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Midpoint type Fractional Integral Inequalities for convex and positive symmetric Increasing functions

Muhammad Amer Latif
Department of Basic Sciences,
Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia,
Email: m_amer_latif@hotmail.com; mlatif@kfu.edu.sa
Sabir Hussain
Department of Mathematics,
University of Engineering and Technology, Lahore, Pakistan
Email: sabirhus@gmail.com

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Abstract.: In this study, some midpoint type Hermite-Hadamard fractional integral inequalities and related results for a class of convex functions with respect to an increasing function incorporating a positive-weighted symmetric function generalizing some classical results are discussed.

AMS (MOS) Subject Classification Codes: Primary 26A33, 26A51; Secondary 26D10, 26D15
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1. Introduction

In the study and applications of fractional integral inequalities, convexity has great importance, particularly during the past couple of years. Mathematicians have observed a strongly intimate correlation between convexity and symmetry theories and many integral inequalities for convex functions have been formulated in literature [4, 5, 7, 8, 11, 16, 17]. The most common comprehensive are Hermite-integral Hadamard's inequalities:

$$\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\bar{\tau}}\int_{\bar{\tau}}^{\upsilon}\jmath(\alpha)\,d\alpha \leq \frac{\jmath(\bar{\tau})+\jmath(\upsilon)}{2}, \tag{1. 1}$$

where the function $\jmath : \hbar \rightarrow \mathcal{R}$ is convex on $\hbar$ and $\jmath \in L^1([\bar{\tau}, \upsilon])$.

The fractional integral inequalities of H-H type are proved by Sarikiya in [15] by using Riemann-Liouville fractional integrals (RL-fractional integrals) [3, 12]. The extended inequalities for (1. 1) and fractional integral inequalities of Hermite-Hadamard type proved in [15] are the Fejér [5] and Hermite-Hadamard-Fejér type fractional integral inequalities

[9]. Various forms of fractional derivatives including RL, Hadamard, Caputo, Caputo–Hadamard, Riesz, ψ-RL, Prabhakar, and weighted versions [10, 12, 14, 18, 6] have been developed to date. Most of these versions are described in the RL sense based on the corresponding fractional integral. Left- and right-sided RL-fractional integrals are generalized in the definition given below:

Definition 1.1. *Let $(\bar{\tau}, \upsilon) \subseteq \mathcal{R}$ and $\pi : (\bar{\tau}, \upsilon] \to \mathcal{R}^+$ an increasing monotone function with a continuous derivative π' on the interval $(\bar{\tau}, \upsilon)$. Then, the left- and right-side of the weighted fractional integrals of a function $\jmath$ with respect to π on $[\bar{\tau}, \upsilon]$ of order $\varepsilon > 0$ are defined by* [10]*:*

$$\left({}_{\varsigma}\lambda^{\varepsilon;\pi}_{\bar{\tau}+}\jmath\right)(\alpha) = \frac{\varsigma^{-1}(\alpha)}{\Gamma(\varepsilon)} \int_{\bar{\tau}}^{\alpha} \pi'(\mathfrak{u}) \left(\pi(\alpha) - \pi(\mathfrak{u})\right)^{\varepsilon-1} \jmath(\mathfrak{u}) \varsigma(\mathfrak{u})\, d\mathfrak{u} \tag{1. 2}$$

$$\left({}_{\varsigma}\lambda^{\varepsilon;\pi}_{\upsilon-}\jmath\right)(\alpha) = \frac{\varsigma^{-1}(\alpha)}{\Gamma(\varepsilon)} \int_{\alpha}^{\upsilon} \pi'(\mathfrak{u}) \left(\pi(\mathfrak{u}) - \pi(\alpha)\right)^{\varepsilon-1} \jmath(\mathfrak{u}) \varsigma(\mathfrak{u})\, d\mathfrak{u}, \tag{1. 3}$$

provided that: $\varsigma^{-1}(\alpha) = \frac{1}{\varsigma(\alpha)}$ for $\varsigma(\alpha) \neq 0$.

The following observations are obvious from the above definition:

- If $\pi \equiv I$, identity operator, and $\varsigma \equiv 1$, then the weighted fractional integral operators in the Definition 1.1 reduce to the classical RL-fractional integral operators.
- If $\varsigma \equiv 1$, we get the fractional integral operators of a function $\jmath$ with respect to another function $\pi(\alpha)$ of order $\varepsilon > 0$ which are defined in [1, 14].

The study analyzes several inequalities of the Hermite-Hadamard-Fejér type through weighted fractional operators with positive symmetrical weight function in the kernel. This paper is organized as follows: After this Introduction in Section 2 some assumptions are discussed, and in Section 3 main results related to the topic are presented.

2. Assumptions

Throughout the whole discussion, we denote $\mathcal{R}$, the set of all real numbers; $\mathcal{R}^+$, the set of all nonnegative real numbers; k a positive integer; $0 \leq \lambda \leq k$; $\varepsilon > 0$; $\hbar \subset \mathcal{R}$, an interval such that $\bar{\tau}, \upsilon \in \hbar^{\circ}$, interior of $\hbar$, with $\bar{\tau} < \upsilon$. let $\theta_{k,\bar{\tau},\upsilon}, \theta^*_{k,\bar{\tau},\upsilon} : [0, k] \to \mathcal{R}$ be two functions defined by:

$$\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u}) = \frac{k+\mathfrak{u}}{2k}\bar{\tau} + \frac{k-\mathfrak{u}}{2k}\upsilon; \quad \theta^*_{k,\bar{\tau},\upsilon}(\mathfrak{u}) = \frac{k-\mathfrak{u}}{2k}\bar{\tau} + \frac{k+\mathfrak{u}}{2k}\upsilon. \tag{2. 4}$$

$$h_1 := \frac{1}{\Gamma(\varepsilon)} \int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)} \int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}} \pi'(\alpha) \left(\frac{\bar{\tau}+\upsilon}{2} - \pi(\alpha)\right)^{\varepsilon-1} (\varsigma \circ \pi)(\alpha) \times \tag{2. 5}$$

$$\left(\jmath' \circ \pi\right)(\mathfrak{u})\, \pi'(\mathfrak{u})\, d\alpha d\mathfrak{u} \tag{2. 6}$$

$$h_2 := \frac{1}{\Gamma(\varepsilon)} \int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)} \int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)} \pi'(\alpha) \left(\pi(\alpha) - \frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1} (\varsigma \circ \pi)(\alpha) \times \tag{2. 7}$$

$$\left(\jmath' \circ \pi\right)(\mathfrak{u})\, \pi'(\mathfrak{u})\, d\alpha d\mathfrak{u}. \tag{2. 8}$$

$$\left({}_{\varsigma}\lambda_{\bar{\tau}+}^{\varepsilon}\jmath\right)(\alpha) := \frac{1}{\Gamma(\varepsilon)}\int_{\bar{\tau}}^{\alpha}(\alpha-\mathfrak{u})^{\varepsilon-1}\jmath(\mathfrak{u})\varsigma(\mathfrak{u})\,d\mathfrak{u}, \quad \varepsilon>0. \tag{2. 9}$$

$$\left({}_{\varsigma}\lambda_{\upsilon-}^{\varepsilon;\pi}\jmath\right)(\alpha) := \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{\upsilon}(\mathfrak{u}-\alpha)^{\varepsilon-1}\jmath(\mathfrak{u})\varsigma(\mathfrak{u})\,d\mathfrak{u}, \quad \varepsilon>0. \tag{2. 10}$$

$$\mathfrak{K}(\bar{\tau},\upsilon;q) := \sqrt[q]{\frac{(\varepsilon+3)\left|\jmath'(\bar{\tau})\right|^{q}+(3\varepsilon+5)\left|\jmath'(\upsilon)\right|^{q}}{4(\varepsilon+2)}} \tag{2. 11}$$

$$\mathfrak{u}_{\Delta} := k.\frac{\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)}{\frac{\bar{\tau}+\upsilon}{2}-\Delta}, \quad \Delta\in\{\bar{\tau},\upsilon\}. \tag{2. 12}$$

3. MAIN RESULTS

Lemma 3.1. *Let* $\varsigma:[\bar{\tau},\upsilon]\to\mathcal{R}^{+}$ *be an integrable function such that* $\varsigma(\bar{\tau}+\upsilon-\alpha)=\varsigma(\alpha)$, *then*

$$\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)=\varsigma\left(\theta_{k,\bar{\tau},\upsilon}^{*}(\mathfrak{u})\right) \tag{3. 13}$$

$$\left(\lambda_{\pi^{-1}(\bar{\tau})+}^{\varepsilon;\pi}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)=\left(\lambda_{\pi^{-1}(\upsilon)-}^{\varepsilon;\pi}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right) \tag{3. 14}$$

Proof. Setting $\alpha\to\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})$, obviously $\alpha\in[\bar{\tau},\upsilon]$ for each $\mathfrak{u}\in[0,k]$ and

$$\bar{\tau}+\upsilon-\alpha=\bar{\tau}+\upsilon-\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})=\theta_{k,\bar{\tau},\upsilon}^{*}(\mathfrak{u}),$$

hence by the definition of symmetry, we obtain

$$\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)=\varsigma(\alpha)=\varsigma(\bar{\tau}+\upsilon-\alpha)=\varsigma\left(\theta_{k,\bar{\tau},\upsilon}^{*}(\mathfrak{u})\right).$$

By using the symmetric property of ς, we have

$$(\varsigma\circ\pi)(\mathfrak{u})=\varsigma(\pi(\mathfrak{u}))=\varsigma(\bar{\tau}+\upsilon-\pi(\mathfrak{u})),\ \text{for all } \mathfrak{u}\in\left[\pi^{-1}(\bar{\tau}),\pi^{-1}(\upsilon)\right].$$

From this and by setting $\pi(\alpha)=\bar{\tau}+\upsilon-\pi(\mathfrak{u})$, it follows that

$$\begin{aligned}
&\left(\lambda_{\pi^{-1}(\bar{\tau})+}^{\varepsilon;\pi}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\\
&=\frac{1}{\Gamma(\varepsilon)}\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}(\varsigma\circ\pi)(\alpha)\pi'(\alpha)\,d\alpha\\
&=\frac{1}{\Gamma(\varepsilon)}\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}(\varsigma\circ\pi)(\alpha)\pi'(\alpha)\,d\alpha\\
&=\frac{1}{\Gamma(\varepsilon)}\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\left(\pi(\mathfrak{u})-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\varsigma(\bar{\tau}+\upsilon-\pi(\mathfrak{u}))\pi'(\mathfrak{u})\,d\mathfrak{u}\\
&=\frac{1}{\Gamma(\varepsilon)}\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\left(\pi(\mathfrak{u})-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}(\varsigma\circ\pi)(\mathfrak{u})\pi'(\mathfrak{u})\,d\mathfrak{u}\\
&=\left(\lambda_{\pi^{-1}(\upsilon)-}^{\varepsilon;\pi}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right).
\end{aligned}$$

□

Lemma 3.2. *Let* $\jmath : [\bar{\tau}, \upsilon] \subseteq \mathcal{R}^{+} \to \mathcal{R}$ *be a function such that* $\jmath, \jmath^{'} \in L^{1}[\bar{\tau}, \upsilon]$; *let* $\varsigma : [\bar{\tau}, \upsilon] \to \mathcal{R}^{+}$ *be an integrable function such that* $\varsigma(\bar{\tau} + \upsilon - \alpha) = \varsigma(\alpha)$. *If* $\pi : [\bar{\tau}, \upsilon) \to \mathcal{R}^{+}$ *is an increasing function and continuous on* $(\bar{\tau}, \upsilon)$, *then*

$$
{}_{\varsigma}\hbar^{\varepsilon;\pi}(\bar{\tau}, \upsilon) := \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right) - \varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right) \times
\frac{\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\jmath \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right) + \left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\jmath \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)}{2}. \quad (3.\ 15)
$$

Proof. By use of Definition 1.1, identity (3.14) and integrating by parts (2.5), the following holds:

$$
\begin{aligned}
\hbar_1 &= \frac{1}{\Gamma(\varepsilon)} \int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)} \int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}} \pi^{'}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2} - \pi(\alpha)\right)^{\varepsilon-1} (\varsigma \circ \pi)(\alpha)\, d\alpha . d\left[(\jmath \circ \pi)(\mathfrak{u})\right] \\
&= \frac{\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{\Gamma(\varepsilon)} \int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)} \pi^{'}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2} - \pi(\alpha)\right)^{\varepsilon-1} (\varsigma \circ \pi)(\alpha)\, d\alpha \\
&- \frac{1}{\Gamma(\varepsilon)} \int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)} \pi^{'}(\mathfrak{u})\left(\frac{\bar{\tau}+\upsilon}{2} - \pi(\mathfrak{u})\right)^{\varepsilon-1} (\varsigma \circ \pi)(\mathfrak{u}) (\jmath \circ \pi)(\mathfrak{u})\, d\mathfrak{u} \\
&= \frac{\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{\Gamma(\varepsilon)} \int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)} \pi^{'}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2} - \pi(\alpha)\right)^{\varepsilon-1} (\varsigma \circ \pi)(\alpha)\, d\alpha \\
&- \varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right) \frac{(\varsigma \circ \pi)^{-1}\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)}{\Gamma(\varepsilon)} \\
&\times \int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)} \pi^{'}(\mathfrak{u})\left(\frac{\bar{\tau}+\upsilon}{2} - \pi(\mathfrak{u})\right)^{\varepsilon-1} (\varsigma \circ \pi)(\mathfrak{u}) (\jmath \circ \pi)(\mathfrak{u})\, d\mathfrak{u} \\
&= \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right) \\
&- \varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right).
\end{aligned}
$$

Analogously

$$
\begin{aligned}
\hbar_2 &= -\frac{1}{\Gamma(\varepsilon)} \int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)} \int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)} \pi^{'}(\alpha)\left(\pi(\alpha) - \frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1} (\varsigma \circ \pi)(\alpha)\, d\alpha . d\left[(\jmath \circ \pi)(\mathfrak{u})\right] \\
&= \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\varsigma \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right) \\
&- \varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\varsigma \circ \pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right).
\end{aligned}
$$

$$\begin{aligned}\Rightarrow \hbar_1+\hbar_2 &= \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\\ &-\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\\ &+\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\\ &-\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right). \quad (3.16)\end{aligned}$$

A combination of (2.7) and (3.14) yields the desired result (3.15). □

Theorem 3.3. *Let $\jmath:[\bar{\tau},\upsilon]\subseteq\mathcal{R}^+\to\mathcal{R}$ be an L^1 convex function with $0<\bar{\tau}<\upsilon$ and $\varsigma:[\bar{\tau},\upsilon]\to\mathcal{R}^+$ integrable such that $\varsigma(\bar{\tau}+\upsilon-\alpha)=\varsigma(\alpha)$; let $\pi:[\bar{\tau},\upsilon)\to\mathcal{R}^+$ be an increasing and continuous function on $(\bar{\tau},\upsilon)$, then*

$$\begin{aligned}&\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\times\\ &\left[\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)+\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\right]\\ &\leq\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\jmath\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\\ &+\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left({}_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\jmath\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\\ &\leq\frac{\jmath(\bar{\tau})+\jmath(\upsilon)}{2}\left[\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}(\varsigma\circ\pi)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\right)\right.\\ &\left.+\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\varsigma\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)\right]. \quad (3.17)\end{aligned}$$

Proof. By convexity of $\jmath$ on $[\bar{\tau},\upsilon]$

$$\jmath\left(\frac{\alpha+\lambda}{2}\right)\leq\frac{\jmath(\alpha)+\jmath(\lambda)}{2},\quad \alpha,\lambda\in[\bar{\tau},\upsilon],$$

and for $\alpha\to\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right);\ \lambda\to\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)$, we have

$$2\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\leq\jmath\left(\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)+\jmath\left(\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right).$$

Multiplication, on either side, by $\mathfrak{u}^{\varepsilon-1}\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)$ and integration over $[0,k]$, yields:

$$\begin{aligned}2\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\int_0^k\mathfrak{u}^{\varepsilon-1}\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)d\mathfrak{u}&\leq\int_0^k\mathfrak{u}^{\varepsilon-1}\jmath\left(\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)d\mathfrak{u}\\ &+\int_0^k\mathfrak{u}^{\varepsilon-1}\jmath\left(\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)d\mathfrak{u}. \quad (3.18)\end{aligned}$$

But, by identity (3.14) in Lemma 3.1

$$\Gamma(\varepsilon)k^{\varepsilon}\frac{\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}\varsigma\circ\pi\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)+\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}\varsigma\circ\pi\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)}{2\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)^{\varepsilon}}$$
$$=\Gamma(\varepsilon)k^{\varepsilon}\frac{\left(\lambda^{\varepsilon;\pi}_{\pi^{-1}(\bar{\tau})+}\varsigma\circ\pi\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)}{\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)^{\varepsilon}}$$
$$=\frac{k^{\varepsilon}}{\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)^{\varepsilon}}\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}(\varsigma\circ\pi)(\alpha).\pi'(\alpha)\,d\alpha$$
$$=\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(k\frac{\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)}{\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}}\right)^{\varepsilon-1}(\varsigma\circ\pi)(\alpha)\,\pi'(\alpha)\,\frac{kd\alpha}{\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}}$$
$$=\int_0^k\wp^{\varepsilon-1}\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\wp)\right)d\wp, \quad (3.\ 19)$$

But, by (2.12)-(3.13) and weighted fractional operator, the followings hold:

$$\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(_{\pi^{-1}(\bar{\tau})+}\lambda^{\varepsilon;\pi}_{\varsigma\circ\pi}(\jmath\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)$$
$$+\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left(_{\varsigma\circ\pi}\lambda^{\varepsilon;\pi}_{\pi^{-1}(\upsilon)-}(\jmath\circ\pi)\right)\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)$$
$$=\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\frac{(\varsigma\circ\pi)^{-1}\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)}{\Gamma(\varepsilon)}\times$$
$$\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}(\jmath\circ\pi)(\alpha)(\varsigma\circ\pi)(\alpha)\,\pi'(\alpha)\,d\alpha$$
$$+\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\frac{(\varsigma\circ\pi)^{-1}\left(\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right)}{\Gamma(\varepsilon)}\times$$
$$\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}(\jmath\circ\pi)(\alpha)(\varsigma\circ\pi)(\alpha)\,\pi'(\alpha)\,d\alpha$$
$$=\frac{1}{\Gamma(\varepsilon)}\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}(\jmath\circ\pi)(\alpha)(\varsigma\circ\pi)(\alpha)\,\pi'(\alpha)\,d\alpha$$
$$+\frac{1}{\Gamma(\varepsilon)}\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}(\jmath\circ\pi)(\alpha)(\varsigma\circ\pi)(\alpha)\,\pi'(\alpha)\,d\alpha$$
$$=\frac{\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)^{\varepsilon}}{k^{\varepsilon}\Gamma(\varepsilon)}\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(k\frac{\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)}{\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}}\right)^{\varepsilon-1}$$
$$\times(\jmath\circ\pi)(\alpha)(\varsigma\circ\pi)(\alpha)\,\pi'(\alpha)\,\frac{kd\alpha}{\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)}+\frac{\left(\upsilon-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon}}{k^{\varepsilon}\Gamma(\varepsilon)}$$

$$\times\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\left(k\frac{\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}}{\upsilon-\frac{\bar{\tau}+\upsilon}{2}}\right)^{\varepsilon-1}(\jmath\circ\pi)(\alpha)(\varsigma\circ\pi)(\alpha)\pi'(\alpha)\frac{kd\alpha}{\upsilon-\frac{\bar{\tau}+\upsilon}{2}}$$

$$=\frac{\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)^{\varepsilon}}{k^{\varepsilon}\Gamma(\varepsilon)}\int_0^k \mathfrak{u}_{\bar{\tau}}^{\varepsilon-1}\jmath\left(\varsigma\left(\theta_{k,\bar{\tau},\upsilon}\left(\mathfrak{u}_{\bar{\tau}}\right)\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}\left(\mathfrak{u}_{\bar{\tau}}\right)\right)d\mathfrak{u}_{\bar{\tau}}$$

$$+\frac{\left(\upsilon-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon}}{k^{\varepsilon}\Gamma(\varepsilon)}\int_0^k \mathfrak{u}_{\upsilon}^{\varepsilon-1}\jmath\left(\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}\left(\mathfrak{u}_{\upsilon}\right)\right)\right)\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}\left(\mathfrak{u}_{\upsilon}\right)\right)d\mathfrak{u}_{\upsilon}$$

$$=\frac{\left(\frac{\bar{\tau}+\upsilon}{2}-\bar{\tau}\right)^{\varepsilon}}{k^{\varepsilon}\Gamma(\varepsilon)}\int_0^k \hbar^{\varepsilon-1}\jmath\left(\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\hbar)\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\hbar)\right)d\hbar$$

$$+\frac{\left(\upsilon-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon}}{k^{\varepsilon}\Gamma(\varepsilon)}\int_0^k \hbar^{\varepsilon-1}\jmath\left(\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}(\hbar)\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\hbar)\right)d\hbar$$

$$=\int_0^k \Im^{\varepsilon-1}\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\Im)\right)d\Im \quad (3.\ 20)$$

A combination of (3.18)-(3.19) and (3.20) yields the first inequality in (3.17). For the second inequality in (3.17), by convexity of $\jmath$

$$\jmath\left(\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)+\jmath\left(\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)\le\jmath(\bar{\tau})+\jmath(\upsilon). \quad (3.\ 21)$$

Multiplication, on either side, by $\mathfrak{u}^{\varepsilon-1}\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)$ and integration over $[0,k]$, yields:

$$\int_0^k \mathfrak{u}^{\varepsilon-1}\jmath\left(\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)d\mathfrak{u}+$$

$$\int_0^k \mathfrak{u}^{\varepsilon-1}\jmath\left(\varsigma\left(\theta^*_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)\right)\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)d\mathfrak{u}$$

$$\le\left[\jmath(\bar{\tau})+\jmath(\upsilon)\right]\int_0^k \mathfrak{u}^{\varepsilon-1}\varsigma\left(\theta_{k,\bar{\tau},\upsilon}(\mathfrak{u})\right)d\mathfrak{u}. \quad (3.\ 22)$$

A combination of (3.19)-(3.20), (3.13) and (3.22) yields the second inequality in (3.17). This completes the proof. □

The following remark gives us some consequences in Theorem 3.3 and some new identities in Lemma 3.2.

Remark 3.4. • *For $\pi\equiv I$, identity operator, inequality (3.17) reduces to*

$$\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left[\left(\lambda^{\varepsilon}_{\bar{\tau}+}\varsigma\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)+\left(\lambda^{\varepsilon}_{\upsilon-}\varsigma\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right]$$

$$\le\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\left[\left({}_{\varsigma}\lambda^{\varepsilon}_{\bar{\tau}+}\jmath\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)+\left({}_{\varsigma}\lambda^{\varepsilon}_{\upsilon-}\jmath\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right]$$

$$\le\frac{\jmath(\bar{\tau})+\jmath(\upsilon)}{2}\left[\left(\lambda^{\varepsilon}_{\bar{\tau}+}\varsigma\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)+\left(\lambda^{\varepsilon}_{\upsilon-}\varsigma\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)\right], \quad (3.\ 23)$$

provided that the left and right weighted RL-fractional operators ${}_{\varsigma}\lambda^{\varepsilon}_{\bar{\tau}+}$ and ${}_{\varsigma}\lambda^{\varepsilon}_{\upsilon-}$ of order $\varepsilon>0$ are defined by (2.9) *and* (2.10) *respectively.*

- *For* $\pi \equiv I$, *identity operator, and* $\varepsilon = 1$ *the inequality* (3.17) *reduces to*

$$\frac{\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\bar{\tau}}^{\upsilon}\varsigma(\mathfrak{u})\,d\mathfrak{u} \leq \int_{\bar{\tau}}^{\upsilon}\jmath(\mathfrak{u})\varsigma(\mathfrak{u})\,d\mathfrak{u} \leq \frac{\jmath(\bar{\tau})+\jmath(\upsilon)}{2\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\bar{\tau}}^{\upsilon}\varsigma(\mathfrak{u})\,d\mathfrak{u}. \tag{3. 24}$$

- *For* $\pi \equiv I$, *identity operator, and* $\varsigma \equiv 1$ *inequality* (3.17) *reduces to*

$$\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) \leq \Gamma(\varepsilon+1)\frac{\left({}_{\varsigma}\lambda_{\bar{\tau}+}^{\varepsilon}\jmath\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)+\left({}_{\varsigma}\lambda_{\upsilon-}^{\varepsilon}\jmath\right)\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2^{1-\varepsilon}(\upsilon-\bar{\tau})^{\varepsilon}} \leq \frac{\jmath(\bar{\tau})+\jmath(\upsilon)}{2}. \tag{3. 25}$$

- *For* $\pi \equiv I$, *identity operator,* $\varsigma \equiv 1$, $\varepsilon = 1$ (3.17) *reduces to inequality* (1.1).
- *For* $\pi \equiv I$, *identity operator, identity* (3.15) *reduces to*

$$\begin{aligned}
&\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)\ \lambda_{\bar{\tau}+}^{\varepsilon}\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)-\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right)\frac{{}_{\varsigma}\lambda_{\bar{\tau}+}^{\varepsilon}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)+{}_{\varsigma}\lambda_{\upsilon-}^{\varepsilon}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2}\\
&\quad=\frac{1}{2\Gamma(\varepsilon)}\int_{\bar{\tau}}^{\frac{\bar{\tau}+\upsilon}{2}}\int_{\bar{\tau}}^{\mathfrak{u}}\left(\frac{\bar{\tau}+\upsilon}{2}-\alpha\right)^{\varepsilon-1}\varsigma(\alpha)\jmath'(\mathfrak{u})\,d\alpha d\mathfrak{u}\\
&\qquad-\frac{1}{2\Gamma(\varepsilon)}\int_{\frac{\bar{\tau}+\upsilon}{2}}^{\upsilon}\int_{\mathfrak{u}}^{\upsilon}\left(\alpha-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\varsigma(\alpha)\jmath'(\mathfrak{u})\,d\alpha d\mathfrak{u}.
\end{aligned} \tag{3. 26}$$

- *For* $\pi \equiv I$, *identity operator, and* $\varsigma \equiv 1$ *identity* (3.15) *reduces to*

$$\begin{aligned}
&\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)-\Gamma(\varepsilon+1)\frac{\lambda_{\bar{\tau}+}^{\varepsilon}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)+\lambda_{\upsilon-}^{\varepsilon}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2^{1-\varepsilon}(\upsilon-\bar{\tau})^{\varepsilon}}=\frac{\upsilon-\bar{\tau}}{4}\int_0^1(1-\mathfrak{u}^{\varepsilon})\times\\
&\qquad\left[\jmath'\left(\frac{1+\mathfrak{u}}{2}\bar{\tau}+\frac{1-\mathfrak{u}}{2}\upsilon\right)-\jmath'\left(\frac{1-\mathfrak{u}}{2}\bar{\tau}+\frac{1+\mathfrak{u}}{2}\upsilon\right)\right]d\mathfrak{u}.
\end{aligned} \tag{3. 27}$$

- *For* $\pi \equiv I$, *identity operator,* $\varsigma \equiv 1$ *and* $\varepsilon = 1$ *identity* (3.15) *reduces to*

$$\begin{aligned}
&\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)-\frac{1}{\upsilon-\bar{\tau}}\int_{\bar{\tau}}^{\upsilon}\jmath(\alpha)\,d\alpha\\
&=\frac{\upsilon-\bar{\tau}}{4}\int_0^1(1-\mathfrak{u})\left[\jmath'\left(\frac{1+\mathfrak{u}}{2}\bar{\tau}+\frac{1-\mathfrak{u}}{2}\upsilon\right)-\jmath'\left(\frac{1-\mathfrak{u}}{2}\bar{\tau}+\frac{1+\mathfrak{u}}{2}\upsilon\right)\right]d\mathfrak{u}.
\end{aligned} \tag{3. 28}$$

Theorem 3.5. *Let* $\jmath : [\bar{\tau}, \upsilon] \subseteq \mathcal{R}^+ \to \mathcal{R}$ *be a function for which* $\jmath, \jmath' \in L^1[\bar{\tau}, \upsilon]$ *and* $\varsigma : [\bar{\tau}, \upsilon] \to \mathcal{R}^+$ *an integrable such that* $\varsigma(\bar{\tau}+\upsilon-\alpha) = \varsigma(\alpha)$; *let* $\pi : [\bar{\tau}, \upsilon) \to \mathcal{R}^+$ *be an increasing and continuous function on* $(\bar{\tau}, \upsilon)$. *Moreover, if* $\left|\jmath'\right|$ *is convex on* $[\bar{\tau}, \upsilon]$, *then*

$$\left|{}_{\varsigma}\hbar^{\varepsilon;\pi}(\bar{\tau},\upsilon)\right| \leq \frac{(\bar{\tau}-\upsilon)^{\varepsilon+1}\left[\left|\jmath'(\bar{\tau})\right|+\left|\jmath'(\upsilon)\right|\right]}{2^{\varepsilon+2}(\varepsilon+1)\Gamma(\varepsilon)}\left\|\varsigma\circ\pi\right\|_{\infty}. \tag{3. 29}$$

Proof. By properties of modulus to identity (3.15) in Lemma 3.2

$$
\begin{aligned}
&\left|{}_{\varsigma}\hbar^{\varepsilon;\pi}\left(\bar{\tau},\upsilon\right)\right| \\
&\leq \frac{1}{2\Gamma\left(\varepsilon\right)}\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\mathfrak{u}}\left|\pi^{'}\left(\alpha\right)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi\left(\alpha\right)\right)^{\varepsilon-1}\left(\varsigma\circ\pi\right)\left(\alpha\right)\right|\times \\
&\qquad\left|\left(\jmath^{'}\circ\pi\right)\left(\mathfrak{u}\right)\right|\left|\pi^{'}\left(\mathfrak{u}\right)\right|d\alpha d\mathfrak{u} \\
&+\frac{1}{2\Gamma\left(\varepsilon\right)}\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}\left(\upsilon\right)}\int_{\mathfrak{u}}^{\pi^{-1}\left(\upsilon\right)}\left|\pi^{'}\left(\alpha\right)\left(\pi\left(\alpha\right)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\left(\varsigma\circ\pi\right)\left(\alpha\right)\right|\times \\
&\qquad\left|\left(\jmath^{'}\circ\pi\right)\left(\mathfrak{u}\right)\right|\left|\pi^{'}\left(\mathfrak{u}\right)\right|d\alpha d\mathfrak{u} \\
&\leq \frac{\left\|\varsigma\circ\pi\right\|_{\infty}}{2\Gamma\left(\varepsilon\right)}\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left[\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\mathfrak{u}}\pi^{'}\left(\alpha\right)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi\left(\alpha\right)\right)^{\varepsilon-1}d\alpha\right] \\
&\qquad\times\left|\left(\jmath^{'}\circ\pi\right)\left(\mathfrak{u}\right)\right|\pi^{'}\left(\mathfrak{u}\right)d\mathfrak{u} \\
&+\frac{\left\|\varsigma\circ\pi\right\|_{\infty}}{2\Gamma\left(\varepsilon\right)}\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}\left(\upsilon\right)}\left[\int_{\mathfrak{u}}^{\pi^{-1}\left(\upsilon\right)}\pi^{'}\left(\alpha\right)\left(\pi\left(\alpha\right)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}d\alpha\right] \\
&\qquad\times\left|\left(\jmath^{'}\circ\pi\right)\left(\mathfrak{u}\right)\right|\pi^{'}\left(\mathfrak{u}\right)d\mathfrak{u}. \qquad (3.\ 30)
\end{aligned}
$$

But, the convexity of $\left|\jmath^{'}\right|$ on $[\bar{\tau},\upsilon]$ for $\mathfrak{u}\in\left[\pi^{-1}\left(\bar{\tau}\right),\pi^{-1}\left(\upsilon\right)\right]$, yields

$$
\left|\left(\jmath^{'}\circ\pi\right)\left(\mathfrak{u}\right)\right|\leq\frac{\upsilon-\pi\left(\mathfrak{u}\right)}{\upsilon-\bar{\tau}}\left|\jmath^{'}\left(\bar{\tau}\right)\right|+\frac{\pi\left(\mathfrak{u}\right)-\bar{\tau}}{\upsilon-\bar{\tau}}\left|\jmath^{'}\left(\upsilon\right)\right|. \qquad (3.\ 31)
$$

Application of (3.31) yields the following:

$$
\begin{aligned}
&\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\mathfrak{u}}\pi^{'}\left(\alpha\right)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi\left(\alpha\right)\right)^{\varepsilon-1}\left|\left(\jmath^{'}\circ\pi\right)\left(\mathfrak{u}\right)\right|\pi^{'}\left(\mathfrak{u}\right)d\alpha d\mathfrak{u} \\
&=\int_{\pi^{-1}\left(\bar{\tau}\right)}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\left(\frac{\upsilon-\bar{\tau}}{2}\right)^{\varepsilon}-\left(\frac{\bar{\tau}+\upsilon}{2}-\pi\left(\mathfrak{u}\right)\right)^{\varepsilon}\right)\pi^{'}\left(\mathfrak{u}\right)\times \\
&\qquad\frac{\left[\upsilon-\pi\left(\mathfrak{u}\right)\right]\left|\jmath^{'}\left(\bar{\tau}\right)\right|+\left[\pi\left(\mathfrak{u}\right)-\bar{\tau}\right]\left|\jmath^{'}\left(\upsilon\right)\right|}{\varepsilon(\upsilon-\bar{\tau})}d\mathfrak{u} \\
&=\frac{2^{-\varepsilon-3}\left(\upsilon-\bar{\tau}\right)^{\varepsilon+1}\left[\left(\varepsilon+3\right)\left|\jmath^{'}\left(\bar{\tau}\right)\right|+\left(3\varepsilon+5\right)\left|\jmath^{'}\left(\upsilon\right)\right|\right]}{\left(\varepsilon+1\right)\left(\varepsilon+2\right)}. \qquad (3.\ 32)
\end{aligned}
$$

Analogously:

$$\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)}\pi^{'}(\alpha)\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\left|\left(\jmath^{'}\circ\pi\right)(\mathfrak{u})\right|\pi^{'}(\mathfrak{u})\,d\alpha d\mathfrak{u}$$

$$=\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\left(\left(\frac{\upsilon-\bar{\tau}}{2}\right)^{\varepsilon}-\left(\pi(\mathfrak{u})-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon}\right)\pi^{'}(\mathfrak{u})\times \frac{[\upsilon-\pi(\mathfrak{u})]\left|\jmath^{'}(\bar{\tau})\right|+[\pi(\mathfrak{u})-\bar{\tau}]\left|\jmath^{'}(\upsilon)\right|}{\varepsilon(\upsilon-\bar{\tau})}d\mathfrak{u}$$

$$=\frac{2^{-\varepsilon-3}(\upsilon-\bar{\tau})^{\varepsilon+1}\left[(\varepsilon+3)\left|\jmath^{'}(\upsilon)\right|+(3\varepsilon+5)\left|\jmath^{'}(\bar{\tau})\right|\right]}{(\varepsilon+1)(\varepsilon+2)}. \quad (3.\ 33)$$

A combination of (3.30), (3.32)-(3.33) yields the desired inequality (3.29). □

Theorem 3.6. *Let* $\jmath:[\bar{\tau},\upsilon]\subseteq\mathcal{R}^{+}\to\mathcal{R}$ *be a function for which* $\jmath,\jmath^{'}\in L^{1}[\bar{\tau},\upsilon]$ *and* $\varsigma:[\bar{\tau},\upsilon]\to\mathcal{R}^{+}$ *an integrable such that* $\varsigma(\bar{\tau}+\upsilon-\alpha)=\varsigma(\alpha)$; *let* $\pi:[\bar{\tau},\upsilon)\to\mathcal{R}^{+}$ *be an increasing and continuous function on* $(\bar{\tau},\upsilon)$. *Moreover, if* $\left|\jmath^{'}\right|^{q}$ *is convex on* $[\bar{\tau},\upsilon]$ *for* $q\geq 1$, *then*

$$\left|{}_{\varsigma}\hbar^{\varepsilon;\pi}(\bar{\tau},\upsilon)\right|\leq\frac{\|\varsigma\circ\pi\|_{\infty}(\upsilon-\bar{\tau})^{\varepsilon+1}\left[\mathfrak{K}(\bar{\tau},\upsilon;q)+\mathfrak{K}(\upsilon,\bar{\tau};q)\right]}{2^{\varepsilon+2}(\varepsilon+1)\Gamma(\varepsilon)} \quad (3.\ 34)$$

Proof. By properties of modulus to identity (3.15) in Lemma 3.2 and power-mean inequality

$$\left|{}_{\varsigma}\hbar^{\varepsilon;\pi}(\bar{\tau},\upsilon)\right|$$

$$\leq\frac{\|\varsigma\circ\pi\|_{\infty}}{2\Gamma(\varepsilon)}\sqrt[q-1]{\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}}\pi^{\prime}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}\pi^{\prime}(\mathfrak{u})\,d\alpha d\mathfrak{u}}$$

$$\times\sqrt[q]{\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}}\pi^{\prime}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}\left|(\jmath^{\prime}\circ\pi)(\mathfrak{u})\right|^{q}\pi^{\prime}(\mathfrak{u})\,d\alpha d\mathfrak{u}}$$

$$+\frac{\|\varsigma\circ\pi\|_{\infty}}{2\Gamma(\varepsilon)}\sqrt[q-1]{\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)}\pi^{\prime}(\alpha)\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\pi^{\prime}(\mathfrak{u})\,d\alpha d\mathfrak{u}}\times$$

$$\sqrt[q]{\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)}\pi^{\prime}(\alpha)\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\left|(\jmath^{\prime}\circ\pi)(\mathfrak{u})\right|^{q}\pi^{\prime}(\mathfrak{u})\,d\alpha d\mathfrak{u}} \quad (3.\ 35)$$

But, by convexity of $\left|\jmath^{'}\right|^{q}$

$$\left|\left(\jmath^{'}\circ\pi\right)(\mathfrak{u})\right|^{q}\leq\frac{\upsilon-\pi(\mathfrak{u})}{\upsilon-\bar{\tau}}\left|\jmath^{'}(\bar{\tau})\right|^{q}+\frac{\pi(\mathfrak{u})-\bar{\tau}}{\upsilon-\bar{\tau}}\left|\jmath^{'}(\upsilon)\right|^{q}. \quad (3.\ 36)$$

It may be observed that:

$$\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}}\pi^{'}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}\pi^{'}(\mathfrak{u})\,d\alpha d\mathfrak{u}$$
$$=\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)}\pi^{'}(\alpha)\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\pi^{'}(\mathfrak{u})\,d\alpha d\mathfrak{u}$$
$$=\frac{(\upsilon-\bar{\tau})^{\varepsilon+1}}{2^{\varepsilon+1}(\varepsilon+1)} \quad (3.37)$$

Application of (3.36) and (3.37) yields the following:

$$\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}}\pi^{'}(\alpha)\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}\left|\left(\jmath^{'}\circ\pi\right)(\mathfrak{u})\right|^{q}\pi^{'}(\mathfrak{u})\,d\alpha d\mathfrak{u}$$
$$\leq\int_{\pi^{-1}(\bar{\tau})}^{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}\int_{\pi^{-1}(\bar{\tau})}^{\mathfrak{u}}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}\pi^{'}(\alpha)\pi^{'}(\mathfrak{u})\times$$
$$\frac{[\upsilon-\pi(\mathfrak{u})]\left|\jmath^{'}(\bar{\tau})\right|^{q}+[\pi(\mathfrak{u})-\bar{\tau}]\left|\jmath^{'}(\upsilon)\right|^{q}}{\upsilon-\bar{\tau}}d\alpha d\mathfrak{u}$$
$$=\frac{(\upsilon-\bar{\tau})^{\varepsilon+2}\left[(\varepsilon+3)\left|\jmath^{'}(\bar{\tau})\right|^{q}+(3\varepsilon+5)\left|\jmath^{'}(\upsilon)\right|^{q}\right]}{2^{\varepsilon+3}(\upsilon-\bar{\tau})(\varepsilon+1)(\varepsilon+2)} \quad (3.38)$$

Analogously:

$$\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)}\pi^{'}(\alpha)\left(\pi(\alpha)-\frac{\bar{\tau}+\upsilon}{2}\right)^{\varepsilon-1}\left|\left(\jmath^{'}\circ\pi\right)(\mathfrak{u})\right|^{q}\pi^{'}(\mathfrak{u})\,d\alpha d\mathfrak{u}$$
$$\leq\int_{\pi^{-1}\left(\frac{\bar{\tau}+\upsilon}{2}\right)}^{\pi^{-1}(\upsilon)}\int_{\mathfrak{u}}^{\pi^{-1}(\upsilon)}\left(\frac{\bar{\tau}+\upsilon}{2}-\pi(\alpha)\right)^{\varepsilon-1}\pi^{'}(\alpha)\pi^{'}(\mathfrak{u})\times$$
$$\frac{[\upsilon-\pi(\mathfrak{u})]\left|\jmath^{'}(\bar{\tau})\right|^{q}+[\pi(\mathfrak{u})-\bar{\tau}]\left|\jmath^{'}(\upsilon)\right|^{q}}{\upsilon-\bar{\tau}}d\alpha d\mathfrak{u}$$
$$=\frac{(\upsilon-\bar{\tau})^{\varepsilon+2}\left[(\varepsilon+3)\left|\jmath^{'}(\upsilon)\right|^{q}+(3\varepsilon+5)\left|\jmath^{'}(\bar{\tau})\right|^{q}\right]}{2^{\varepsilon+3}(\upsilon-\bar{\tau})(\varepsilon+1)(\varepsilon+2)}. \quad (3.39)$$

A combination of (3.35), (3.37)-(3.39) yields the desired inequality (3.34). □

The following remark provides some consequences of Theorems 3.5 and 3.6.

Remark 3.7. • *For* $\pi \equiv I$, *identity operator, inequalities* (3.29) *and* (3.34), *respectively, reduce to*

$$\left| \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) \lambda^{\varepsilon}_{\bar{\tau}+}\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right) - \varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right) \frac{{}_{\varsigma}\lambda^{\varepsilon}_{\bar{\tau}+}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) + {}_{\varsigma}\lambda^{\varepsilon}_{\upsilon-}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2} \right| \le \frac{(\bar{\tau}-\upsilon)^{\varepsilon+1}\|\varsigma\|_{\infty}\left[\left|\jmath'(\bar{\tau})\right| + \left|\jmath'(\upsilon)\right|\right]}{2^{\varepsilon+2}(\varepsilon+1)\Gamma(\varepsilon)}. \quad (3.\ 40)$$

$$\left| \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) \lambda^{\varepsilon}_{\bar{\tau}+}\varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right) - \varsigma\left(\frac{\bar{\tau}+\upsilon}{2}\right) \frac{{}_{\varsigma}\lambda^{\varepsilon}_{\bar{\tau}+}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) + {}_{\varsigma}\lambda^{\varepsilon}_{\upsilon-}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2} \right| \le \frac{\|\varsigma\|_{\infty}(\upsilon-\bar{\tau})^{\varepsilon+1}\left[\mathfrak{K}(\bar{\tau},\upsilon;q) + \mathfrak{K}(\upsilon,\bar{\tau};q)\right]}{2^{\varepsilon+2}(\varepsilon+1)\Gamma(\varepsilon)} \quad (3.\ 41)$$

• *For* $\pi \equiv I$, *identity operator, and* $\varsigma \equiv 1$ *inequalities* (3.29) *and* (3.34), *respectively, reduce to*

$$\left| \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) - 2^{\varepsilon}\Gamma(\varepsilon+1) \frac{\lambda^{\varepsilon}_{\bar{\tau}+}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) + \lambda^{\varepsilon}_{\upsilon-}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2(\upsilon-\bar{\tau})^{\varepsilon}} \right| \le \frac{(\bar{\tau}-\upsilon)^{\varepsilon+1}\left[\left|\jmath'(\bar{\tau})\right| + \left|\jmath'(\upsilon)\right|\right]}{2^{\varepsilon+2}(\varepsilon+1)\Gamma(\varepsilon)}. \quad (3.\ 42)$$

$$\left| \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) - \Gamma(\varepsilon+1) \frac{\lambda^{\varepsilon}_{\bar{\tau}+}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) + \lambda^{\varepsilon}_{\upsilon-}\jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right)}{2^{1-\varepsilon}(\upsilon-\bar{\tau})^{\varepsilon}} \right| \le \frac{(\upsilon-\bar{\tau})^{\varepsilon+1}\left[\mathfrak{K}(\bar{\tau},\upsilon;q) + \mathfrak{K}(\upsilon,\bar{\tau};q)\right]}{2^{\varepsilon+2}(\varepsilon+1)\Gamma(\varepsilon)} \quad (3.\ 43)$$

• *For* $\pi \equiv I$, *identity operator,* $\varsigma \equiv 1$ *and* $\varepsilon = 1$ *inequalities* (3.29) *and* (3.34), *respectively, reduce to*

$$\left| \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) - \frac{1}{\upsilon-\bar{\tau}}\int_{\bar{\tau}}^{\upsilon}\jmath(\alpha)\,d\alpha \right| \le \frac{(\bar{\tau}-\upsilon)^{2}\left[\left|\jmath'(\bar{\tau})\right| + \left|\jmath'(\upsilon)\right|\right]}{16}. \quad (3.\ 44)$$

$$\left| \jmath\left(\frac{\bar{\tau}+\upsilon}{2}\right) - \frac{1}{\upsilon-\bar{\tau}}\int_{\bar{\tau}}^{\upsilon}\jmath(\alpha)\,d\alpha \right| \le \frac{(\upsilon-\bar{\tau})^{2}\left[\mathfrak{K}(\bar{\tau},\upsilon;q) + \mathfrak{K}(\upsilon,\bar{\tau};q)\right]}{16} \quad (3.\ 45)$$

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