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Analysis of Multi Term Fractional Differential Equations using Variational Iteration Method

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Abstract.: In this paper, we have obtained an approximate solution of multi-term Caputo fractional differential equations (MFDEs) using the Variational iteration method (VIM). Further, we have obtained the convergence criteria and error approximation of VIM for solving generalized multi term fractional differential equations. The obtained results are shown using plots to demonstrated the efficiency and accuracy of the VIM.

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1. Introduction

The study of fractional calculus has become an active and vital area of research due to its demonstrated applications in engineering, applied science, diffusion processes, fluid flow and many other fields [28, 1, 36, 30]. Recently the notable focus has been given by many researchers to investigate and develop a new concept in the theory of fractional calculus. Many numerical methods applied to find an approximate solution of fractional differential equations (FDEs). The purpose to utilize fractional differential equations to obtain approximate solution is to improve and generalize several ordinary differential systems. Hence, demonstrating some real world phenomena using fractional derivative operator has fascinated several researchers in the field of applied mathematics. [34, 20, 33, 8].

Over the last two decades, research scholars and authors in mathematics have engaged themselves in developing mathematical modelling of various biological processes and diseases in order to develop the qualitative behaviour and stability. Several authors have succeeded also in this enterprise. The foremost models of this kind, which studies optimal control of diabetes, tuberculosis and control strategy for the outbreak of dengue fever was jointly introduced by Jajarmi et al. [16, 17]. Baleanu et al. analysed optimal control of a tumor-immune surveillance with non-singular derivative operator [2]. Fractional SIRS-SI malaria disease model with application of vaccines, anti-malarial drugs, and spraying was investigated by Kumar et al. in [23].

A fractional differential equation consist of more than one differential operator is known as a multi-term fractional differential equation and possesses numerous applications in applied sciences. Many researchers in the area of applied sciences and engineering have considerable attention to seek with this type of problems and proposed computationally effective algorithms for simulating analytical and approximate solutions of these equations [19, 5, 10, 29, 3, 18].

Some of the most used and efficient analytical or numerical methods for solving these fractional differential equations are given as the Finite difference method [21, 39], Adomian decomposition method (ADM) [24], Homotopy analysis method [25, 27], Adams-Bashforth- Moulton method [4], Iterative Laplace transform method [31, 32], Spectral collocation method [40], Homotopy perturbation method [11, 9, 6, 7] and New iterative method [22, 35]. One such frequently used method known as Variational iteration method (VIM) introduced by He [12] is the most accurate and effective technique to get the solutions of linear and nonlinear differential equations [26]. In [37] an algorithm is proposed to convert the MFDEs into a system of FDEs which is further solved by using VIM. But this technique has limitations if the order of equation is very high. To overcome this difficulty in recent years Yang et.al. [38] studied convergence of the VIM and obtained analytical solutions of MFDE. Motivated by this work, in this paper we have obtained approximate solutions of several MFDEs easily by using VIM. Moreover, convergence analysis and error estimate of generalized MFDEs using VIM is also investigated. Furthermore, The numerical results are obtained by utilizing variational iteration method alongwith Mathematica software, and results are demonstrated using graphs. The parameters and initial conditions are allocated arbitrary values to verify our results.

The remaining part of the paper is designed as follows. Some basic definitions and properties of Fractional calculus are mentioned in Section 2. In Section 3, a brief explanation of VIM is presented. In Section 4, we study convergence analysis and theorem for error estimate of generalizing MFDEs by using VIM. In Section 5, we present the effectiveness of the proposed method by taking several types of MFDEs. The conclusion of the study is drawn in Section 6.

2. Preliminaries

In this section, we recall some basic definitions and properties of Fractional calculus.

Definition 2.1. *([28]) The Riemann-Liouville (R-L) fractional integral of order $\vartheta(\vartheta \geq 0)$ is defined by*

$$J^{\vartheta} f(t) = \frac{1}{\Gamma(\vartheta)} \int_a^t (t-x)^{\vartheta-1} f(x)dx, \;\; t > a, \quad \vartheta > 0. \tag{2. 1}$$

Definition 2.2. *([28]) The fractional derivative operator of R-L of order $\vartheta(\vartheta \geq 0)$, and $n \in \mathbb{N} \cup \{0\}$ is defined as*

$${}^{R}D_t^{\vartheta} f(t) = D^n J^{n-\vartheta} f(t) = \frac{1}{\Gamma(n-\vartheta)} \frac{d^n}{dt^n} \int_a^t (t-x)^{n-\vartheta-1} f(x)dx, \; t > a, \; n-1 < \vartheta \leq n. \tag{2. 2}$$

Definition 2.3. *([28]) The Caputo fractional derivative of order $\vartheta(\vartheta \geq 0)$ and $n \in \mathbb{N}\cup\{0\}$ is given by*

$${}^{c}D_t^{\vartheta} f(t) = \begin{cases} J^{n-\vartheta} D^n f(t) = \frac{1}{\Gamma(n-\vartheta)} \int_a^t (t-x)^{n-\vartheta-1} \dfrac{d^n}{dx^n} f(x)dx, \; t > a, \; n-1 < \vartheta < n, & ; \\ \dfrac{d^n}{dt^n} f(t) \; \vartheta = n, \end{cases} \tag{2. 3}$$

where D^n is the classical derivative of order n.

Next, we state some properties of the operators $J^{\vartheta}, {}^{R}D_t^{\vartheta}, {}^{c}D_t^{\vartheta} f(t)$

For $f(t) \in C^m[a,b], \; \vartheta, \; \varpi \geq 0, \; n-1 < \vartheta \leq n, \; \vartheta + \varpi \leq m, a \geq 0$ and $\delta \geq -1$

$$J^{\vartheta} J^{\varpi} f(t) = J^{\varpi} J^{\vartheta} f(t) = J^{\vartheta+\varpi} f(t) \tag{2. 4}$$

$${}^{c}D_t^{\vartheta} J^{\varpi} f(t) = J^{\varpi} D_t^{\vartheta} f(t) = {}^{c} D_t^{\vartheta-\varpi} f(t) = J^{\varpi-\vartheta} f(t) \tag{2. 5}$$

$${}^{R}D_t^{\vartheta} f(t) = D_t^{\vartheta} f(t), \text{ for } f^k(a) = 0, \; k = 0, 1 \cdots, m-1 \tag{2. 6}$$

$${}^{R}D_t^{\vartheta} (t-a)^{\delta} = D_t^{\vartheta} (t-a)^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\delta-\vartheta+1)} (t-a)^{\delta-\vartheta} \tag{2. 7}$$

$$D_t^{\vartheta}(t^k) = 0, \; \forall k = 0, 1 \cdots, n-1 \tag{2. 8}$$

The Caputo's fractional differential operator is linear

$$D_t^{\vartheta}(c_1 f(t) + c_2 g(t)) = c_1 D_t^{\vartheta} f(t) + c_2 D_t^{\vartheta} g(t) \tag{2. 9}$$

3. Variational iteration method

To illustrate the notions of VIM, we examine the differential equation of the form

$$Ly(t) + Ry(t) + Ny(t) = f(t), \tag{3. 10}$$

where Ly(t) is a linear operator, Ry(t) denote the reminder linear term, Ny(t) and f(t) respectively denotes a nonlinear operator and nonhomogeneous source term subject to the initial conditions

$$y^k(0) = b_k,\ k = 0, 1, \cdots, n-1. \tag{3. 11}$$

and linear operator L is taking as the highest integer order of differentiation.
For example if $D_t^\alpha y\,(t)\,; m-1 < \alpha \leq m,$ then $Ly(t) = D_t^m y\,(t)$. In the light of VIM, a correction functional of Eq. (3. 10) can be written as:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)\big(Ly_n(s) + R\tilde{y}_n(s) + N\tilde{y}_n(s) - f(s)\big)ds, \tag{3. 12}$$

where the general Lagrangian multiplier denoted by λ and can be obtained by taking successive approximation of $y_j,\ j \geq 0$. $\tilde{y}_n$ is a restricted value hence it becomes a constant, therefore $\delta\tilde{y}_n = 0$, δ represent the variational derivative. Now, applying restricted variations to Ny, the nonlinear term, so that Lagrange multiplier can be easily obtained by selecting proper initial function $y_0(t)$.

Considering the variation of Eq. (3. 12) with respect to y_n, the independent variable, we get

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda(s) L y_n(s) ds, \tag{3. 13}$$

we use integration by parts to determine Lagrange multiplier $\lambda(s)$ [15]. Whenever, we take $Ry_n(s) = 0$ or we consider $Ry_n(s)$ as a nonlinear term then by the choice of L and also applying the VIM method [13, 14] we get,

$$\lambda = \frac{(-1)^m}{(m-1)!}(s-t)^{(m-1)}. \tag{3. 14}$$

Substituting equation (3. 14) into (3. 12), gives the approximate solution as below after omitting the restrictions,

$$y_{n+1}(t) = y_n(t) + \int_0^t \frac{(-1)^m}{(m-1)!}(s-t)^{(m-1)}\Big(Ly_n(s) + R\tilde{y}_n(s) + N\tilde{y}_n(s) - f(s)\Big)ds. \tag{3. 15}$$

The solution of equation Eq. (3. 10) will be calculated as

$$y(t) = \lim_{n\to\infty} y_n(t). \tag{3. 16}$$

4. ANALYSIS OF VIM FOR SOLVING MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS

The convergence analysis of VIM to solve generalized MFDEs is studied in this section. The MFDE consider as

$$\begin{cases} D_t^{\alpha}x(t)=\sum_{i=1}^{n-1}a_i(t)D_t^{\beta_i}x(t)+a_0(t)x(t)+N[t,x(t),D_t^{\beta_1}x(t),\cdots,D_t^{\beta_{n-1}}x(t)]+g(t), \\ x^k(0)=b_k,\ k=0,1,\cdots,n-1. \end{cases} \quad . \tag{4. 17}$$

where $n-1<\alpha\leq n\in\mathbb{N}$ and $0<\beta_1<\beta_2<\cdots<\beta_{n-1}<\alpha$, N is non linear function of $t,x(t),D_t^{\beta_1}x(t),\cdots,D_t^{\beta_{n-1}}x(t)$, whereas g(t) and $a_i(t)$ are functions of t.
We consider $N:[0,T]\times\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\to\mathbb{R}$ as continuous function. Let $N(t,y_0,y_1,\cdots,y_{n-1})$ exists and $\dfrac{\partial N}{\partial y_k}$ is continuous and bounded partial derivatives with $\gamma_k=\sup\limits_{0\leq t\leq T}\left|\dfrac{\partial N}{\partial y_k}\right|,\ \forall k=0,1,\cdots n-1.$

Define the norm $\|y(t)\|_\infty=\max\limits_{0\leq t\leq T}|y(t)|,\ \forall y(t)\in C[0,T].$

If we put $\bar{x}(t)=x(t)-\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k$, then equation Eq.(4. 17) is written as

$$\begin{cases} D_t^{\alpha}\bar{x}(t)=\sum_{i=1}^{n-1}a_i(t)D_t^{\beta_i}\left(\bar{x}(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\right)+a_0(t)\left(\bar{x}(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\right)+g(t) \\ \qquad +N\Big[t,\bar{x}(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k,D_t^{\beta_1}\left(\bar{x}(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\right), \\ \qquad\qquad \cdots,D_t^{\beta_{n-1}}\left(\bar{x}(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\right)\Big], \\ x^k(0)=0,\ k=0,1,\cdots,m-1. \end{cases} \tag{4. 18}$$

from definition (2.2) and property (2.5), we have $D_t^{\alpha}\bar{x}(t)=D_t^nJ^{n-\alpha}\bar{x}(t)$. Let $J^{n-\alpha}\bar{x}(t)=u(t)$ then $\bar{x}(t)=D^{n-\alpha}u(t)$ and hence Eq.(4. 18) becomes

$$\begin{cases} D_t^{(n)}u(t)=\sum_{i=1}^{n-1}a_i(t)D_t^{\beta_i}\left(D_t^{n-\alpha}u(t)+\sum_{k=0}^{n-1}\dfrac{b_k}{\Gamma(1+k)}t^k\right)+a_0(t)\big(D_t^{n-\alpha}u(t) \\ +\sum_{k=0}^{n-1}\dfrac{b_k}{\Gamma(1+k)}t^k\Big)+N\Big[t,D_t^{n-\alpha}u(t)+\sum_{k=0}^{m-1}\dfrac{b_k}{\Gamma(1+k)}t^k,D_t^{\beta_1}\big(D_t^{n-\alpha}u(t) \\ +\sum_{k=0}^{m-1}\dfrac{b_k}{\Gamma(1+k)}t^k\Big),\cdots,\ D_t^{\beta_{n-1}}\left(D_t^{n-\alpha}u(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\right)\Big]+g(t). \end{cases} \tag{4. 19}$$

The correction functional is constructed using VIM as below

$$u_{m+1}(t)=u_m(t)+\int_0^t\lambda(s)\Bigg\{D_s^{(n)}u_m(s)-\sum_{i=1}^{n-1}a_i(s)D_s^{\beta_i}\left(D_s^{n-\alpha}\tilde{u}_m(s)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right)$$

$$
\begin{aligned}
&- a_0(s)\left(D_s^{n-\alpha}\tilde{u}_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) - N\Bigg[s, D_s^{n-\alpha}\tilde{u}_m(s) \\
&+ \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k,\ D_s^{\beta_1}\left(D_s^{n-\alpha}\tilde{u}_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right), \cdots, \\
&D_s^{\beta_{n-1}}\left(D_s^{n-\alpha}\tilde{u}_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right)\Bigg] - g(s)\Bigg\}ds. \qquad (4.\ 20)
\end{aligned}
$$

The above-mentioned correction functional is made stationary and observing that $\delta\tilde{u}_m(t) = 0$,

$$
\delta u_{m+1}(t) = \delta u_m(t) + \delta\int_0^t \lambda(s) D_s^{(n)} u_m(s)ds. \qquad (4.\ 21)
$$

This gives the stationary conditions as

$$
\lambda(s)|_{s=t} = 0,\ \lambda'(s)|_{s=t} = 0, \cdots, 1 + (-1)^{n-1}\lambda^{n-1}(s)|_{s=t} = 0,\ \lambda^n(s)|_{s=t} = 0,
$$

this implies $\lambda(s) = -\dfrac{(t-s)^{n-1}}{(n-1)!}$. By taking $n = 1,$ we get $\lambda(s) = -1,$ and taking $n = 2,$ we get $\lambda(s) = -(t-s)$. Therefore, iteration formula is expressed as

$$
\begin{aligned}
u_{m+1}(t) = u_m(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\Bigg\{&D_s^{(n)}u_m(s) - \sum_{i=1}^{n-1} a_i(s) D_s^{\beta_i}\Bigg(D_s^{n-\alpha}u_m(s) \\
&+ \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg) - a_0(s)\left(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) - g(s) \\
&- N\Bigg[s, D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k, D_s^{\beta_1}\left(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right), \\
&\cdots, D_s^{\beta_{n-1}}\left(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right)\Bigg]\Bigg\}ds. \qquad (4.\ 22)
\end{aligned}
$$

The successive iterations are calculated, starting with an initial approximation $u_0(t) = 0$, which provides the exact solution as

$$
u(t) = \lim_{m\to\infty} u_m(t). \qquad (4.\ 23)
$$

Therefore exact solution of problem Eq. (4. 17) is obtained as

$$
x(t) = D_t^{n-\alpha}u(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k. \qquad (4.\ 24)
$$

In next theorem, we show that the sequence $\{u_m(t)\}_{m=1}^{\infty}$ defined by Eq. (4. 22) with $u_0(t) = u_0$ converges to the solution of Eq. (4. 19).

Theorem 4.1. *Let $u(t), u_i(t) \in C^n[0,T],\ i = 0, 1, \cdots$ The sequence given in Eq.* (4. 22) *with $u_0(t) = u_0$ converges to u(t). That is, the exact solution of Eq.* (4. 19) *with the error estimate is given by*

$$\|\eta_{m+1}(t)\|_\infty \leq \|\eta_0(t)\|_\infty \frac{(\tau\ n\ T^\alpha)^{m+1}}{\Gamma\big((m+1)(\alpha-\beta_{n-1})+1\big)}. \tag{4. 25}$$

where $\tau = \max_{0\leq i\leq n-1} (a_i + \gamma_i)$, $\eta_j(t) = u_j(t) - u(t),\ j = 1, 2, \cdots$

Proof. From equation Eq. (4. 19) we get

$$\begin{aligned} u(t) = u(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\Bigg\{ & D_s^{(n)}u(s) - \sum_{i=1}^{n-1} a_i(s) D_s^{\beta_i}\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) \\ & - a_0(s)\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) - g(s) - N\Bigg[s, D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k, \\ & D_s^{\beta_1}\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right), \cdots, D_s^{\beta_{n-1}}\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right)\Bigg]\Bigg\} ds. \end{aligned} \tag{4. 26}$$

Eq. (4. 22) and Eq. (4. 26) gives,

$$\begin{aligned} \eta_{m+1}(t) = \eta_m(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\Bigg\{ & D_s^{(n)}\eta_m(s) - \Bigg\{\Bigg[\sum_{i=1}^{n-1} a_i(s) D_s^{\beta_i}\Bigg(D_s^{n-\alpha}u_m(s) \\ & + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg) + a_0(s)\left(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) \\ & + N\Bigg(s, D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k, D_s^{\beta_1}\left(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right), \\ & \cdots, D_s^{\beta_{n-1}}\left(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right)\Bigg) + g(s)\Bigg] - \Bigg[\sum_{i=1}^{n-1} a_i(s) D_s^{\beta_i} \\ & \left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) + a_0(s)\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right) \\ & + N\Bigg(s, D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k, D_s^{\beta_1}\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right), \cdots, \\ & D_s^{\beta_{n-1}}\left(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\right)\Bigg) + g(s)\Bigg]\Bigg\}\Bigg\} ds \end{aligned} \tag{4. 27}$$

where $\eta_j(t) = u_j(t) - u(t),\ j = 1, 2, \cdots$
By utilizing the fact that $\eta_m^k(0) = 0,\ m = 0, 1, \cdots,\ k = 0, 1, \cdots, n-1$, and integration

by parts

$$\eta_{m+1}(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\Bigg\{\Bigg[\sum_{i=1}^{n-1} a_i(s) D_s^{\beta_i}\Bigg(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg)$$
$$+ a_0(s)\Bigg(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg) + N\Bigg(s, D_s^{n-\alpha}u_m(s)$$
$$+ \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k, D_s^{\beta_1}\Bigg(D_s^{n-\alpha}u_m(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg),$$
$$\cdots, D_s^{\beta_{n-1}}\Bigg(D_s^{n-\alpha}u_m(s) \quad + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg)\Bigg) + g(s)\Bigg]$$
$$- \Bigg[\sum_{i=1}^{n-1} a_i(s) D_s^{\beta_i}\Bigg(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg)$$
$$+ a_0(s)\Bigg(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg)$$
$$+ N\Bigg(s, D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k, D_s^{\beta_1}\Bigg(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg), \cdots,$$
$$D_s^{\beta_{n-1}}\Bigg(D_s^{n-\alpha}u(s) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}s^k\Bigg)\Bigg) + g(s)\Bigg]\Bigg\} ds. \qquad (4.\ 28)$$

Observing that $N(t, y_0, y_1, \cdots, y_{n-1})$ exists and $\dfrac{\partial N}{\partial y_k}$ is continuous and bounded partial derivatives $\forall k = 0, 1, \cdots n-1$. Next we use the R-L concept of integration to simplify the terms, we get

$$\Big|\eta_{m+1}(t)\Big| = \Bigg| J^n \sum_{i=1}^{n-1} a_i(t) D_t^{n-\alpha+\beta_i}\big(u_m(t) - u(t)\big) + J^n a_0(t) D_t^{n-\alpha}\big(u_m(t) - u(t)\big)$$
$$+ J^n N\Bigg[t, D_t^{n-\alpha}u_m(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k, D_t^{\beta_1}\Bigg(D_t^{n-\alpha}u_m(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\Bigg), \cdots,$$
$$D_t^{\beta_{n-1}}\Bigg(D_t^{n-\alpha}u_m(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\Bigg)\Bigg] - J^n N\Bigg[t, D_t^{n-\alpha}u(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k,$$
$$D_t^{\beta_1}\Bigg(D_t^{n-\alpha}u(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\Bigg), \cdots, D_t^{\beta_{n-1}}\Bigg(D_s^{n-\alpha}u(t) + \sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k\Bigg)\Bigg]\Bigg|$$

Now, we define the following function for $0 \le \phi \le 1$ by taking $\tau = \max_{0 \le i \le n-1}(a_i + \gamma_i)$ and using Lagrange's mean value theorem with N_i' as the partial derivative of function N for the i^{th} variable.

$$\xi(t)=\Big(t, D_t^{n-\alpha}u(t)+\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k+\phi\big(D_t^{n-\alpha}\eta_m(t)\big),\ D_t^{n-\alpha+\beta_1}u(t)+D_t^{\beta_1}\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k$$

$$\phi\big(D_t^{n-\alpha+\beta_1}\eta_m(t)\big),\cdots, D_t^{n-\alpha+\beta_{n-1}}u(t)+D_t^{\beta_{n-1}}\sum_{k=0}^{n-1}\frac{b_k}{\Gamma(1+k)}t^k+\phi\big(D_t^{n-\alpha+\beta_{n-1}}\eta_m(t)\big)\Big)$$

$$\begin{aligned}\Big|\eta_{m+1}(t)\Big| &= \Bigg|J^n\sum_{i=1}^{n-1}a_i(t)D_t^{n-\alpha+\beta_i}\eta_m(t)+J^n a_0(t)D_t^{n-\alpha}\eta_m(t)+J^nN_0'(\xi(t))D_t^{n-\alpha}\eta_m(t)+\cdots\\
&\qquad +J^nN_{n-1}'(\xi(t))D_t^{n-\alpha+\beta_{n-1}}\eta_m(t)\Bigg|\\
&\le a_0\big|J^nD_t^{n-\alpha}\eta_m(t)\big|+a_1\big|J^nD_t^{n-\alpha+\beta_1}\eta_m(t)\big|+\cdots+a_{n-1}\big|J^nD_t^{n-\alpha+\beta_{n-1}}\eta_m(t)\big|\\
&\quad +\gamma_0\big|J^nD_t^{n-\alpha}\eta_m(t)\big|+\gamma_1\big|J^nD_t^{n-\alpha+\beta_1}\eta_m(t)\big|+\cdots+\gamma_{n-1}\big|J^nD_t^{n-\alpha+\beta_{n-1}}\eta_m(t)\big|\\
&\le a_0J^\alpha\big|\eta_m(t)\big|+a_1J^{\alpha-\beta_1}\big|\eta_m(t)\big|+\cdots+a_{n-1}J^{\alpha-\beta_{n-1}}\big|\eta_m(t)\big|+\gamma_0J^\alpha\big|\eta_m(t)\big|\\
&\qquad +\gamma_1J^{\alpha-\beta_1}\big|\eta_m(t)\big|+\cdots+\gamma_{n-1}J^{\alpha-\beta_{n-1}}\big|\eta_m(t)\big|\\
&= \Big(a_0J^\alpha+a_1J^{\alpha-\beta_1}+\cdots+a_{n-1}J^{\alpha-\beta_{n-1}}+\gamma_0J^\alpha+\gamma_1J^{\alpha-\beta_1}+\cdots+\gamma_{n-1}J^{\alpha-\beta_{n-1}}\Big)\big|\eta_m(t)\big|\\
&= \Big((a_0+\gamma_0)J^\alpha+(a_1+\gamma_1)J^{\alpha-\beta_1}+\cdots+(a_{n-1}+\gamma_{n-1})J^{\alpha-\beta_{n-1}}\Big)\big|\eta_m(t)\big|\\
&\ \ \vdots\\
&\le \Big((a_0+\gamma_0)J^\alpha+(a_1+\gamma_1)J^{\alpha-\beta_1}+\cdots+(a_{n-1}+\gamma_{n-1})J^{\alpha-\beta_{n-1}}\Big)^{m+1}\big|\eta_0(t)\big|\\
&\le \Big((a_0+\gamma_0)J^\alpha+(a_1+\gamma_1)J^{\alpha-\beta_1}+\cdots+(a_{n-1}+\gamma_{n-1})J^{\alpha-\beta_{n-1}}\Big)^{m+1}\max_{0\le\sigma\le T}\big|\eta_0(\sigma)\big|\\
&\le \tau^{m+1}\Big(J^\alpha+J^{\alpha-\beta_1}+\cdots+J^{\alpha-\beta_{n-1}}\Big)^{m+1}\max_{0\le\sigma\le T}\big|\eta_0(\sigma)\big|\\
&\le \tau^{m+1}\max_{0\le\sigma\le T}\big|\eta_0(\sigma)\big|n^{m+1}\frac{1}{\Gamma\big((m+1)(\alpha-\beta_{n-1})\big)}\int_0^t(t-s)^{(m+1)\alpha-1}ds\\
&= \tau^{m+1}\max_{0\le\sigma\le T}\big|\eta_0(\sigma)\big|n^{m+1}\frac{t^{(m+1)\alpha}}{\Gamma\big((m+1)(\alpha-\beta_{n-1})\big)\big((m+1)\alpha\big)}.
\end{aligned}\tag{4. 29}$$

Where $\tau=\max\limits_{0\le i\le n-1}(a_i+\gamma_i)$. Using the fact that $T,\ \tau,\ \alpha,\ \|\eta_0(t)\|_\infty,\ n$ are constants, $\alpha-\beta_{n-1}>0$ and property of Gamma function

$$\begin{aligned}\frac{1}{\Gamma\big((m+1)(\alpha-\beta_{n-1})\big)\big((m+1)\alpha\big)} &\le \frac{1}{\Gamma\big((m+1)(\alpha-\beta_{n-1})\big)\big((m+1)\alpha-\beta_{n-1}\big)}\\
&= \frac{1}{\Gamma\big((m+1)(\alpha-\beta_{n-1})+1\big)}.\end{aligned}$$

Now using convergence property of Mittag-Leffler functions, we get

$$\|\eta_{m+1}(t)\|_\infty \leq \|\eta_0(t)\|_\infty \frac{(\tau\, n\, T^\alpha)^{m+1}}{\Gamma\big((m+1)(\alpha-\beta_{n-1})+1\big)} \to 0, \text{ as } m \to \infty.$$

This ends the proof. □

5. Illustrative Examples

To demonstrate the accuracy of VIM to find an approximate solution of MFDEs, we study four examples. The Mathematica software is used to perform all the computations.
Example 5.1. *First, we consider the Bageley-Torvik initial value problem (IVP)*

$$\begin{cases} D_t^\alpha x(t) + D_t^2 x(t) - 2\sqrt{\pi} D_t^\beta x(t) + 4x(t) = 4t^9 - \dfrac{131072\, t^{8.5}}{12155} + 72\, t^7 + \dfrac{49152\, t^{6.5}}{143\sqrt{\pi}} \quad ; \\ x(0) = x'(0) = x''(0) = 0\ \alpha \in (2,3),\ \beta \in (0,1). \end{cases} \tag{5. 30}$$

If we take $\alpha = 2.5,\ \beta = 0.5$ and $x(t) = D_t^{0.5} u(t)$, then we get

$$D_t^3 u(t) + D_t^{2.5} u(t) - 2\sqrt{\pi} D_t u(t) + 4 D_t^{0.5} u(t) = 4t^9 - \frac{131072\, t^{8.5}}{12155} + 72\, t^7 + \frac{49152\, t^{6.5}}{143\sqrt{\pi}}. \tag{5. 31}$$

The iteration formula to find solution of Eq. (5. 31) is

$$u_{n+1}(t) = u_n(t) - \int_0^t \frac{(t-s)^2}{2} \varphi(s) ds, \tag{5. 32}$$

where

$$\varphi(s) = D_s^3 u_n(s) + D_s^{2.5} u_n(s) - 2\sqrt{\pi} D_s u_n(s) + 4\, D_s^{0.5} u_n(s) - 4s^9 + \frac{131072\, s^{8.5}}{12155} - 72\, s^7 - \frac{49152\, s^{6.5}}{143\sqrt{\pi}} \tag{5. 33}$$

we take initial approximation as

$$u_0(t) = 0 \tag{5. 34}$$

By using the iteration formula Eq. (5. 32) successively , we get remaining iterations as follows

$$
\begin{aligned}
u_1(t) &= 0.3202037589\, t^{9.5} + 0.1\, t^{10} - 0.0094003542\, t^{11.5} + 0.0030303030\, t^{12} \\
u_2(t) &= 0.320204t^{9.5} - 0.0000173399t^{14.5} - 0.000197472t^{13.5} - 0.00169715t^{12.5} \\
&\quad - \text{5.724587470723463*^-17} t^{11.5} - 0.0304956t^{10.5} + 0.0000590228t^{14.} \\
&\quad - 0.000344767t^{12} - 0.1t^{10.} + 0.0000590228t^{14} + 0.00571584t^{12} + 0.1t^{10} \\
u_3(t) &= 0.3202037589\, t^{9.5} + 0.0999999999\, t^{10} + 2.28983 \times 10^{-16}\, t^{10.5} \\
&\qquad + 2.18636 \times 10^{-18}\, t^{11.5} + 0.0030303030\, t^{12} + 6.93817 \times 10^{-18}\, t^{12} \\
&\qquad + 0.0004662004\, t^{13} - 6.42389 \times 10^{-18}\, t^{13.5} + 0.0001046151\, t^{14} \\
&\quad - 0.0000614684\, t^{14.5} + 5.55001 \times 10^{-6}\, t^{15} + 6.35364 \times 10^{-22}\, t^{16} - \cdots , \\
u_4(t) &= 0.3202037589\, t^{9.5} - \text{small terms} \\
&\vdots \\
u_n(t) &= 0.3202037589\, t^{9.5} - \text{small terms.}
\end{aligned}
$$

For $n \to \infty$, the small terms are neglected. It is observed that, the convergence occurred at $u(t) = 0.3202037589\, t^{9.5}$ i.e.

$$u(t) = \lim_{n\to\infty} u_n(t) = 0.3202037589\, t^{9.5.} \tag{5. 35}$$

and the exact solution of example Eq. (5. 30) is

$$x(t) = D_t^{0.5} u(t) = \frac{1}{3} t^9. \tag{5. 36}$$

It can be shown that $x(t) = \frac{1}{3} t^9$ is the exact solution of Bageley-Torvik IVP.

Example 5.2. *Here, we consider the linear Cauchy IVP*

$$\begin{cases} D_t^{\alpha} x(t) + 2D_t x(t) + 3\sqrt{t} D_t^{\beta} x(t) + (1-t)x(t) = 4t + \dfrac{2}{\Gamma(1.5)} t^{0.5} + \dfrac{4}{\Gamma(1.5)} t^2 + (1-t)t^2 \quad ; \\ \\ x(0) = x'(0) = 0 \ \alpha \in (1,2),\ \beta \in (0,1). \end{cases} \tag{5. 37}$$

If we take $\alpha = 1.5,\ \beta = 0.5$ and $x(t) = D_t^{0.5} u(t)$ then we get

$$D_t^2 u(t) + 2D_t^{1.5} u(t) + 3\sqrt{t} D_t u(t) + (1-t) D_t^{0.5} u(t) = 4t + \frac{2}{\Gamma(1.5)} t^{0.5} + \frac{4}{\Gamma(1.5)} t^2 + (1-t)t^2. \tag{5. 38}$$

The iteration formula to find solution of Eq. (5. 38) is

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\varphi(s) ds, \tag{5. 39}$$

where

$$\varphi(s) = D_s^2 u_n(s) + 2D_s^{1.5} u_n(s) + 3\sqrt{s} D_s u_n(s) + (1-s) D_s^{0.5} u_n(s) - 4s - \frac{2}{\Gamma(1.5)} s^{0.5} - \frac{4}{\Gamma(1.5)} s^2 - (1-s)s^2. \quad (5.\ 40)$$

we take initial approximation as

$$u_0(t) = 0. \quad (5.\ 41)$$

By using the iteration formula Eq. (5. 39) successively , we get remaining iterations as follows

$$u_1(t) = 0.6018022225\, t^{2.5} + 0.6666666666\, t^3 - 0.2927930556\, t^4 + 0.05\, t^5$$

$$\begin{aligned} u_2(t) = {} & 0.6018022225\, t^{2.5} + 0.6666666666\, t^3 - 0.6877739685\, t^{3.5} - 0.2927930556\, t^4 \\ & + 0.2685005891\, t^{4.5} - 0.05\, t^5 + 0.0416832708\, t^{5.5} + 0.00375525299\, t^{6.5} \\ & - 0.0023263314\, t^{7.5} - 0.0003269276\, t^{8.5} \end{aligned}$$

$$\begin{aligned} u_3(t) = {} & 0.6018022225\, t^{2.5} + 0.6666666666\, t^3 - 0.0007161446\, t^{3.5} - 0.2927930556\, t^4 \\ & + 0.9939750309\, t^{4.5} + 0.4425153192\, t^5 + 0.2379420042\, t^{5.5} - 0.2343482417\, t^6 \\ & - 0.0000288932\, t^{6.5} - 0.0486537838\, t^7 - 0.0057357154\, t^{7.5} - 0.0008616721\, t^8 \\ & - 0.0004864794\, t^{8.5} + 0.0011734071\, t^9 + 0.00004853176\, t^{9.5} \\ & + 0.0009262949\, t^{10} - \text{small terms}, \end{aligned}$$

$$u_4(t) = 0.6018022225\, t^{2.5} - \text{small terms}$$

$$\vdots$$

$$u_n(t) = 0.6018022225\, t^{2.5} \ - \text{small terms}.$$

For $n \to \infty$, the small terms are neglected. It is observed that, the convergence occurred at $u(t) = 0.6018022225\, t^{2.5}$ i.e.

$$u(t) = \lim_{n\to\infty} u_n(t) = 0.6018022225\, t^{2.5}. \quad (5.\ 42)$$

and the exact solution of (5. 37) is

$$x(t) = D_t^{0.5} u(t) = t^2. \quad (5.\ 43)$$

Example 5.3. *The IVP consider here as follows*

$$\begin{cases} D_t^{\alpha} x(t) + D_t^2 x(t) + (D_t^{\beta} x(t))^2 + (x(t))^3 = 2t \ + \frac{2\, t^{3-\alpha}}{\Gamma(4-\alpha)} \ + \left(\frac{2\, t^{3-\beta}}{\Gamma(4-\beta)}\right)^2 + \left(\frac{t^3}{3}\right)^3 \quad ; \\ x(0) = x'(0) = 0 \ \alpha \in (1,2),\ \beta \in (0,1). \end{cases} \quad (5.\ 44)$$

If we take $\alpha = 1.5,\ \beta = 0.5$ and $x(t) = D_t^{0.5} u(t)$ then

$$D_t^2 u(t) + D_t^{2.5} u(t) + (D_t u(t))^2 + (D_t^{0.5} u(t))^3 = 2t \ + \frac{2\, t^{1.5}}{\Gamma(2.5)} \ + \left(\frac{2\, t^{2.5}}{\Gamma(3.5)}\right)^2 + \left(\frac{t^3}{3}\right)^3. \quad (5.\ 45)$$

The iteration formula to find solution of Eq. (5. 45) is

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\varphi(s)ds, \tag{5. 46}$$

where

$$\varphi(s) = D_s^2 u(s) + D_s^{2.5} u(s) + (D_s u(s))^2 + (D_s^{0.5} u(s))^3 - 2s - \frac{2\, s^{1.5}}{\Gamma(2.5)} - \left(\frac{2\, s^{2.5}}{\Gamma(3.5)}\right)^2 - \left(\frac{s^3}{3}\right)^3. \tag{5. 47}$$

we take initial approximation as

$$u_0(t) = 0. \tag{5. 48}$$

By using the iteration formula Eq. (5. 46) successively , we get remaining terms as follows

$$u_1(t) = 0.1719434921\, t^{3.5} + 0.3333333333\, t^3 + 0.0086229979\, t^7 + 0.0003367003\, t^{11}$$

$$u_2(t) = 0.1719434921\, t^{3.5} - 0.6018022224\, t^{2.5} - 0.0333333333\, t^6 - 0.0568922724 t^{6.5}$$
$$-0.0026990991 t^{9.5} - 0.0053654209\, t^{10} - 0.0027393633\, t^{10.5} - 0.0001495343\, t^{13.5}$$
$$-0.0000407000\, t^{14} - \text{small terms}$$

$$u_3(t) = 0.1719434921\, t^{3.5} + 0.3333333333\, t^3 - 0.1131768484\, t^5 + 0.0833665416\, t^{5.5}$$
$$+0.2082219716\, t^6 - 0.0094400348\, t^{8.5} - 0.0070252884\, t^9 + 0.0201613432\, t^{9.5}$$
$$+0.0139282901\, t^{10} - \text{small terms}$$

$$u_4(t) = 0.1719434921\, t^{3.5} - \text{small terms}$$

$$\vdots$$

$$u_n(t) = 0.1719434921\, t^{3.5} \; - \text{small terms}$$

For $n \to \infty$, the small terms are neglected. It is observed that, the convergence occurred at $u(t) = 0.1719434921\, t^{3.5}$ i.e.

$$u(t) = \lim_{n\to\infty} u_n(t) = 0.1719434921\, t^{3.5}. \tag{5. 49}$$

and the exact solution of Eq. (5. 44) is

$$x(t) = D_t^{0.5} u(t) = \frac{1}{3} t^3. \tag{5. 50}$$

Example 5.4. *Consider the following IVP*

$$\begin{cases} D_t^{\alpha} x(t) + D_t^{\beta} x(t) D_t^{\gamma} x(t) = \dfrac{\Gamma(4)\, t^{0.5}}{\Gamma(1.5)} + \dfrac{(\Gamma(4))^2\, t^4}{\Gamma(1.5)\Gamma(1.5)} \\ x(0) = x'(0) = x''(0) = 0 \;\alpha \in (2,3),\; \beta \in (1,2),\; \gamma \in (0,1). \end{cases} \tag{5. 51}$$

Taking $\alpha = 2.5,\ \beta = 1.5,\ \gamma = 0.5$, and $x(t) = D_t^{0.5}u(t)$ then

$$D_t^3 u(t) + D_t^2 u(t) D_t u(t) = \frac{\Gamma(4)\ t^{0.5}}{\Gamma(1.5)} + \frac{(\Gamma(4))^2\ t^4}{\Gamma(1.5)\Gamma(1.5)}. \tag{5. 52}$$

The iteration formula to find solution of Eq. (5. 52) as

$$u_{n+1}(t) = u_n(t) - \int_0^t \frac{(t-s)^2}{2}\varphi(s)ds, \tag{5. 53}$$

where

$$\varphi(s) = D_s^3 u(s) + D_s^2 u(s) D_s u(s) = \frac{\Gamma(4)\ s^{0.5}}{\Gamma(1.5)} + \frac{(\Gamma(4))^2\ s^4}{\Gamma(1.5)\Gamma(1.5)} \tag{5. 54}$$

we take initial approximation as

$$u_0(t) = 0. \tag{5. 55}$$

By using the iteration formula Eq. (5. 53) successively , we get remaining terms as follows

$$u_1(t) = 0.5158304763\ t^{3.5} + 0.0582052363\ t^7$$

$$u_2(t) = 0.5158304763t^{3.5} + 0.0582052363t^7 - 0.0388034908t^{7.} - 0.0073743243t^{10.5} - \cdots$$

$$u_3(t) = 0.5158304763t^{3.5} + 0.0582052363t^7 - 0.0388034908t^{7.} - 0.0024581081t^{10.5}$$
$$-0.0004560566t^{14} + 0.0011734791\ t^{14.} + 0.0000763398\ t^{17.5} - 0.0000112666\ t^{21} - \cdots$$

$$u_4(t) = 0.5158304763\ t^{3.5} + 0.0582052363t^7 - 0.0388034908t^{7.} - 0.0024581081t^{10.5}$$
$$-0.0004560566t^{14} - 0.0000506598\ t^{17.5} - 0.0000159773\ t^{21} - \text{small terms}$$

$$\vdots$$

$$u_n(t) = 0.5158304763\ t^{3.5}\ - \text{small terms}$$

For $n \to \infty$, the small terms are neglected. It is observed that, convergence occurred at $u(t) = 0.5158304763\ t^{3.5}$ i.e.

$$u(t) = \lim_{n\to\infty} u_n(t) = 0.5158304763\ t^{3.5}. \tag{5. 56}$$

and the exact solution of example Eq. (5. 51) is given as

$$x(t) = D_t^{0.5}u(t) = t^3. \tag{5. 57}$$

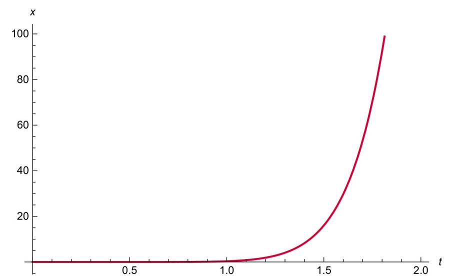

FIGURE 1. Plot for approx. sol of Ex.5.1

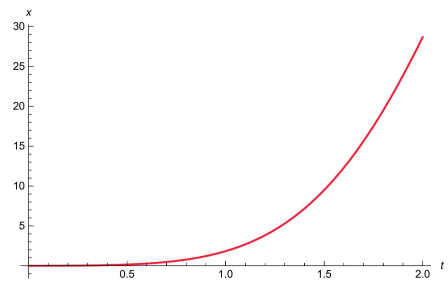

FIGURE 2. Plot for approx. sol of Ex.5.2

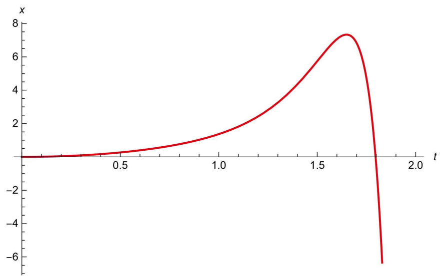

FIGURE 3. Plot for approx. sol of Ex.5.3

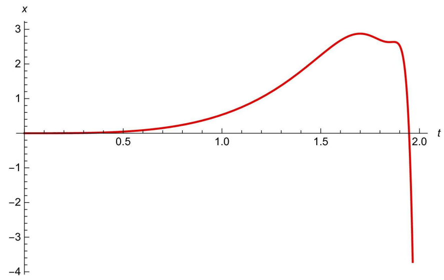

FIGURE 4. Plot for approx. sol of Ex.5.4

Fig. 1 to Fig. 4 shows the 2d plot behavior respectively of fourth term series solution of Examples 5.1 to 5.4 for parameter α and $0 \leq t \leq 2$. using Variational iteration method which infers that VIM can foresee the conduct of said variables precisely for the considered region. It is observed that all the curves of approximate solution are exactly similar with the curves of exact solutions. The figures uncovers that a difference in the esteem influences the dynamics of the MFDE. The non-integer order has negligible effect in the dynamics of the MFDE.

6. CONCLUSIONS:

In this paper, we have applied VIM to get an approximate solution of multi-term nonlinear fractional differential equation successfully, where we overcome the difficulty of converting MFDEs into a system of equations and obtaining solutions. It has been found in the present study that the approximate solutions of MFDEs can be effectively obtained with nonlocal fractional operator such as the Caputo one. Effectively in a sense that the Caputo fractional differential equation proposed here is shown to have better fit in comparison to the equation modeled with local classical derivatives. The convergence and theorem for error approximation are also given. This approach is a direct method to solve the multi-term linear or nonlinear fractional differential equations without any limiting norms and high computations. To demonstrate the tremendous performance of the suggested method, the numerical examples are simulated. Therefore, the present technique is an efficient mathematical tool for many researchers working in the field of applied sciences and engineering to study the solutions of multi-term linear or nonlinear fractional differential equations. In the future work, some newly proposed novel nonlocal fractional operators such as Caputo-Fabrizio, Atangana-Gomez, Atangana-Baleanu and fractal-fractional having memory effects will be tested for thorough investigation of nonlinear mathematical models of fractional differential equations.

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