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Examining Higher Order Aberration in Eyes after DSEK

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Abstract. The fundamental idea of this work is to modify Descemet's Stripping Endothelial Keratoplasty DSEK mathematical model by incorporating the Zernike polynomial to examine Higher Order Aberrations (HOA). This model has been developed in a way that can identify the type of aberration occurring after a keratoplasty. Model variables are compared with the data available in already published literature. Surgically induced Higher Order Aberrations (HOA) and change in corneal power are the main outcome measures. Eye illness and their treatments have always been an interesting topic of discussion for researchers. Mechanical and software engineers have managed to helped ophthalmologists by developing advanced machines that can identify the patients' complaints in seconds. Moreover, several new and advanced surgical procedures have been adopted for eye problems. One amongst them is eye aberration. Only recently has it become possible to detect one type of aberration (Astigmatism & Higher Order Aberrations) with special equipments. However, it is still not easy to identify aberrations as Myopia and Hyperopia. This paper studies the induced eye aberrations caused by DSEK and their detection through a modified DSEK model.

AMS Subject Classification Codes: 00A71; 34Axx; 34C60; 65D17; 33C47
Key Words: Mathematical Model; Zernike Polynomial; Aberration; Stability; Differential Equation.

1. Introduction

Eye illnesses and their treatments have always been an exciting topic of discussion for researchers. Mechanical and software engineers have helped ophthalmologists by developing advanced machines that can diagnose a patient's ailment within seconds. Moreover, several new and advanced surgical procedures have been adopted for various eye problems. One among them is eye aberration. To be specific, when the components of the eye that are involved in refraction, the lens and cornea, have some defect, it prevents light rays from being focused at a single point and results in a distorted or blurred image. This situation is called "Eye Aberration". "Chromatic Aberration" is the one in which distortion of color occurs. There are normally two kinds of eye aberration: "Lower-Order Aberration" caused by abnormal shaped cornea includes nearsightedness (myopia), farsightedness (hyperopia) and astigmatism; "Higher-Order Aberration" (HOA) are more complex then lower ones, caused mainly by the abnormal curvature of the cornea or crystalline lens. Such aberrations can also occur due to scarring of cornea. There are some other highest order aberrations. Among them, only two spherical aberrations, such as trefoil and also coma, are of very much clinical importance in this work.

The cause of increase in myopia might be the spherical aberration after LASIK especially at night time. It also might result in photos of points surrounding halos. Spherical aberration exacerbates low-light myopia (night-time myopia). The pupil constricts under brighter conditions, absorbing the more distant rays and reducing the spherical aberration effect. Many peripheral rays penetrate the eye when the pupil enlarges and the concentration changes anteriorly, hence under low-light settings the individual gets even more myopic. Coma is normal in patients recovering from poor corneal grafts, keratoconus and good laser ablations. Trefoil causes fewer picture content loss relative to coma.

Among total aberrations of the eye, higher-order aberrations comprises of only 10% of these aberrations, and these aberrations increase with ageing.

There are usually three types of aberrations hyperopia, myopia i.e. positive defocus, and the normal type of astigmatism called low-order aberrations. Few of the first order aberrations are very insignificant and do not effect the vision process at all. They are commonly named as zero order aberration piston and prisms. Approximately 90% aberrations of an eye are low-order aberrations.

Measuring the aberrations of the eye has recently become possible and is mostly treated through refractive surgeries. Many researchers have studied various cases and treatments related to aberrations of eye. Jason et al. [13] studied 109 normal human subjects and analyzed that Zernike modes are efficient basis functions for describing the eye's wave aberration. Another group of researchers [10] studied optical aberration measurements using infrared (787 nm) and visible light (543 nm) in a heterogeneous group of subjects to assess whether aberrations are similar in both wavelengths and to estimate experimentally the ocular chromatic focus shift. Sometimes eye aberrations can also occur as a side effect of some refractive surgery. As some ophthalmologists [12] discovered symptomatic postoperative laser refractive surgery with patients having irregular corneas have higher-order aberrations that are 2.3 to 3.5 times greater than asymptomatic postoperative Laser-Assisted In-Situ Keratomileusis (LASIK) and normal preoperative eyes, respectively. The higher-order aberrations seem to correlate with corneal topography. Kohnen et al. [9] determined that Myopic and hyperopic LASIK had different patterns of HOA induction. Myopic LASIK induced positive spherical aberrations and positive secondary astigmatism, whereas hyperopic LASIK induced negative spherical aberrations and negative secondary astigmatism. Hyperopic LASIK induced more third- and fifth-order coma like aberrations than myopic LASIK.

Queirós et al. [14] analyzed the three techniques that increase the wavefront aberrations of the cornea and change the relative contribution of coma like and spherical like aberrations

among them corneal refractive therapy induces more spherical like aberrations than standard and custom LASIK.
Goldstone et al. [3] compared the results of contact lens corneal refractive therapy and LASIK to analyze that both can effectively correct myopia, but both these techniques increase higher order aberrations to a similar degree for 6mm pupils. However, the statistics in this paper show that spherical aberrations were significantly higher after Corneal Refractive Therapy (CRT) than after LASIK. Barbero et al. [2] studied ocular and corneal aberrations, before and after cataract surgery, in a group of eyes with spherical Intraocular Lens (IOL) and observed that pseudophakic eyes, healthy eyes having aberration problems, and eyes before cataract surgery of the same age. They noticed that all of them have same type of aberrations. However, aberrations in pseudophakic eyes are significantly higher in younger people, they found a slight increase of corneal aberrations after surgery.
In this work, by modifying a mathematical model, higher-order aberrations are studied as a consequence of Endothelial Keratoplasty. Since such aberrations can also be caused by scarring of the cornea, the best way to describe them mathematically is Zernike Polynomials [11]. Zernike's has the ability to express wave front data in polynomial form. Therefore, the term describing aberrations occurring in vision is represented by Zernike polynomial in this work.
L. Vasudevan and F. Andre [19] provided the detailed mathematical formulation and discussion of Zernike polynomials. Their work comprises of mathematical basis, orthonormality, recurrence relations, relation with other polynomial sequences, wavefront error and transformation of Zernike polynomials. Also, algebraic expansion of sequence orders one through ten in the form of tables is provided. Most interestingly, different forms of pupil such as scaled, translated, rotated and elliptical are also deliberated upon.
Iskander et al. [5] considered a set of orthogonal basis functions formed by the product of angular functions and radial polynomials to model the anterior corneal surface analytically. Aberrations are mostly defined by the Zernike polynomials, they describe these shapes as the sum of simplicial shapes or as the shape of base functions. Zernike polynomials can be of any order but mostly for aberrations first fifteen polynomials are considered. These orders describe the aberration either positive or negative and will show constantly changing image quality. Few examples of such aberrations are trefoil, spherical aberration, coma, piston etc.
Zernike polynomials are written as Z_n^m in polar coordinates (ρ, θ) where n is the order of aberration, m gives the angular frequency means the number the wavefront pattern replicates himself, ρ is the radial distance for a particular pupil and θ shows the angle.
The main reasons for using Zernike polynomials instead of using any other set of orthogonal polynomials are:

(1) Zernike's are orthogonal in the interior of a unit circle, only for continuous form. They will not be orthogonal for discrete type within a unit circle [15].
(2) Zernike's possess radial symmetry which leads to a product function $R(\rho)G(\theta^{'})$, where $G(\theta^{'})$ is a continuous periodic function, defined as $G(\theta^{'}) = e^{\pm im\theta^{'}}$.
(3) Radial function always has to be a polynomial ρ of degree n and contain no power of ρ less than m i.e. $n \geq m$.

(4)

$$z_n^m(\rho,\phi) = \begin{cases} R_n^m(\rho) \text{ is odd} & \text{if } m \text{ is odd} \\ R_n^m(\rho) \text{ is even} & \text{if } m \text{ is even} \end{cases}$$

$$Zernike\ Polynomials = z_n^m(\rho,\phi) = \begin{cases} Z_n^m(\rho,\phi) = R_n^m(\rho)\cos(m\phi) & for\ m \geq 0 \\ Z_n^{-m}(\rho,\phi) = R_n^m(\rho)\sin(m\phi) & for\ m < 0 \end{cases}$$

where

$$R_n^m(\rho) = \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^k (n-k)!}{k!(\dfrac{n+m}{2}-k)!(\dfrac{n-m}{2}-k)!}\rho^{n-2k}$$

$\forall m, n > 0$ with $n \geq m, \phi$ is the azimuthal angle, ρ is the radial distance $0 \leq \rho \leq 1$.

For one-dimensional plane such as cornea the radial polynomials are given as

TABLE 1. Zernike radial polynomials to be utilized in modified DSEK mathematical model.

m	n	Zernike Radial Polynomial
0	0	$R_n^m(\rho) = 1$
1	1	$R_n^m(\rho) = \rho$
0	2	$R_n^m(\rho) = 2\rho^2 - 1$
2	2	$R_n^m(\rho) = \rho^2$
1	3	$R_n^m(\rho) = 3\rho^3 - 2\rho$
3	3	$R_n^m(\rho) = \rho^3$
0	4	$R_n^m(\rho) = 6\rho^4 - 6\rho^2 + 1$
2	4	$R_n^m(\rho) = 4\rho^4 - 3\rho^2$
4	4	$R_n^m(\rho) = \rho^4$

2. MODIFIED DSEK MATHEMATICAL MODEL

Khalid & Fareeha [8] developed a DSEK mathematical model through the consideration of relationships of ocular parameters among themselves to analyze the behaviour of ocular parameters after DSEK. Also the change in refractive power of eye through DSEK model. Now, in this paper, a modified DSEK model is developed to find out the reason of the induction of HOA after DSEK. The facts for the formation of modified DSEK model are the same as Khalid & Fareeha [8] suggested. In this work, it is assumed that the vision of eye from the front side mainly depends on four parameters, refractive index $p(t)$, axial length $q(t)$, corneal curvature $r(t)$ and central corneal thickness $s(t)$. Moreover, the radial polynomials are considered here as cornea is considered to be a one dimensional plane for simulation. Their governing differential equations are therefore given as

$$\begin{aligned} p'(t) =& \alpha\frac{r(t)}{s(t)} + \gamma q(t) + \delta + R_n^m(\rho) \\ q'(t) =& -\delta - \beta\frac{r(t)}{s(t)} - \gamma q(t) - s(t)q(t) \\ r'(t) =& -\beta\frac{r(t)}{s(t)} - q(t) - s(t)p(t) \\ s'(t) =& s(t)p(t) + s(t)q(t) + q(t) - R_n^m(\rho) \end{aligned} \tag{2.1}$$

The dynamics of the eye after Descemet's Stripping Endothelial Keratoplasty can be observed through the following equation $v(t) = p(t) + q(t) + r(t) + s(t)$. Change in this dynamics can be observed from the following equation

$$\frac{d}{dt}v(t) = \frac{d}{dt}p(t) + \frac{d}{dt}q(t) + \frac{d}{dt}r(t) + \frac{d}{dt}s(t)$$

$$\frac{d}{dt}v(t) = (\alpha - 2\beta)\frac{r(t)}{s(t)}$$

If $\alpha = 2\beta$ then $\frac{d}{dt}v(t) = 0$ and hence show no change in vision after DSEK. In this work, let us consider that $\alpha > 2\beta$. In order to understand the behaviour of this system presume a set ω for modified DSEK mathematical model with initial conditions

$$\omega = \{(p, q, r, s)|p = p_\circ > 0, q = q_\circ > 0, r = r_\circ \geq 0, s = s_\circ > 0\}. \tag{2.2}$$

Consider an autonomous system $\frac{dv}{dt} = G(v), \quad G : D \subset R^n \rightarrow R^n$. For modified DSEK mathematical model Eq.2.1 be

$$\frac{dv}{dt} = F(t, v(t)) \tag{2.3}$$

Where F and G are locally Lipschitz in $v \in R$ and their solution exists $\forall t > 0$. If $F(t, v(t)) \rightarrow G(v)$ as $t \rightarrow \infty$ uniformly for $v \in D$ then system in Eq.2.3 is said to be asymptotically autonomous with limits system of Eq.2.2.

Lemma 2.1. *Suppose F and G are locally Lipschitz in $v \in D$ see [4,6]. If any solutions of system Eq.2.1 are bounded and the equilibria E_{equil} of Eq.2.2 is globally asymptotically stable, then any solution of $v(t)$ of system in Eq.2.2 such that $lim_{t\rightarrow\infty}v(t) = E_{equil}$*

Theorem 2.2. *All feasible solution $v = v(t) = \Big(p(t), q(t), r(t), s(t)\Big)$ of DSEK model in Eq. (2.1) are bounded and enters the region by $\omega = \{(p, q, r, s)|p = p_\circ > 0, q = q_\circ > 0, r = r_\circ \geq 0, s = s_\circ > 0\}$ if and only if $\alpha = 2\beta$ then $v(t)$ is closed and bounded on region Ω.*

Proof. Given $v = \Big(p(t), q(t), r(t), s(t)\Big)$, then $v(t) = F\Big(p(t), q(t), r(t), s(t)\Big)$. On differentiating it becomes $\dfrac{dv}{dt} = \dfrac{d}{dt}(p+q+r+s)$

$$\begin{aligned}\frac{dv}{dt} = \frac{d}{dt}\Big(&\alpha\frac{\int r(t)dt}{s(t)} + \alpha\int\frac{\int r(t)dt}{s(t)^2}dt + \gamma\int q(t)dt + (\delta + R_n^m(\rho))t - \beta\frac{\int r(t)dt}{s(t)} - \\ &\beta\int\frac{\int r(t)dt}{s(t)^2}dt - \int q(t)dt - s(t)\int p(t)dt + \int s'(t)\Big(\int p(t)dt\Big)dt - \\ &\delta t - \beta\frac{\int r(t)dt}{s(t)} - \beta\int\frac{\int r(t)dt}{s(t)^2}dt - \gamma\int q(t)dt - s(t)\int q(t)dt + \\ &\int s'(t)\Big(\int q(t)dt\Big)dt + s(t)\int p(t)dt - \int s'(t)\big(p(t)dt\big)dt + \\ &s(t)\int q(t)dt + \int s'(t)\Big(\int q(t)dt\Big)dt + \int q(t)dt - R_n^m(\rho)t\Big)\end{aligned} \tag{2.4}$$

$$\frac{dv}{dt} = (\alpha - 2\beta)\frac{r(t)}{s(t)} \tag{2.5}$$

Since $\alpha, \beta, \gamma, \delta > 0$, the condition $\alpha > 2\beta$ is proved to be compulsory for closed and bounded region. In this case practically $s(t) \neq 0$ so this rational function is continuous everywhere. therefore it is established that

$$\frac{d}{dt}(p+q+r+s) = (\alpha - 2\beta)\frac{r(t)}{s(t)} \tag{2.6}$$

It is deduced that Eq. (2.1) is bounded. □

3. Qualitative and Stability Analysis

For the qualitative analysis, by taking derivative terms of modified DSEK mathematical model equal to zero and upon solving Eq.2.1 and considering $\alpha > 2\beta$ gives the only equilibrium point as $\Big(\dfrac{(R_n^m(\rho)+\delta)^2}{R_n^m(\rho)\gamma^2}, \dfrac{(R_n^m(\rho)+\delta)}{\gamma}, 0, \dfrac{R_n^m(\rho)\gamma}{R_n^m(\rho)+\delta}\Big)$. Modified DSEK model cannot have a trivial equilibrium $(0,0,0,0)$ since physically this trivial equilibrium has no meaning for Eq.2.1. Jacobian matrix for system Eq.2.1 given as

$$\frac{\partial(f,g,h,i)}{\partial(p,q,r,s)} = \begin{bmatrix} 0 & \gamma & \dfrac{\alpha}{s(t)} & -\dfrac{\alpha r(t)}{s(t)^2} \\ 0 & -\gamma - s(t) & -\dfrac{\beta}{s(t)} & \dfrac{\beta r(t)}{s(t)^2} - q(t) \\ -s(t) & -1 & -\dfrac{\beta}{s(t)} & \dfrac{\beta r(t)}{s(t)^2} - p(t) \\ s(t) & 1 + s(t) & 0 & p(t) + q(t) \end{bmatrix} \tag{3.7}$$

Theorem 3.1. *System 2.1 has* $\Big(\dfrac{(R_n^m(\rho)+\delta)^2}{R_n^m(\rho)\gamma^2}, \dfrac{(R_n^m(\rho)+\delta)}{\gamma}, 0, \dfrac{R_n^m(\rho)\gamma}{R_n^m(\rho)+\delta}\Big)$ *as a locally stable and globally asymptotically stable equilibrium if and only if Routh Hurwitz criteria is satisfied otherwise* E *is an unstable equilibrium.*

Proof. The local stability of this equilibrium solution can be given by linearizing the system 2.1 around $\left(\frac{(R_n^m(\rho)+\delta)^2}{R_n^m(\rho)\gamma^2}, \frac{(R_n^m(\rho)+\delta)}{\gamma}, 0, \frac{R_n^m(\rho)\gamma}{R_n^m(\rho)+\delta}\right)$ then by using parameters values given in Table 2 and equilibrium point, the Jacobian becomes

$$J_{Eq} = \begin{bmatrix} 0 & 3.32 & -30.120 - \frac{0.451}{R_n^m(\rho)} & 0 \\ 0 & -\frac{0.0498}{0.015+R_n^m(\rho)} & 15.060 + \frac{0.225}{R_n^m(\rho)} & 0.0045 + 0.301 R_n^m(\rho) \\ \frac{3.32 R_n^m(\rho)}{0.015+R_n^m(\rho)} & -1 & 15.060 + \frac{0.225}{R_n^m(\rho)} & 0.0027 + \frac{0.0000204}{R_n^m(\rho)} + 0.09072 R_n^m(\rho) \\ -\frac{3.32 R_n^m(\rho)}{0.015+R_n^m(\rho)} & 1 - \frac{3.32 R_n^m(\rho)}{0.015+R_n^m(\rho)} & 0 & -0.00723 - \frac{0.0000204}{R_n^m(\rho)} - 0.39192 R_n^m(\rho) \end{bmatrix} \tag{3.8}$$

The obtained characteristic polynomial is given as

$$x^4 + ax^3 + bx^2 + cx + d$$

where

$$a = \frac{1}{R_n^m(\rho)\gamma^2(R_n^m(\rho)+\delta)}\Big(R_n^m(\rho)^2(1+\gamma) + \delta^2(-\beta\gamma+\delta) + R_n^m(\rho)\delta$$
$$(-2\beta\gamma + \gamma^2 + 3\delta + \gamma\delta) + R_n^m(\rho)^2(-\beta\gamma + (3+2\gamma)\delta)\Big)$$
$$b = \frac{1}{R_n^m(\rho)^2\gamma^3(R_n^m(\rho)+\delta)}\Big(- (R_n^m(\rho)+\delta)\big(R_n^m(\rho)^2(\beta + \beta\gamma + \gamma^2 - \gamma^3) + \beta\delta^2 -$$
$$R_n^m(\rho)^2\big(\gamma^2(\alpha+\delta) + \beta(\gamma^2 - 3\delta - 2\gamma\delta)\big) + R_n^m(\rho)\delta\big(- \gamma^2\delta + \beta(-\gamma^2 + \gamma^3 + 3\delta + \gamma\delta)\big)\Big)$$
$$c = \frac{1}{R_n^m(\rho)^2\gamma^2(R_n^m(\rho)+\delta)}\Big(R_n^m(\rho)^4(3\beta + \alpha\gamma - \beta\gamma + \gamma^2) - R_n^m(\rho)\beta\gamma\delta^2 - \beta\delta^4 +$$
$$R_n^m(\rho)^3\big(\gamma(2\alpha+\gamma^2)\delta - \beta(\gamma^3 - 8\delta + 3\gamma\delta)\big) + R_n^m(\rho)^2\delta\big(\alpha\gamma(\gamma^2+\delta) - \beta(\gamma^3 - 6\delta + 3\gamma\delta)\big)\Big)$$
$$d = (\alpha - 2\beta)(R_n^m(\rho)+\delta)$$

By applying Routh Hurwitz stability criteria from [7] on the characteristic polynomial equation, stability of modified DSEK system 2.1 can be determined as follows

$$\begin{array}{llll} S^4 & : & 1 & b & d \\ S^3 & : & a & c & 0 \\ S^2 & : & \frac{ab-c}{a} & d & \\ S^1 & : & c - \frac{a^2 d}{ab-c} & 0 & \\ S^\circ & : & d & 0 & \end{array}$$

As Routh Hurwitz criteria states that for a system to be stable the first column of table must have positive terms or similar sign terms therefore modified DSEK mathematical model will only be stable if $\frac{a^2d}{ab-c} < c < ab$ otherwise unstable. Also for the stability of system requires $ab \neq c$. Otherwise modified DSEK mathematical model is unstable in region elsewhere. □

Therefore, the result stated in Lemma 2.1 is proved. After mathematically proving the stability, uniqueness and existence of the modified DSEK mathematical model, it is now necessary for readers other than mathematicians to demonstrate the applicability and effectiveness of this model.

4. NUMERICAL SIMULATION

For numerical simulation, accurate data collection is most important aspect. Hence, we will examine this model for two different ethnic populations i.e. Hispanic and Chinese. To verify the results of eye surgery on Hispanic population, data have been collected from previously published research papers see Table 2. In San Francisco glaucoma clinic from June 2002 to April 2004 around 116 Hispanic race eyes were studied Aghaian et al. [1] for central corneal thickness and calculated to be 548.1μmm but for ease of our calculation we converted it into 0.548mm. Twelker et al. [18] analyzed 4881 eyes among them was 22.9 % were Hispanic and among those results we picked refractive power of Hispanic eyes as 7.62 mm, axial length as 23.99 mm and 7.78 mm as corneal curvature.

To verify the results of eye surgery on Chinese population, data has been collected

TABLE 2. Description of values for constants and variables according to Hispanic eyes [16, 17].

Constants & Variables	Values
α	100
β	50
γ	3.32 mm
δ	15 μm
p(0)	7.62 mm
q(0)	23.99 mm
r(0)	7.78mm
s(0)	0.5481mm

from previously published research papers see Table 3. For this model the mean value of axial length was taken from Yin et al. [20] because their research study involved 3468 persons in which 1963 were women approx 56.6% with a mean age of 64.6$\pm$9.8 years, the individuals age range was between 50 to 93 years. And as this study suggested the mean of axial length was 23.25$\pm$1.14mm. Since this study was only to observe the axial length so for the estimation of Corneal Curvature for Chinese race we looked into [21]. In San Francisco glaucoma clinic from June 2002 to April 2004 around 157 Chinese race eyes were studied by [1] for Central Corneal Thickness and calculated to be 555.6μm but for ease of our calculation we converted it into 0.5556mm.

Now, two cases of aberration will be discussed for both ethnic groups using the details in Table 2 and 3 and a thorough discussion of the results will be given. Note that for the simulation a Zernike radial polynomial will be used.

4.1. **Piston.** The first case of aberration is called Piston in this the Zernike polynomial considered for Zernike is $Z_n^m = 1$. Piston is also called the zeroth order Zernike polynomial. This surface is constant over the entire plane and hence show no error or variance.

TABLE 3. Description of values for constants and variables according to Chinese eyes [16, 17].

Constants & Variables	Values
α	100
β	50
γ	3.32 mm
δ	15 μm
p(0)	7.50 mm
q(0)	23.25 mm
r(0)	7.84mm
s(0)	0.5556mm

Usually among the Zernike polynomials Piston is ignored because it do not disturb the image quality but is used in the calculating the mean value of an ocular wavefront. Commonly this means a normal vision that show no aberration in vision. Hence the whole plane is of same colour and have no pattern in it.
Based on this description the simulation for Chinese population shows disturbance in vision and hence a huge oscillation can be seen in Fig.1 but after 3-4 days it become smooth and normal in one color plane. Color bar also depicts the zeroth order that is Piston at $n = 0$. This oscillation is due to the surgery impact on eye but as soon the eye heals completely the vision becomes normal.
Now, by simulating the modified DSEK model for Hispanic population shows disturbance in vision and hence a huge oscillation can be seen in Fig.2 but after 3-4 days it become smooth and normal in one color plane. Color bar also depicts the zeroth order that is Piston at $n = 0$. This oscillation is due to the surgery impact on eye but as soon the eye heals completely the vision becomes normal.

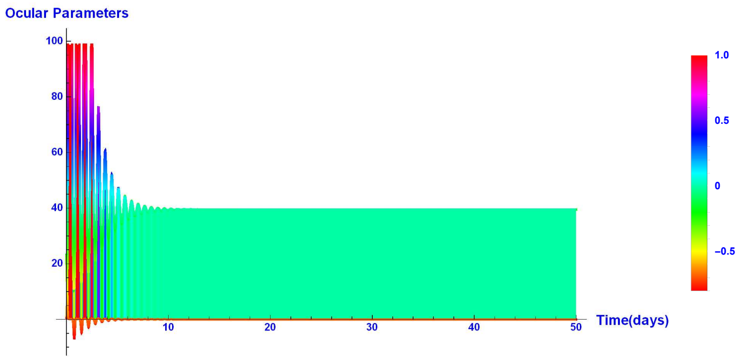

FIGURE 1. A graphical illustration of a modified DSEK model demonstrating the existence of Piston after DSEK in Chinese population eyes.

4.2. **Astigmatism.** Astigmatism is a popular issue in vision induced by a corneal-shaped defect. With astigmatism, there is an irregular curve to the lens of the eye or cornea, which is the front surface of the eye. This will adjust the way light travels to the retina, or refracts. This allows vision to be distorted, fuzzy or blurry. Basically in astigmatism the light ray do not focus on retina instead have two focal points this is why the image appears to be blurry. Further this astigmatism is divided into four cases. First one is Simple Hyperopic Astigmatism: in this astigmatism one focus line ends up on retina but the second focus line ends up behind the retina. Second case is of Simple Myopic Astigmatism: in this type of astigmatism one focus line is on retina whereas the second focus line ends in front of the retina. then comes the third case of astigmatism it is called Compound Hyperopic Astigmatism: in this case both focal lines ends up behind the retina and in the last case

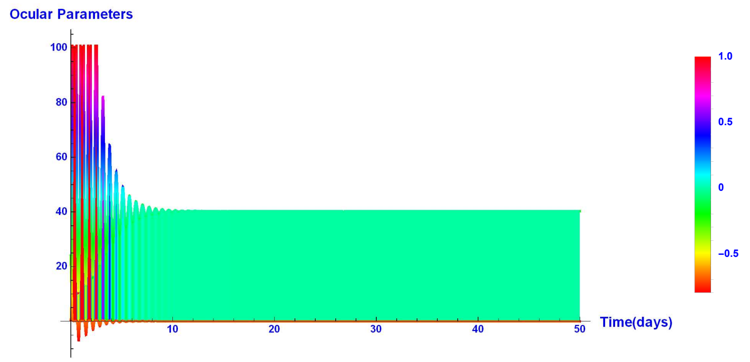

FIGURE 2. A simulated illustration of the modified DSEK model showing the appearance of Piston in the Hispanic population's eyes after DSEK.

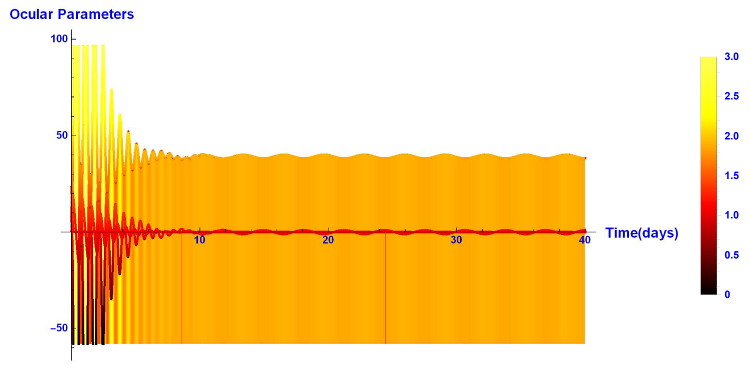

FIGURE 3. A graphical illustration of a modified DSEK model demonstrating the existence of Astigmatism after DSEK in Chinese population eyes.

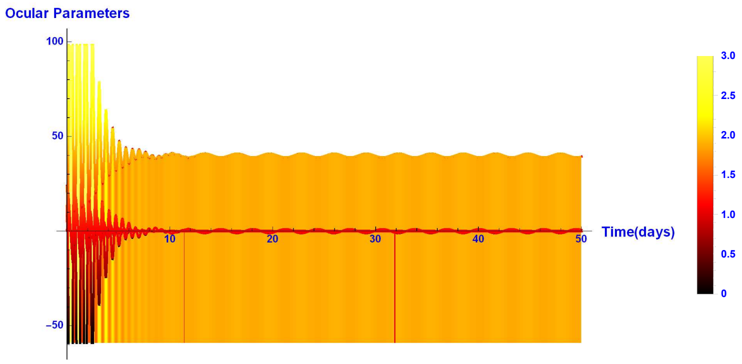

FIGURE 4. A simulated illustration of the modified DSEK model showing the appearance of Astigmatism in the Hispanic population's eyes after DSEK.

that is called Compound Myopic Astigmatism: both the focal lines ends up in front of the retina. Zernike polynomial which describes astigmatism is Z^2_{-2} and Z^2_2 so on colour scale it will be astigmatism if region has colour of $n = 2$.
In Fig.3 Chinese race is simulated by modified DSEK model for forty days time by using data in Table 2. Since astigmatism is blurred and fuzzy vision so this model depicts its vision in positive and negative plane. This yellow orangish area in Fig.3 describes the distorted vision and the vertical colour bar represents the order of aberration. Since astigmatism is a second order aberration which means $n = 2$ so the colour bar clearly shows the astigmatism range. Hence, the Chinese race has a fair chance of occurring astigmatism. Among four types of astigmatism it is Compound Myopic Astigmatism as two focal bars can be seen in the Fig.3.

In Fig.4 Hispanic race is simulated by modified DSEK model for forty days time by using data in Table 3. Since astigmatism is blurred and fuzzy vision so this model depicts its vision in positive and negative plane. This yellow orangish area in Fig.4 describes the distorted vision and the vertical colour bar represents the order of aberration. Since astigmatism is a second order aberration which means $n = 2$ so the colour bar clearly shows the astigmatism range. Hence, the Hispanic race has a fair chances of occurring astigmatism. Among four types of astigmatism it is Compound Myopic Astigmatism as two focal bars can be seen in the Fig.4.

5. CONCLUSION

In this paper, we modified the DSEK model successfully by introducing the Zernike polynomial in the model. The stability, uniqueness, and existence of this modified system is proved. This model is simulated using software to detect two types of aberrations in two ethnicities, Chinese and Hispanic. Data was collected from previous published research papers on DSEK.

The aim of this paper was to utilize a system of ordinary differential equations for detecting the types of aberration that occur after DSEK. Results show that by using this modified DSEK model, we can detect the aberration and its types in both ethnicities, Chinese and Hispanic. Both ethnicities can have normal vision or astigmatism. Similarly, by varying the Zernike polynomial we can detect all types of aberrations, whether they exist in a patient after DSEK or not.
In the future, these aberrations and their types can be explored further by modified DSEK mathematical model. Such as horizontal/vertical coma, horizontal/vertical tilt and horizontal/vertical trefoil etc.

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