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Analytical Method for Solving Inviscid Burger Equation

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Abstract: In this paper, we use the natural decomposition method (NDM) for solving inviscid Burger equation (BE). The NDM is associated with the Adomain decomposition method (ADM) and the natural transform method. Applying the analytic method, we solved successfully both lin-ear and non-linear partial differential equations. By applying the NDM, we compute the best approximation solution of linear and non-linear par-tial differential equations. In our experiments, we report comparisons with the exact solution.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09

Key Words: Natural decomposition method,Adomain polynomial, Inviscid Burger equation

1. INTRODUCTION

The BE observable scalar partial differential equation can be denoted by following expression [11]

$$u_t + u_{xx} = \varepsilon u_{xx} \tag{1. 1}$$

where ε, $\varepsilon > 0$ and $\varepsilon = 0$ represent the viscosity of fluid dynamics, viscous BE and inviscid BE. The inviscid BE discontinuities may be coming out infinite part, such as discontinuities cannot be removed by any classical method, for this purpose optimal control perspective has been used [17, 21]. The simplest type of nonlinear partial differential equation (PDE's) is the BE. It can be applied in different fields i.e. cancer growth, modeling of gas dynamics, and traffic flow [4, 13]. Bateman derived [8] BE in physical circumstances. Witham [29] introduced that BE combines diffusive and nonlinear effects. Cole [5] solved the BE initially. Kuo and Lee [23] found the solution of the BE. Bendaas [12] used the method of characteristics to get the exact solution of the inviscid BE. Burger's derived the viscous BE as the model of turbulence flow. There are many methods to solve the viscous and inviscid BE such as the Wood [30] and also found the solution of viscous BE. Kudryavtsev and Sapozhnikov [22] used the Darboux transformation to derived the viscous BE. Aswin et al [3] calculated the solution of the viscous BE by using differential quadrature method (DQM). Vijitha et al [27] solved the viscous by novel numerical method and the non-standard finite difference method [14] used to solve the viscous BE and nonlinear PDEs applied to explain the phenomena in the field of mechanics, physics, and biology. Numerous PDEs applied in the field of engineering, fluid mechanics, plasma physics, and chemical physics. Partial differential equations [28] used to explain the process of wave propagation and heat flow. Currently, a large number of researchers pay attention to solve the nonlinear PDEs. In this paper, we solved the inviscid BE by using NDM. Khan derived [20] an integral transform for the N-transform. Belgacem and Silambarasan [5, 6] discovered the name natural transform and also applied the NDM method to solved many non-linear differential equations like that, Maxwells equation [7], nonlinear PDEs [25], fractional telegraph equation [9], Time-Fractional coupled Korteweg-de vries equation [10], Sine-Gordon equation [26], nonlinear fractional-order differential equations [16]. There are many other similar transform and decomposition methods which are used to solved differential and integral equations such as Natural Daftardar-Jafari Method (NDJM) [18], Gupta integral transform[15], Adomain Decomposition Method [24], Laplace Decomposition Method[2], and Differential Transform Method[1]. The paper is arranged as follows: the Sec. 2, discuss the basic idea of definition and some properties of the natural transform method. The NDM use to study the BE solution in Sec. 3. In this method, we solve the three problems to show the simplicity and accuracy of this method Sec. 4. The last Sec. 5, we deal the conclusion and graphs.

2. BASIC DEFINITION AND PROPERTIES OF N-TRANSFORM

The natural transform of the function

$$f(t) \geq 0, t\epsilon(-\infty, \infty)$$

defined by [5]

$$N^{+}[f(t)] = R(s,u) = \frac{1}{u}\int_{0}^{\infty} e^{-st} f(ut)dt, (s,u)\epsilon(-\infty,\infty)$$

where s and u are the variables of N-transform.
Some properties of N- transform are given below,
i. When $R(s,u)$ is N- transform and the Laplace transform is $L(s)$, then

$$N^{+}[f(t)] = R(s,u) = \frac{1}{u}\int_{0}^{\infty} e^{\frac{-st}{u}} f(t)dt = \frac{1}{u}L(\frac{s}{u}).$$

ii. When $R(s,u)$ is N- transform and Sumudu transform is $G(s)$,then

$$N^{+}[f(t)] = R(s,u) = \frac{1}{s}\int_{0}^{\infty} e^{-t} f(\frac{ut}{s})dt = \frac{1}{u}G(\frac{s}{u}).$$

iii. If $N^{+}[f(t)] = R(s,u)$, then

$$N^{+}[f(at)] = \frac{1}{a}R(s,u).$$

iv. If $N^{+}[f(t)] = R(s,u)$, then

$$N^{+}[f^{'}(t)] = \frac{s}{u}R(s,u) - \frac{f(0)}{u}.$$

v. If $N^{+}[f(t)] = R(s,u)$, then

$$N^{+}[f^{''}(t)] = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f^{'}(o)}{u}.$$

vi. If $f_n(t)$ is the nth order derivative w.r.t t then N-transform of the function is given,

$$N^{+}[f_n(t)] = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^{k}(0).$$

vii. Let the function $f(t)$ associated a set A multiplied by weight function then,

$$N^{+}[e^{\pm t}f(t)] = \frac{s}{s\pm u}R[\frac{su}{s\mp u}]$$

Table 1: N-Transform and Conversion to the Sumudu and Laplace Transform

2. The natural decomposition method for inviscid Burgers equation
The NDM demonstrated in the following inviscid BE

$$u_t + uu_x = 0 \qquad (2.\ 2)$$

f(t)	N[f(t)]	S[f(t)]	L[f(t)]
1	$\frac{1}{s}$	1	$\frac{1}{s}$
T	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{s-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3..$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\cos t$	$\frac{s}{s^2+u^2}$	$\frac{1}{1+u^2}$	$\frac{s}{s^2+1}$
$\sin t$	$\frac{s}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{s}{s^2+1}$

with the initial value

$$u(x, 0) = g(x) \tag{2. 3}$$

By taking the natural transform to the Eq. (2.2)

$$N^+[u_t] + N^+[uu_x] = 0$$

$$N^+(u) = R(x, s) = \frac{g(x)}{s} - \frac{u}{s}N^+[uu_x]. \tag{2. 4}$$

Applying the inverse transform of equation (2.4)

$$N^-[R(x, s)] = N^-[\frac{g(x)}{s} - \frac{u}{s}N^+[uu_x]]. \tag{2. 5}$$

From equation (2.5) we have

$$u(x, t) = G(x, t) - N^-[\frac{u}{s} - N^+[uu_x]]. \tag{2. 6}$$

Now, we find the non-linear term in series form

$$u(x, t) = \sum_{n=o}^{\infty} u_n(x, t). \tag{2. 7}$$

The nonlinear term uu_x decomposed as

$$uu_x = \sum_{n=o}^{\infty} A_n \tag{2. 8}$$

where A_n are the Adomian polynomials and calculated as

$$A_n = \frac{1}{n!}\frac{d^n}{dx^n}[F(\sum_{i=o}^{n} \lambda^i u_i(x, t))]_{\lambda=0} \tag{2. 9}$$

where n=0, 1, 2
Substituting Eq. (2.7) and Eq. (2.8) into equation (2.6), we get

$$\sum_{n=o}^{\infty} u_n(x,t) = G(x,t) + N^-[\frac{u}{s}N^+[\sum_{n=o}^{\infty} A_n]] \tag{2. 10}$$

By comparing Eq. (2.10) on both sides, we concluded that

$$u_0(x,t) = G(x,t)$$

,

$$u_1(x,t) = N^-[\frac{u}{s}N^+[A_0]]$$
$$u_2(x,t) = N^-[\frac{u}{s}N^+[A_1]]$$
$$:$$
$$:$$
$$u_{n+1} = N^-[\frac{u}{s}N^+[A_n]] \tag{2. 11}$$

In the general relation, we computed the remaining terms of $u(x,t)$ as $u_1, u_1, ...u_n$where u_o represent the initial condition.
The series solution can be concluded as follows,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{2. 12}$$

APPLICATION

We solved the numerical solution of BE by applying the NDM. Our examples show that the NDM solution converges rapidly to the exact solution. All the figures drawn by using MATLAB software.
Example 1
Consider the following inviscid BE,

$$u_t + uu_x = 0 \tag{2. 13}$$

subject to the condition:

$$u(x,0) = x \tag{2. 14}$$

By taking the natural transform to the Eq. (2.13), we obtain

$$N^+[u_t] + N^+[uu_x] = 0$$
$$R(x,s) = \frac{x}{s} - \frac{u}{s}N^+[uu_x] \tag{2. 15}$$

By taking the inverse transform of Eq. (2.15), we have

$$N^-[R(x,s)] = N^-[\frac{x}{s} - \frac{u}{s}N^+[uu_x]]$$

$$u(x,t) = x - N^{-}[\frac{u}{s}N^{+}[uu_x]]$$

Now, we assume the series solution of the following form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{2. 16}$$

The non-linear term represent as

$$uu_x = \sum_{n=0}^{\infty} A_n$$

Here,

$$A_0 = u_0 u_{0_x}$$
$$A_1 = u_{0_x} u_1 + u_0 u_{1_x}$$
$$A_2 = u_{0_x} u_2 + u_1 u_{1_x} + u_0 u_{2_x}$$
$$A_2 = u_{0_x} u_3 + u_2 u_{1_x} + u_1 u_{2_x} + u_0 u_{2_x}$$

therefore

$$u_0(x,t) = x \tag{2. 17}$$
$$u_1(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_0]]$$
$$u_1(x,t) = -xt \tag{2. 18}$$
$$u_2(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_1]]$$
$$u_2(x,t) = xt^2 \tag{2. 19}$$
$$u_3(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_2]]$$
$$u_3(x,t) = -xt^3 \tag{2. 20}$$

so,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)...$$
$$u(x,t) = x - xt + xt^2 - xt^3 + ... \tag{2. 21}$$

Similarly $u_4, u_5, u_6, ...$ can be calculated by NDM

$$u(x,t) = x(1 - t + t^2 - t^3 + ...)$$
$$u(x,t) = \frac{x}{1+t}, t < 1 \tag{2. 22}$$

It is required result.
Let $u_t = u_t(x,t)$, where $0.1 \leq t \leq 0.5$, be the exact solution from NDM. Table 2 demonstrates that the comparison of the exact solution obtained by NDM with the approximate result by ADM.

Example 2

Consider the inviscid BE (2.13) with initial condition

$$u(x,0) = -x \tag{2. 23}$$

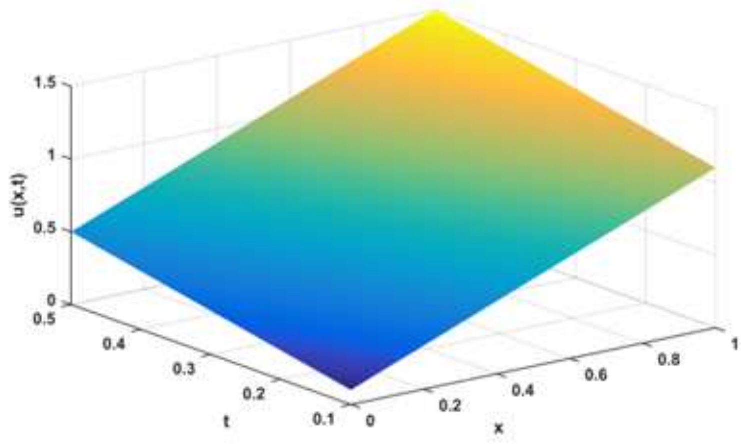

FIGURE 1. Analytic result for Example 1 for $0 \leq t \leq 1$ and $0.1 \leq t \leq 0.5$

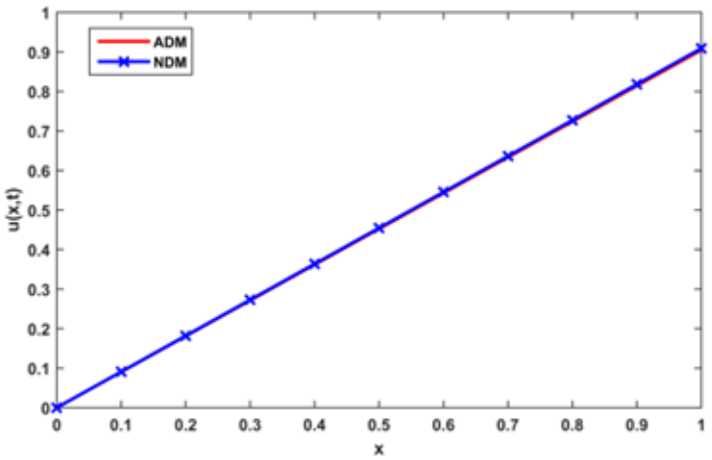

FIGURE 2. Comparison between the results of NDM and Laplace decomposition method(LDM) for $t = 0.1$

By taking the transform of natural in the Eq. (2.13) , we obtain

$$N^{+}[u_t] + N^{+}[uu_x] = 0$$

$$R(x,s) = \frac{x}{s} - \frac{u}{s} N^{+}[uu_x] \tag{2. 24}$$

By taking the inverse transform of Eq. (2.24), we have

$$N^{-}[R(x,s)] = N^{-}[\frac{x}{s} - \frac{u}{s}N^{+}[uu_x]]$$
$$u(x,t) = -x - N^{-}[\frac{u}{s}N^{+}[uu_x]]$$

The series solution of the problem can be written as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{2. 25}$$

The non-linear term expressed as

$$uu_x = \sum_{n=0}^{\infty} A_n$$

Therefore,

$$u_0(x,t) = -x \tag{2. 26}$$
$$u_1(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_1]]$$
$$u_1(x,t) = -xt \tag{2. 27}$$
$$u_2(x,t) = -xt^2 \tag{2. 28}$$
$$u_2(x,t) = -xt^3 \tag{2. 29}$$

For general form, we expressed as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)...$$
$$u(x,t) = -x - xt - xt^2 - xt^3 - ... \tag{2. 30}$$

Similarly $u_4, u_5, u_6, ...$ can be calculated by NDM

$$u(x,t) = \frac{x}{t-1}, \tag{2. 31}$$

it is require result.
Let $u_t = u_t(x,t)$, where $0.1 \leq t \leq 0.5$, be the exact solution from NDM. Table 3 demonstrates that the comparison of the exact solution obtained by NDM with the approximate result by ADM.

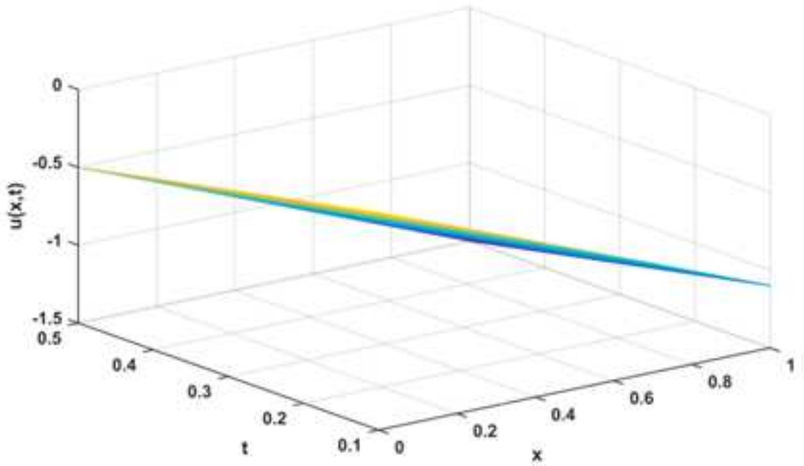

FIGURE 3. Analytic result for Example 2 for $0 \leq t \leq 1$ and $0.1 \leq t \leq 0.5$

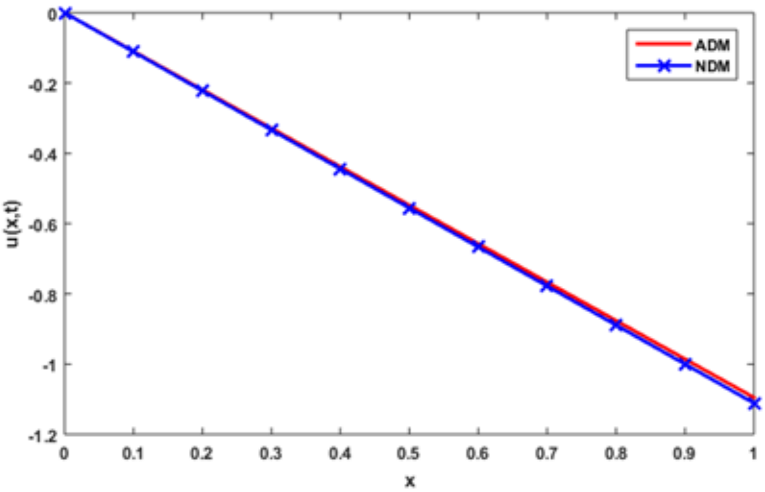

FIGURE 4. Comparison between the results of NDM and Laplace decomposition method(LDM) for $t = 0.1$

Example 3
Consider the inviscid BE (2.13) with initial condition

$$u(x, 0) = 2x \tag{2. 32}$$

Applying natural transform of Eq. (2.13) with the initial value

$$N^{+}u_t + N^{+}uu_x = 0$$

$$\frac{s}{u}R(x,s) - \frac{u(x,0)}{u} + N^{+}[uu_x] = 0$$

$$R(x,s) = \frac{2x}{s} - \frac{u}{s} + N^{+}[uu_x] \tag{2. 33}$$

Taking the inverse transform of Eq. (2.33)

$$N^{-}R(x,s) = N^{-}[\frac{2x}{s} - \frac{u}{s}N^{+}[uu_x]]$$

$$u(x,t) = 2x - N^{-}[\frac{u}{s}N^{+}[uu_x]] \tag{2. 34}$$

Series solution of the problem can be written as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{2. 35}$$

The non-linear term expressed as

$$uu_x = \sum_{n=0}^{\infty} A_n$$

Therefore,

$$u_0(x,t) = 2x \tag{2. 36}$$

$$u_1(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_0]]$$

$$u_1(x,t) = -4xt \tag{2. 37}$$

$$u_2(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_1]]$$

$$u_2(x,t) = 8xt^2 \tag{2. 38}$$

$$u_3(x,t) = -N^{-}[\frac{u}{s}N^{+}[A_2]]$$

$$u_3(x,t) = -16xt^2 \tag{2. 39}$$

In general form, we expressed as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)...$$

$$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + ... \tag{2. 40}$$

Similarly,$u_4, u_5, u_6, ...$ can be calculated by NDM

$$u(x,t) = 2x(1 - 2t + 4t^2 - 8t^3 + ...)$$

After simplification, it can be written as,

$$u(x,t) = 2x(1+2t)^{-1}$$

$$u(x,t) = \frac{2x}{1+2t}, \tag{2. 41}$$

it is require result.

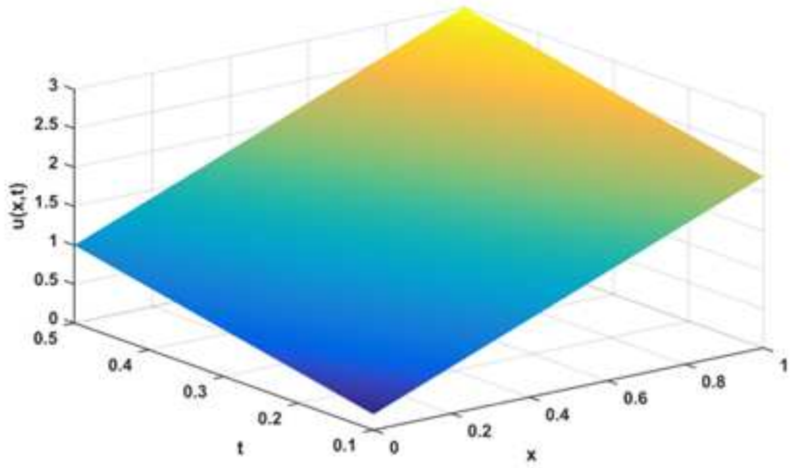

FIGURE 5. Analytic result for Example 3 for $0 \leq t \leq 1$ and $0.1 \leq t \leq 0.5$

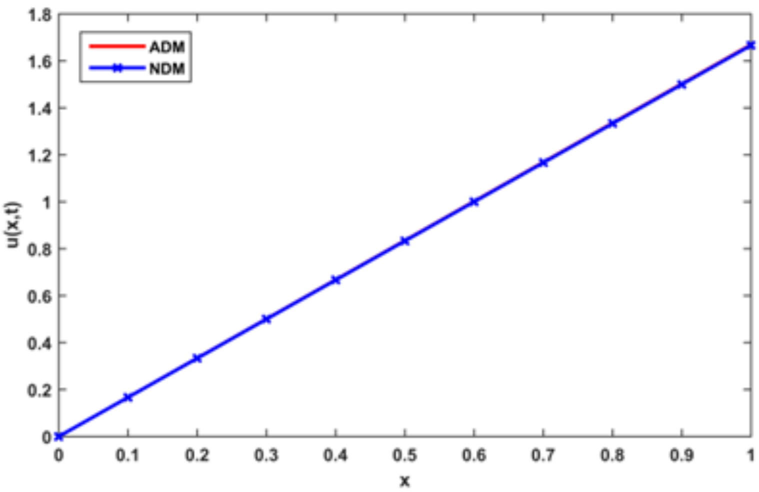

FIGURE 6. Comparison between the results of NDM and Laplace decomposition method(LDM) for $t = 0.1$

Let $u_t = u_t(x,t)$, where $0.1 \leq t \leq 0.5$, be the exact solution from NDM. Table 4 demonstrates that the comparison of the exact solution obtained by NDM with the approximate result by ADM.

Table 2: Comparison of the results for inviscid BE Problem 1

x	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]
0	0	0	0	0	0	0	0	0	0	0
0.2	0.1818	0.1810	0.1667	0.1637	0.1538	0.1482	0.1429	0.1341	0.1500	0.1213
0.4	0.3636	0.3619	0.3333	0.3275	0.3077	0.2963	0.2857	0.2681	0.2781	0.2426
0.6	0.5455	0.5429	0.5000	0.4912	0.4615	0.4445	0.4286	0.4022	0.4111	0.3639
0.8	0.7273	0.7239	0.6667	0.6550	0.6134	0.5927	0.5714	0.5363	0.5444	0.4852
1	0.9091	0.9048	0.8333	0.8187	0.7692	0.7408	0.7143	0.6703	0.6703	0.6065

Table 3: Comparison of the results for inviscid BE Problem 2

X	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]
0	0	0	0	0	0	0	0	0	0	0
0.2	-0.222	-0.219	-0.250	-0.236	-0.286	-0.253	-0.333	-0.266	-0.400	-0.279
0.4	-0.444	-0.438	-0.500	-0.473	-0.571	-0.504	-0.667	-0.532	-0.800	-0.557
0.6	-0.667	-0.657	-0.750	-0.709	-0.857	-0.756	-1.000	-0.798	-1.200	-0.836
0.8	-0.889	-0.876	-1..000	-0.945	-1.143	-1.007	-1.333	-1.064	-1.600	-1.115
1	-1.111	-1.095	-1.250	-1.181	-1.429	-1.259	-1.667	-1.330	-2.000	-1.393

Table 4: Comparison of the results for inviscid BE Problem 3

x	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]	OURS	ADM[25]
0	0	0	0	0	0	0	0	0	0	0
0.2	0.3333	0.3341	0.2857	0.2899	0.2500	0.2602	0.2222	0.2403	0.2000	0.2270
0.4	0.6667	0.6681	0.5714	0.5797	0.5000	0.5205	0.4444	0.4807	0.4000	0.4540
0.6	1.0000	1.0022	0.8571	0.8696	0.7500	0.7807	0.6667	0.7210	0.6000	0.6811
0.8	1.3333	1.3363	1.1429	1.1591	1.0000	1.0410	0.8889	0.9613	0.8000	0.9081
1	1.6667	1.6703	1.4286	1.4493	1.2500	1.3012	1.1111	1.2017	1.0000	1.1351

3. CONCLUSION

In this study, we applied the NDM to solve the inviscid BE and compare our results with the already existing numerical results in the literature. This analytic method has studied a numerical solution that converges rapidly to the exact solution. This method computes more accuracy and reduces the errors. We proposed an analytical approach to solve several nonlinear differential equations due to high efficiency. The NDM can be used to solve other non-linear equations that exist in many other fields such as science and engineering in the future.

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