r}$. It is important to note that, the local series starts from $\binom{n-k}{1}$ and not $\binom{n-k}{0}=1$. So, while interpreting generalized formula based on combination operator, care must be practiced. This is more concretely expressed by the fact that, while the term before first term of local series, which becomes zeroth term for local series and k^{th} term of globalized local series, is obviously zero, $-(-1)^{k-k}\binom{n-k}{k-k}x^{n-k} = -x^{n-k} \neq 0$. So, the generalization outlined above breaks down at the aforementioned condition, and hence, for the present purpose, it is convenient to assume $\binom{a}{b}=0$,if $b\leq 0$ or $b>a$. This can be treated as an informal "abuse of notation". Now, it is clear that, in general

$$\begin{bmatrix}\binom{n-k}{1} & -\binom{n-k}{2} & \ldots & -(-1)^r\binom{n-k}{r} & \ldots & -(-1)^{n-k}\binom{n-k}{n-k}\end{bmatrix}\begin{bmatrix}\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-1}\\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-2}\\ \vdots\\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-r}\\ \vdots\\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^0\end{bmatrix} = \begin{bmatrix}x^{n-k}\end{bmatrix}$$

Equation can be cast into global from as

$$\begin{bmatrix} 0 & 0 & \dots & \binom{n-k}{1} & -\binom{n-k}{2} & \dots & -(-1)^r\binom{n-k}{r} & \dots & -(-1)^{n-k}\binom{n-k}{n-k} \end{bmatrix} \begin{bmatrix} \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-1} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-2} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k-r} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{0} \end{bmatrix} = \begin{bmatrix} x^{n-k} \end{bmatrix}$$

So, if wished, the coefficient matrix (A) of n by n matrix can be constructed as

$$\begin{bmatrix} \binom{n}{1} & -\binom{n}{2} & \binom{n}{3} & \dots & -(-1)^r\binom{n}{r} & \dots & -(-1)^n \\ 0 & \binom{n-1}{1} & -\binom{n-1}{2} & \dots & -(-1)^{r-1}\binom{n-1}{r-1} & \dots & -(-1)^{n-1} \\ 0 & 0 & \binom{n-2}{1} & \dots & -(-1)^{r-2}\binom{n-2}{r-2} & \dots & -(-1)^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -(-1)^{r-k}\binom{n-k}{r-k} & \dots & -(-1)^{n-k} \\ \dots & \dots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-3} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{0} \end{bmatrix} = \begin{bmatrix} x^n \\ x^{n-1} \\ x^{n-2} \\ \vdots \\ x^{n-k} \\ \vdots \\ x \end{bmatrix}$$

So,

$$\begin{bmatrix} \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-2} \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-3} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-r} \\ \vdots \\ \sum_{\substack{x=1\\x\in N}}^{x=x} x^{0} \end{bmatrix} = \begin{bmatrix} \binom{n}{1} & -\binom{n}{2} & \binom{n}{3} & \dots & -(-1)^r\binom{n}{r} & \dots & -(-1)^n \\ 0 & \binom{n-1}{1} & -\binom{n-1}{2} & \dots & -(-1)^{r-1}\binom{n-1}{r-1} & \dots & -(-1)^{n-1} \\ 0 & 0 & \binom{n-2}{1} & \dots & -(-1)^{r-2}\binom{n-2}{r-2} & \dots & -(-1)^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -(-1)^{r-k}\binom{n-k}{r-k} & \dots & -(-1)^{n-k} \\ \dots & \dots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} x^n \\ x^{n-1} \\ x^{n-2} \\ \vdots \\ x^{n-k} \\ \vdots \\ x \end{bmatrix}$$

As the coefficient matrix is an upper triangular matrix, the determinant of the matrix is the product of only principal diagonal elements, which equals to =$n!$.

The equation can be solved for any $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-k}$ $(1 \le k \le n)$ by standard matrix methods. Solving this equation for $\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1}$ by Cramer's rule, we find

$$\sum_{\substack{x=1\\x\in N}}^{x=x} x^{n-1} = \frac{\begin{vmatrix} x^n & -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ x^{n-1} & \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ x^{n-2} & 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ x^{n-3} & 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ x^{n-4} & 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ x^{n-5} & 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ x^{n-6} & 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} \binom{n}{1} & -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}}$$

The denominator matrix is the coefficient matrix. So, the denominator is the determinant of coefficient matrix (A) i.e., $n!$.

We will now concentrate to evaluate the numerator. The numerator matrix is similar to the coefficient matrix (A) except the 1st column, which is replaced by the column vector. To evaluate the determinant of the numerator, we will expand by the first column.

$$x^n \begin{vmatrix} \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$-\,x^{n-1} \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$+\,x^{n-2}\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$-\,x^{n-3}\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

+ ... ...

So, the coefficients of x^n and x^{n-1} in the numerator are determinants of two upper triangular matrices which are the product of their main diagonal elements. So, the coefficients of x^n and x^{n-1} in the numerator are $(n-1)!$ and $n!/2$ respectively. So, the coefficients of x^n and x^{n-1} of the series are $\frac{1}{n}$ and $\frac{1}{2}$ respectively.

The coefficient of x^{n-2} in the numerator is,

$$=\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$${R_2}^{'} = R_2 - \frac{(n-1)}{A_{(1,1)}}R_1 = R_2 + \frac{(n-1)}{\binom{n}{2}}R_1$$

This will transform the 2^{nd} row elements as $A^{'}_{(2,j)} = C_{(2,j)}A_{(2,j)}$. Where, $C_{(2,j)} = 1 - \frac{2}{j+1}$.

So, $C_{(2,1)} = 0, C_{(2,2)} = \frac{1}{3}, C_{(2,3)} = \frac{1}{2}, C_{(2,4)} = \frac{3}{5}, C_{(2,5)} = \frac{2}{3}, C_{(2,6)} = \frac{5}{7}$.

$$=\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

So, the coefficient of x^{n-2} in the numerator is $= C_{(2,2)}\binom{n}{2}\binom{n-1}{2}(n-3)! = C_{(2,2)}\frac{n!\binom{n}{2}.2!}{n.2^2}$ $= \frac{n!(n-1)}{12}$

The coefficient of x^{n-3} in the numerator is,

$$= -\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$R_4{}' = R_4 - \frac{(n-2)}{A'_{(2,2)}} R_2$

This will transform the 4^{th} row elements as $A'_{(4,j)} = C_{(4,j)} A_{(4,j)}$. Where, $C_{(4,j)} = 1 - \frac{C_{(2,j)}}{C_{(2,2)}} \times \frac{2}{j} = \frac{(j-2)(j-3)}{j(j+1)}$.

So, $C_{(4,2)} = 0, C_{(4,3)} = 0, C_{(4,4)} = \frac{1}{10}, C_{(4,5)} = \frac{1}{5}, C_{(4,6)} = \frac{2}{7}, C_{(4,7)} = \frac{5}{14}$.

$$= \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

That is an upper triangular matrix with one zero diagonal element. So, the coefficient of x^{n-3} is 0.

The coefficient of x^{n-4} in the numerator is,

$$\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

So, the coefficient of x^{n-4} in the numerator is = $-C_{(2,2)}C_{(4,4)}\ \binom{n}{2}\binom{n-1}{2}\binom{n-2}{3}(n-3)(n-5)!$

$= -C_{(2,2)}C_{(4,4)}\ \frac{n!\binom{n}{4}.4!}{n.2^3.3}$

The coefficient of x^{n-5} in the numerator is,

$$= -\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \cdots & -(-1)^{n-1} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & \cdots & -(-1)^{n-2} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & \cdots & -(-1)^{n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$R_6{}' = R_6 - \frac{(n-4)}{A'_{(4,4)}}R_4$

This will transform the 6^{th} row elements as $A'_{(6,j)} = C_{(6,j)}A_{(6,j)}$. Where, $C_{(6,j)} =$

$1 - \frac{C_{(4,j)}}{C_{(4,4)}} \times \frac{6}{(j-1)(j-2)}$
So, $C_{(6,4)} = 0, C_{(6,5)} = 0, C_{(6,6)} = \frac{1}{7}, C_{(6,7)} = \frac{2}{7}, C_{(6,8)} = \frac{17}{42}, \ldots$.

$$= - \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & \cdots & -(-1)^{n-6} \\ 0 & 0 & 0 & 0 & 0 & C_{(6,6)}\binom{n-4}{3} & \cdots & -(-1)^{n-4}C_{(6,n-4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

That is an upper triangular matrix with one zero diagonal element. So, the coefficient of x^{n-5} is 0.
The coefficient of x^{n-6} in the numerator is,

$$= \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \cdots & -(-1)^n \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \binom{n-1}{7} & \cdots & -(-1)^{n-1} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & -\binom{n-2}{6} & \cdots & -(-1)^{n-2} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & - & \binom{n-3}{5} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & -\binom{n-4}{4} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-7}{1} & \cdots & -(-1)^{n-7} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \cdots & -(-1)^n \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & C_{(2,7)}\binom{n-1}{7} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \binom{n-3}{5} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & -C_{(4,7)}\binom{n-2}{6} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & C_{(6,6)}\binom{n-4}{3} & -C_{(6,7)}\binom{n-4}{4} & \cdots & -(-1)^{n-4}C_{(6,n-4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-7}{1} & \cdots & -(-1)^{n-7} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

So, the coefficient of x^{n-6} in the numerator is =

$$C_{(2,2)}C_{(4,4)}C_{(6,6)} \binom{n}{2}\binom{n-1}{2}\binom{n-2}{3}\binom{n-4}{3}(n-3)(n-5)(n-7)!$$
$$= C_{(2,2)}C_{(4,4)}C_{(6,6)} \frac{n!\binom{n}{6}.6!}{n.2^4.3^2}$$

The coefficient of x^{n-7} in the numerator is,

$$= -\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \binom{n}{9} & \cdots & -(-1)^{n} \\ \binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \binom{n-1}{7} & -\binom{n-1}{8} & \cdots & -(-1)^{n-1} \\ 0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & -\binom{n-2}{6} & \binom{n-2}{7} & \cdots & -(-1)^{n-2} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & - & \binom{n-3}{5} & -\binom{n-3}{6} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & -\binom{n-4}{4} & \binom{n-4}{5} & \cdots & -(-1)^{n-4} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & -\binom{n-5}{4} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & -\binom{n-6}{2} & \binom{n-6}{3} & \cdots & -(-1)^{n-6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-8}{1} & \cdots & -(-1)^{n-8} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

=

$$\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \binom{n}{9} & \cdots & -(-1)^{n} \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & C_{(2,7)}\binom{n-1}{7} & -C_{(2,8)}\binom{n-1}{8} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \binom{n-3}{5} & -\binom{n-3}{6} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & -C_{(4,7)}\binom{n-2}{6} & C_{(4,8)}\binom{n-2}{7} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & -\binom{n-5}{4} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & C_{(6,6)}\binom{n-4}{3} & -C_{(6,7)}\binom{n-4}{4} & C_{(6,8)}\binom{n-4}{5} & \cdots & -(-1)^{n-4}C_{(6,n-4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-8}{1} & \cdots & -(-1)^{n-8} \\ 0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & -\binom{n-6}{2} & \binom{n-6}{3} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$R_8{}' = R_8 - \frac{(n-6)}{A'_{(6,6)}} R_6$$

This will transform the 8^{th} row elements as $A'_{(8,j)} = C_{(8,j)} A_{(8,j)}$. Where, $C_{(8,j)} = 1 - \frac{C_{(6,j)}}{C_{(6,6)}} \times \frac{6}{(j-3)(j-4)}$

So, $C_{(8,6)} = 0$, $C_{(8,7)} = 0$, $C_{(8,8)} = \frac{3}{20}$, $C_{(8,9)} = \frac{3}{10}$, $C_{(8,10)} = \frac{14}{33}$.

=

$$\begin{vmatrix} -\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \binom{n}{9} & \cdots & -(-1)^{n} \\ 0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & C_{(2,7)}\binom{n-1}{7} & -C_{(2,8)}\binom{n-1}{8} & \cdots & -(-1)^{n-1}C_{(2,n-1)} \\ 0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \binom{n-3}{5} & -\binom{n-3}{6} & \cdots & -(-1)^{n-3} \\ 0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & -C_{(4,7)}\binom{n-2}{6} & C_{(4,8)}\binom{n-2}{7} & \cdots & -(-1)^{n-2}C_{(4,n-2)} \\ 0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & -\binom{n-5}{4} & \cdots & -(-1)^{n-5} \\ 0 & 0 & 0 & 0 & 0 & C_{(6,6)}\binom{n-4}{3} & -C_{(6,7)}\binom{n-4}{4} & C_{(6,8)}\binom{n-4}{5} & \cdots & -(-1)^{n-4}C_{(6,n-4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-8}{1} & \cdots & -(-1)^{n-8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{(8,8)}\binom{n-6}{3} & \cdots & -(-1)^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

That is an upper triangular matrix with one zero diagonal element. So, the coefficient of $x^{n-7} = 0$.

The coefficient of x^{n-8} in the numerator is,

$$
= \begin{vmatrix}
-\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \binom{n}{9} & -\binom{n}{10} & -(-1)^{n} \\
\binom{n-1}{1} & -\binom{n-1}{2} & \binom{n-1}{3} & -\binom{n-1}{4} & \binom{n-1}{5} & -\binom{n-1}{6} & \binom{n-1}{7} & -\binom{n-1}{8} & \binom{n-1}{9} & -(-1)^{n-1} \\
0 & \binom{n-2}{1} & -\binom{n-2}{2} & \binom{n-2}{3} & -\binom{n-2}{4} & \binom{n-2}{5} & -\binom{n-2}{6} & \binom{n-2}{7} & -\binom{n-2}{8} & -(-1)^{n-2} \\
0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \binom{n-3}{5} & -\binom{n-3}{6} & \binom{n-3}{7} & -(-1)^{n-3} \\
0 & 0 & 0 & \binom{n-4}{1} & -\binom{n-4}{2} & \binom{n-4}{3} & -\binom{n-4}{4} & \binom{n-4}{5} & -\binom{n-4}{6} & -(-1)^{n-4} \\
0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & -\binom{n-5}{4} & \binom{n-5}{5} & -(-1)^{n-5} \\
0 & 0 & 0 & 0 & 0 & \binom{n-6}{1} & -\binom{n-6}{2} & \binom{n-6}{3} & -\binom{n-6}{4} & -(-1)^{n-6} \\
0 & 0 & 0 & 0 & 0 & 0 & \binom{n-7}{1} & -\binom{n-7}{2} & \binom{n-7}{3} & -(-1)^{n-7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-9}{1} & -(-1)^{n-9} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix}
$$

$$
= -
$$

$$
\begin{vmatrix}
-\binom{n}{2} & \binom{n}{3} & -\binom{n}{4} & \binom{n}{5} & -\binom{n}{6} & \binom{n}{7} & -\binom{n}{8} & \binom{n}{9} & -\binom{n}{10} & -(-1)^{n} \\
0 & -C_{(2,2)}\binom{n-1}{2} & C_{(2,3)}\binom{n-1}{3} & -C_{(2,4)}\binom{n-1}{4} & C_{(2,5)}\binom{n-1}{5} & -C_{(2,6)}\binom{n-1}{6} & C_{(2,7)}\binom{n-1}{7} & -C_{(2,8)}\binom{n-1}{8} & C_{(2,9)}\binom{n-1}{9} & -(-1)^{n-1}C_{(2,n-1)} \\
0 & 0 & \binom{n-3}{1} & -\binom{n-3}{2} & \binom{n-3}{3} & -\binom{n-3}{4} & \binom{n-3}{5} & -\binom{n-3}{6} & \binom{n-3}{7} & -(-1)^{n-3} \\
0 & 0 & 0 & C_{(4,4)}\binom{n-2}{3} & -C_{(4,5)}\binom{n-2}{4} & C_{(4,6)}\binom{n-2}{5} & -C_{(4,7)}\binom{n-2}{6} & C_{(4,8)}\binom{n-2}{7} & -C_{(4,9)}\binom{n-2}{8} & -(-1)^{n-2}C_{(4,n-2)} \\
0 & 0 & 0 & 0 & \binom{n-5}{1} & -\binom{n-5}{2} & \binom{n-5}{3} & -\binom{n-5}{4} & \binom{n-5}{5} & -(-1)^{n-5} \\
0 & 0 & 0 & 0 & 0 & C_{(6,6)}\binom{n-4}{3} & -C_{(6,7)}\binom{n-4}{4} & C_{(6,8)}\binom{n-4}{5} & -C_{(6,9)}\binom{n-4}{6} & -(-1)^{n-4}C_{(6,n-4)} \\
0 & 0 & 0 & 0 & 0 & 0 & \binom{n-7}{1} & -\binom{n-7}{2} & \binom{n-7}{3} & -(-1)^{n-7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{(8,8)}\binom{n-6}{3} & -C_{(8,9)}\binom{n-6}{4} & -(-1)^{n-6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{n-9}{1} & -(-1)^{n-9} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix}
$$

So, the coefficient of x^{n-8} in the numerator is $=$

$$
-C_{(2,2)}C_{(4,4)}C_{(6,6)}C_{(8,8)}\ \binom{n}{2}\binom{n-1}{2}\binom{n-2}{3}\binom{n-4}{3}\binom{n-6}{3}(n-3)(n-5)(n-7(n-9)!
$$

$$
= -C_{(2,2)}C_{(4,4)}C_{(6,6)}C_{(8,8)}\ \frac{n!\binom{n}{8}.8!}{n.2^5.3^3}
$$

4. POWER-SUM TO DERIVE THE BERNOULLI NUMBERS

From the previous examples, we can generalize that, for the coefficient of any x^{n-2p+1} (where $p > 1$), the coefficient (reduced row) matrices' element $A_{(2p-1,2p)} = \binom{n-2p}{1}$, so, $A_{(2p-1,2p-1)} = 0$. The other elements are: $A_{(2k,2k)} = C_{(2k,2k)}\binom{n-2k+2}{3}$, for $k \leq p$, $A_{(2k-1,2k-1)} = \binom{n-2k+1}{3}$, for $k < p$, $A_{(k,k)} = n-k$, for $k > 2p$. Elements below the main diagonal are all zero. That is an upper triangular matrix with one zero diagonal element, i.e., $A_{(2p-1,2p-1)} = 0$. So, the coefficient of x^{n-2p+1}= 0.

For the coefficient of any x^{n-2p} (where $p > 1$), the co-efficient (reduced row) matrices' elements are: $A_{(2k,2k)} = C_{(2k,2k)}\binom{n-2k+2}{3}$, for $k \leq p$. Other diagonal elements are $A_{(k,k)} = n-k$. Elements below the main diagonal are all zero. That is an upper triangular matrix with all non-zero diagonal elements.

So, the coefficient of x^{n-2p} in the numerator is the product of the diagonal elements =

$$\left(\frac{1}{3}\prod_{\substack{i=4\\ i\in 2N}}^{i=2p} C_{(i,i)}\right)\frac{(-1)^{p+1}n!\binom{n}{2p}(2p)!}{n.2^{p+1}.3^{p-1}}$$

Where $C_{(i,j)}$ can be written as an iterated fraction,

$$C_{(i,j)} = \frac{(j-2)(j-3)}{j(j+1)}, \text{ (for } i=4,\ j\geq i). \quad (4.\ 4)$$

$$C_{(i,j)} = 1 - \frac{C_{(i-2,j)}}{C_{(i-2,i-2)}} \times \frac{6}{(j-i+4)(j-i+5)}, \text{ (for } i>4,\ j\geq i). \quad (4.\ 5)$$

$$C_{(2,2)} = \frac{1}{3} \quad (4.\ 6)$$

$$\therefore \sum_{\substack{x=1\\ x\in N}}^{x=x} x^{n-1} = \tfrac{1}{n}x^n + \tfrac{1}{2}x^{n-1} + C_{(2,2)}\tfrac{(-1)^2\binom{n}{2}.2!}{n.2^2}x^{n-2} + C_{(2,2)}C_{(4,4)}\tfrac{(-1)^3\binom{n}{4}.4!}{n.2^3.3}x^{n-4}$$
$$+C_{(2,2)}C_{(4,4)}C_{(6,6)}\frac{(-1)^4\binom{n}{6}.6!}{n.2^4.3^2}x^{n-6} + C_{(2,2)}C_{(4,4)}C_{(6,6)}C_{(8,8)}\frac{(-1)^5\binom{n}{8}.8!}{n.2^5.3^3}x^{n-8} + ...$$

$$\therefore \sum_{\substack{x=1\\ x\in N}}^{x=x} x^{n-1} =$$

$$\frac{1}{n}x^n + \frac{1}{2}x^{n-1} + C_{(2,2)}\frac{(-1)^2\binom{n}{2}.2!}{n.2^2}x^{n-2} + \sum_{\substack{k=4\\ k\in 2N}}^{n-1}\left(\frac{1}{3}\prod_{\substack{i=4\\ i\in 2N}}^{i=k} C_{(i,i)}\right)\frac{(-1)^{\frac{k}{2}+1}\binom{n}{k}(k)!}{n.2^{\frac{k}{2}+1}.3^{\frac{k}{2}-1}}x^{n-k}$$

The k^{th} Bernoulli Number B_k is defined as the coefficient of $\frac{1}{n}\binom{n}{k}x^{n-k}$ of the above power sum of $\sum_{\substack{x=1\\ x\in N}}^{x=x} x^{n-1}$. Accordingly, the $2p^{th}$ Bernoulli Number B_{2p} can be written as:

$$B_{2p} = \frac{(-1)^{p+1}(2p)!}{2^{p+1}.3^{p-1}} C_{(1,1)}C_{(2,2)}\prod_{\substack{i=4\\ i\in 2N}}^{i=2p} C_{(i,i)}$$

$$\therefore B_{2p} = \frac{(-1)^{p+1}(2p)!}{2^{p+1}.3^{p}}\prod_{\substack{i=4\\ i\in 2N}}^{i=2p} C_{(i,i)}$$

For the odd order Bernoulli numbers, as the coefficient of x^{n-2p+1}= 0, $B_{2p-1} = 0$ (for $p > 1$). Here we show first few Bernoulli Numbers calculated by this formula using MATLAB and C programming language.[1]

[1] The source codes are provided in the **Appendix** section.

$$
\begin{aligned}
B_0 &= 1 \\
B_1 &= \frac{1}{2} \\
B_2 &= \frac{1}{6} \\
B_4 &= -\frac{1}{30} \\
B_6 &= \frac{1}{42} \\
B_8 &= -\frac{1}{30} \\
B_{10} &= \frac{5}{66} \\
B_{12} &= -\frac{691}{2730} \\
B_{14} &= \frac{7}{6} \\
B_{16} &= -\frac{3617}{510} \\
B_{18} &= \frac{43867}{798} \\
B_{20} &= -\frac{174611}{330}
\end{aligned}
$$

5. Conclusion

Using numerically verbose examples rather than compact formulae and terminologies to develop a new recurrence relation for Bernoulli numbers (and developing code on two platforms) from the fundamentals of Linear Algebra and Binomial Theorem illustrates the authors' thought process as the research idea was conceptualized. Experts in a variety of fields (including but not limited to) mathematics education, philosophy and psychology of learning, and pedagogy are invited to conduct textual and methodological analysis on the authors' work, as well as to write similar articles documenting their own thought processes, in order to gain insight into the learning and research methodologies of future mathematicians and academicians. The findings explicitly mentioned in this article may result in a more elegant or simple Bernoulli Number formula, whereas the findings embedded in the article's methodology and words may provide insight into Abstract Mathematical thinking and the idea developing process of prospective mathematics researchers.

Authors' Contribution

Both of the authors contributed equally in production of this article.

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APPENDIX (A_1)

C source code:

Disclaimer: this function demo is just a concept-of-proof; poor results creep in when the Bernoulli number exceeds 30 or so.

```c
#include <stdio.h>
#include <math.h>
int main ()
{
    long long int i,j,k,n,m,factorial;
    long double B;
    scanf("%lld",&k);
    double C[k][k];
    C[2][2]=1.0/3;
    for(j=4,i=4;j<=k;j=j+2)
    {
        C[i][j]=(j-2)*(j-3)*1.0/(j*(j+1));
    }
    for(i=6;i<=k;i=i+2)
    {
        for(j=i;j<=k;j=j+2)
        {
            C[i][j]=1-(C[i-2][j])*6/((C[i-2][i-2])*(4+j-i)*(5+j-i));
        }
    }
    for(n=2,B=1;n<=k;n=n+2)
    {
        B=B*C[n][n];
    }
    for(m=1,factorial=1;m<=k;m++)
    {
        factorial=factorial*m;
    }
    B=B*factorial*pow(-1,k/2+1)/(pow(2,k/2+1)*pow(3,k/2-1));
    if (k==0)
        B=1;
    else if (k==1)
        B=1/2;
    else if (k%2==1)
        B=0;
    printf("B(%lld)=%llf",k,B);
}
```

APPENDIX (A_2)

MATLAB source code (run on octave online):
Disclaimer: this function demo is just a concept-of-proof; poor results creep in when the Bernoulli number exceeds 30 or so.

```
function bern(k)
format rat;
C=zeros(k,k);
C(2,2)=1/3;
for j=4:2:k
    i=4;
    C(i,j)=((j-2)/j)*((j-3)/(j+1));
end
for i=6:2:k
    for j=i:2:k
        C(i,j)=1-(C(i-2,j)./C(i-2,i-2)).*6/((4+j-i).*(5+j-i));
    end
end
B=1;
for n=2:2:k
    B=B.*C(n,n);
end

B=B*(factorial(k)/(2^(k/2+1)*3^(k/2-1)))*(-1)^(k/2+1);

if k==0
    B=1;
elseif k==1
    B=1/2;
elseif mod(k,2)==1
    B=0;
end
disp(['B',num2str(k),'=']);
disp(B);
format long;
disp(B);
format short;
```