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Decomposition of complete graphs into paths and cycles of distinct lengths

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Abstract.: Let P_k be the path with k edges and C_k be the cycle with k edges. For $r \geq 3$, we exhibit two decompositions of the complete graph K_{2r+3} into edge-disjoint paths and cycles: the first is of the form $\langle P_3, P_4, C_5, C_6, ..., C_{2r-1}, C_{2r+1}, C_{2r+2}, C_{2r+3}\rangle$ and the second $\langle P_3, P_4, P_5, C_6, ..., C_{2r-1}, C_{2r+1}, C_{2r+2}, C_{2r+3}\rangle$.

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1. Introduction

In this paper, all graphs are assumed to be finite and simple. If H is a subgraph of G, we denote by $G \setminus H$ the subgraph of G obtained by removing all edges of H. We denote by K_n the complete graph on n vertices, by P_n the path with n edges, and by C_n the cycle with n edges. The notation $[v_0, v_1, ..., v_k]$ denotes a path with k edges $v_0v_1, v_1v_2, ..., v_{k-1}v_k$ and $(v_0, v_1, ..., v_{k-1})$ denotes a cycle with k edges $v_0v_1, v_1v_2, ..., v_{k-1}v_0$. We say that edge-disjoint subgraphs $H_1, H_2, ..., H_l$ of a graph G decompose G if their edges partition those of G and express this by writing $\langle H_1, ..., H_l|G\rangle$.

In [2], Alspach posed the following problem. Let n be a positive integer and $m_1, ..., m_r \geq 3$ integers such that $m_1 + ... + m_r = \begin{cases} n(n-1)/2 & if\ n\ is\ odd \\ n(n-1)/2 - n/2 & if\ n\ is\ even. \end{cases}$
Then the complete graph K_n (when n is odd) or $K_n - I$ (when n is even and I ia a 1-factor) can be decomposed into m_i-cycles. Interest in this problem led at first to many partial solutions (see [[3], [6], [7], [8], [10], [12]]); it was settled completely in 2014 by Bryant *et al.* [9]. A general survey on cycle decompositions of K_n may be found in [4]. In 1995, Bryant and Adams [5] proved that if $n \geq 7$ is odd, then $\langle C_3, C_4, C_5, C_6, ..., C_{n-4}, C_{n-2}, C_{n-1}, C_n$ $|K_n\rangle$. In this paper, we exhibit, for $r \geq 3$, two decompositions of K_{2r+3}:
$\langle P_3, P_4, C_5, C_6, ..., C_{2r-1}, C_{2r+1}, C_{2r+2}, C_{2r+3}\rangle$ and
$\langle P_3, P_4, P_5, C_6, ..., C_{2r-1}, C_{2r+1}, C_{2r+2}, C_{2r+3}\rangle$.

2. Notations and Preliminaries

A path with k edges is denoted by P_k and a cycle with k edges is denoted by C_k. Let $\mathcal{P} = \{P_{i_1}, P_{i_2}, ..., P_{i_t}\}$ be the set of paths. If the terminal vertices of $P_{i_1}, P_{i_2}, ..., P_{i_t}$ are all distinct, then $\mathcal{P}$ is called the terminal-vertex disjoint.

The following lemma was proved independently by Bryant and Adams [5] and Chin-Mei Fu *et al*. [11]. We give a proof of the lemma, in our own words, as it could be helpful in understanding the construction of paths and cycles in the proofs of Theorem 3.1 and 3.2.

Lemma 2.1. *[[5], [11]] For any $r \epsilon \mathbb{N}$, there exists a path decomposition $\langle P_1, P_2, ..., P_{2r} | K_{2r+1}\rangle$ such that $P_1, P_3, ..., P_{2r-1}$ and $P_2, P_4, ..., P_{2r}$ are terminal-vertex disjoint.*

Proof. Let $V(K_{2r+1}) = \{v_0, v_1, ..., v_{2r-1}\} \cup \{\infty\}$. Consider the decomposition of K_{2r+1} into Hamilton cycles constructed by Walecki [1]:
$H_i = (\infty, v_i, v_{2r-1+i}, v_{1+i}, v_{2r-2+i}, ..., v_{r-1+i}, v_{r+i})$, $(0 \leq i \leq r-1)$, where the subscripts of v are taken modulo $2r$. These Hamilton cycles can be decomposed into paths of distinct lengths whose terminal-vertices (u, v) are as follows:

$$(u,v) = \begin{cases} (v_{2i}, v_{r+2i}) & \text{if } 0 \leq i \leq r-1 \text{ and } r \text{ is odd} \\ (\infty, v_0) & \text{if } i = 0 \text{ and } r \text{ is even} \\ (v_{2i-1}, v_{r+2i-1}) & \text{if } 1 \leq i \leq \frac{r}{2} \text{ and } r \text{ is even} \\ (v_{2i-r}, v_{2i}) & \text{if } \frac{r}{2} < i \leq r-1 \text{ and } r \text{ is even} \end{cases}$$

We observe that ∞ is not a terminal-vertex of any path when r is odd and v_r is not a terminal-vertex of any path when r is even in the above decomposition.

For example, consider the complete graph K_7. Hence $r = 3$. The Hamilton cycle H_0=$(\infty, v_0, v_5, v_1, v_4, v_2, v_3)$ can be decomposed into the paths $P_2 = [v_0, \infty, v_3]$ and $P_5 = [v_0, v_5, v_1, v_4, v_2, v_3]$. The Hamilton cycle H_1=$(\infty, v_1, v_0, v_2, v_5, v_3, v_4)$ can be decomposed into the paths $P_6 = [v_2, v_0, v_1, \infty, v_4, v_3, v_5]$ and $P_1 = [v_2, v_5]$. The Hamilton cycle H_2=$(\infty, v_2, v_1, v_3, v_0, v_4, v_5)$ can be decomposed into the paths $P_4 = [v_1, v_2, \infty, v_5, v_4]$ and $P_3 = [v_1, v_3, v_0, v_4]$.

□

3. Decomposition of K_n, $n \geq 9$ into paths and cycles of distinct lengths

Theorem 3.1. *If $r \geq 3$, then $\langle P_3, P_4, C_5, C_6, ..., C_{2r-1}, C_{2r+1}, C_{2r+2}, C_{2r+3} | K_{2r+3}\rangle$.*

Proof. The obvious edge-divisibility condition is not satisfied when $r < 3$. Consider the subgraph H of K_{2r+3} induced by the vertices $\{\infty, v_0, v_1, ..., v_{2r-1}\}$. From Lemma 2.1, there exists a path decomposition $\langle P_1, P_2, ..., P_{2r} | H\rangle$ such that $P_1, P_3, ..., P_{2r-1}$ and $P_2, P_4, ..., P_{2r}$ are terminal-vertex disjoint. Construct edge disjoint cycles $C_4, C_6, ..., C_{2r+2}$ in K_{2r+3} by joining the endpoints of each of the paths $P_2, P_4, ..., P_{2r}$ in H to the vertex v_{2r}. Likewise, construct edge disjoint cycles $C_3, C_5, ..., C_{2r+1}$ in K_{2r+3} by joining the endpoints of each of the paths $P_1, P_3, ..., P_{2r-1}$ in H to the vertex v_{2r+1}. We now describe the procedure for constructing our edge decomposition for K_{2r+3}. To get the required decomposition we divide the proof into two cases.
Case I: $r \geq 3$ is odd.

From Lemma 2.1, ∞ is not a terminal-vertex of any path in the $P_1, P_2, ..., P_{2r}$ decomposition of K_{2r+1}. So the edges $\infty v_{2r}, \infty v_{2r+1}$ and $v_{2r}v_{2r+1}$ are not in any cycles of K_{2r+3}. Consider the Hamilton cycles $H_0, H_{\frac{r-1}{2}}$ and $H_{\frac{r+1}{2}}$ in K_{2r+1}. The path decomposition of these Hamilton cycles are $\langle P_2, P_{2r-1}|H_0\rangle$,$\langle P_1, P_{2r}|H_{\frac{r-1}{2}}\rangle$ and $\langle P_3, P_{2r-2}|H_{\frac{r+1}{2}}\rangle$. In K_{2r+3}, we have

$$\begin{aligned}
C_3 &: P_1 \cup [v_{2r-1}, v_{2r+1}, v_{r-1}]\\
C_4 &: P_2 \cup [v_r, v_{2r}, v_0]\\
C_5 &: P_3 \cup [v_{r+1}, v_{2r+1}, v_1]\\
C_{2r} &: P_{2r-2} \cup [v_{r+1}, v_{2r}, v_1]\\
C_{2r+1} &: P_{2r-1} \cup [v_r, v_{2r+1}, v_0]\\
C_{2r+2} &: P_{2r} \cup [v_{2r-1}, v_{2r}, v_{r-1}]
\end{aligned}$$

By using these cycles and the triangle $(\infty, v_{2r}, v_{2r+1})$, now we construct the paths $P_3^{'}, P_4^{'}$ and cycles $C_5^{'}, C_{2r+1}^{'}, C_{2r+2}^{'}$ and $C_{2r+3}^{'}$ and we keep the remaining cycles not taken for construction. The reconstruction is as follows:

$$C_{2r+1}^{'} : P_3 \cup P_{2r-2}$$

$$C_{2r+2}^{'} : (P_{2r-1} \cup P_2 \backslash \infty v_0) \cup [\infty, v_{2r}, v_0]$$

i.e., delete one edge ∞v_0 from P_2 and add 2 edges, ∞v_{2r} from the triangle and $v_{2r}v_0$ from C_4.

$$C_{2r+3}^{'} : (P_1 \cup P_{2r} \backslash v_{2r-1}v_r) \cup [v_{2r-1}, v_{2r}, v_{2r+1}, v_r]$$

i.e., delete one edge $v_{2r-1}v_r$ from P_{2r} and add 3 edges, $v_{2r-1}v_{2r}$ from $C_{2r+2}, v_{2r}v_{2r+1}$ from the triangle and $v_{2r+1}v_r$ from C_{2r+1}.

The remaining paths and edges are $[v_{2r-1}, v_{2r+1}, v_{r-1}]$, $v_r v_{2r}$, $[v_{r+1}, v_{2r+1}, v_1]$, $v_{2r+1}v_0$, $v_{2r}v_{r-1}$, $[v_{r+1}, v_{2r}, v_1]$, ∞v_{2r+1}, $v_{2r-1}v_r$ and ∞v_0. These paths and edges are used to construct $P_3^{'}, P_4^{'}$ and $C_5^{'}$, see Figure 1.

$$\begin{aligned}
P_3^{'} &:[\infty, v_0, v_{2r+1}, v_1]\\
P_4^{'} &:[v_1, v_{2r}, v_{r-1}, v_{2r+1}, \infty]\\
C_5^{'} &:(v_{2r}, v_{r+1}, v_{2r+1}, v_{2r-1}, v_r)
\end{aligned}$$

These newly constructed paths and cycles along with the cycles which were not taken for the construction give the required decomposition.

Case II: $r \geq 4$ is even.

From Lemma 2.1, v_r is not a terminal-vertex of any path in the $P_1, P_2, ..., P_{2r}$ decomposition of K_{2r+1}. So the edges $v_r v_{2r}, v_r v_{2r+1}$ and $v_{2r}v_{2r+1}$ are not in any cycles of K_{2r+3}. Consider the Hamilton cycles H_0, H_1 and $H_{\frac{r}{2}}$ in K_{2r+1}. The path decomposition of these Hamilton cycles are $\langle P_1, P_{2r}|H_0\rangle, \langle P_2, P_{2r-1}|H_1\rangle$ and $\langle P_3, P_{2r-2}|H_{\frac{r}{2}}\rangle$. In K_{2r+3}, we

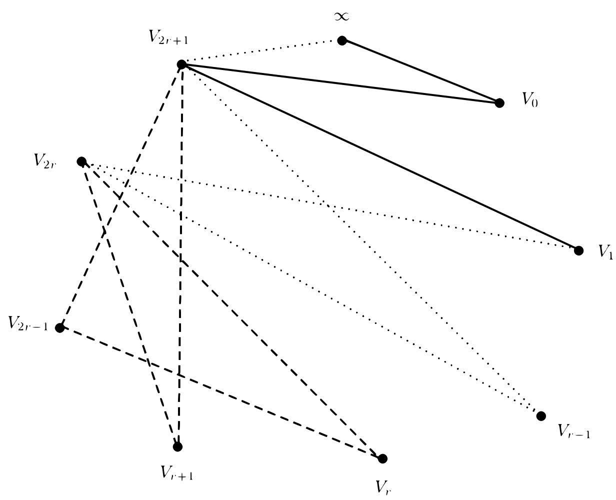

FIGURE 1. $P_3^{'}, P_4^{'}$ and $C_5^{'}$ in K_{2r+3}, r is odd

have

$$
\begin{aligned}
C_3 &: P_1 \cup [\infty, v_{2r+1}, v_0] \\
C_4 &: P_2 \cup [v_{r+1}, v_{2r}, v_1] \\
C_5 &: P_3 \cup [v_{2r-1}, v_{2r+1}, v_{r-1}] \\
C_{2r} &: P_{2r-2} \cup [v_{2r-1}, v_{2r}, v_{r-1}] \\
C_{2r+1} &: P_{2r-1} \cup [v_{r+1}, v_{2r+1}, v_1] \\
C_{2r+2} &: P_{2r} \cup [\infty, v_{2r}, v_0]
\end{aligned}
$$

By using these cycles and the triangle (v_r, v_{2r}, v_{2r+1}), now we construct the paths $P_3^{'}, P_4^{'}$ and cycles $C_5^{'}$, $C_{2r+1}^{'}$, $C_{2r+2}^{'}$ and $C_{2r+3}^{'}$ and we keep the remaining cycles not taken for construction. The reconstruction is as follows:

$$C_{2r+1}^{'} : P_3 \cup P_{2r-2}$$

$$C_{2r+2}^{'} : (P_1 \cup P_{2r} \backslash v_{r-1}v_r) \cup [v_{r-1}, v_{2r+1}, v_r]$$

i.e., delete one edge $v_{r-1}v_r$ from P_{2r} and add 2 edges, $v_{r-1}v_{2r+1}$ from C_5 and $v_{2r+1}v_r$ from the triangle.

$$C_{2r+3}^{'} : (P_{2r-1} \cup P_2 \backslash \infty v_1) \cup [\infty, v_{2r}, v_{2r+1}, v_1]$$

i.e., delete one edge ∞v_1 from P_2 and add 3 edges, ∞v_{2r} from C_{2r+2}, $v_{2r}v_{2r+1}$ from the triangle and $v_{2r+1}v_1$ from C_{2r+1}.

The remaining paths and edges are $[\infty, v_{2r+1}, v_0]$, $[v_{r+1}, v_{2r}, v_1]$, $v_{2r-1}v_{2r+1}$, $v_{2r}v_0$, $v_{r+1}v_{2r+1}$, $[v_{2r-1}, v_{2r}, v_{r-1}]$, ∞v_1, $v_{r-1}v_r$ and v_rv_{2r}. These paths and edges are used to construct $P_3^{'}, P_4^{'}$ and $C_5^{'}$, see Figure 2.

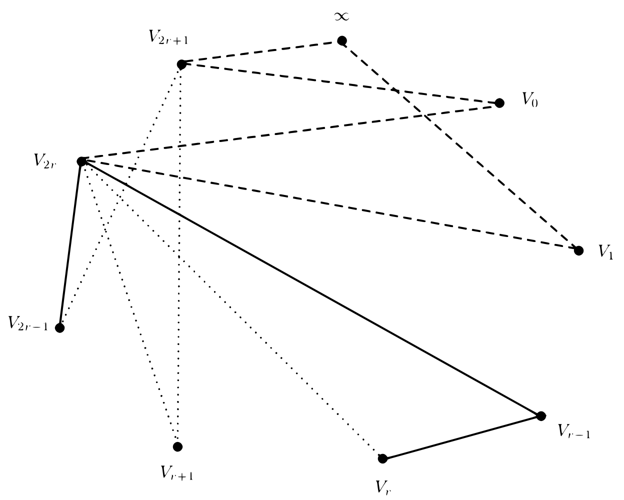

FIGURE 2. $P_3^{'}, P_4^{'}$ and $C_5^{'}$ in K_{2r+3}, r is even

$$P_3^{'} \quad :[v_r, v_{r-1}, v_{2r}, v_{2r-1}]$$
$$P_4^{'} \quad :[v_{2r-1}, v_{2r+1}, v_{r+1}, v_{2r}, v_r]$$
$$C_5^{'} \quad :(\infty, v_{2r+1}, v_0, v_{2r}, v_1)$$

These newly constructed paths and cycles along with the cycles which were not taken for the construction give the required decomposition. □

In Theorem 3.1, we have proved that the complete graph K_{2r+3} can be decomposed into paths and cycles of distinct lengths such that the paths are of lengths 3 and 4 and the cycles are of lengths $5, 6, ..., 2r-1, 2r+1, 2r+2$ and $2r+3$. In the following theorem we prove a similar decomposition of K_{2r+3} in which paths are of lengths 3,4 and 5 and cycles are of lengths $6, 7, ..., 2r-1, 2r+1, 2r+2$ and $2r+3$.

In the proof of Theorem 3.2, the constrction of the cycles $C_3, C_4, ..., C_{2r+2}$ is similar to that of Theorem 3.1. Among these cycles we choose appropriate edges to interchange between them to get the required decomposition.

Theorem 3.2. *If* $r \geq 3$*, then* $\langle P_3, P_4, P_5, C_6, ..., C_{2r-1}, C_{2r+1}, C_{2r+2}, C_{2r+3} | K_{2r+3} \rangle$.

Proof. First we construct the cycles $C_3, C_4, ..., C_{2r+1}, C_{2r+2}$ in K_{2r+3} as in Theorem 3.1.
Case I: $r \geq 3$ is odd.
We construct $C_3, C_4, C_5, C_{2r}, C_{2r+1}, C_{2r+2}$ in K_{2r+3} as in Case I of Theorem 3.1. By using these cycles and the triangle $(\infty, v_{2r}, v_{2r+1})$, we construct the paths $P_3^{'}$, $P_4^{'}$, $P_5^{'}$ and cycles $C_{2r+1}^{'}$, $C_{2r+2}^{'}$ and $C_{2r+3}^{'}$ as follows:

$$C_{2r+1}^{'} : P_1 \cup P_{2r}$$

$$C_{2r+2}^{'} : (P_2 \cup P_{2r-1} \backslash v_{r-1}v_r) \cup [v_{r-1}, v_{2r+1}, v_r]$$

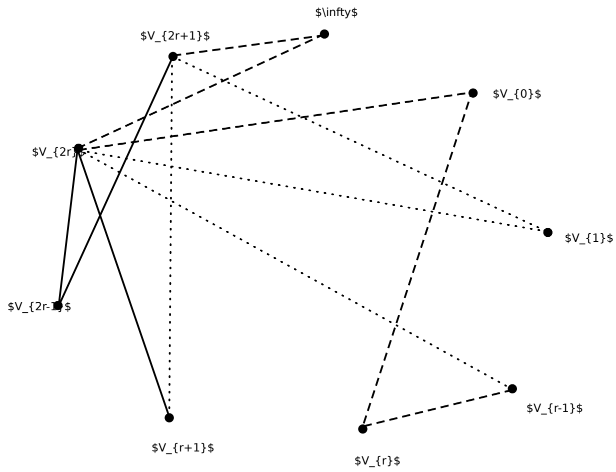

FIGURE 3. $P_3^{'}, P_4^{'}$ and $P_5^{'}$ in K_{2r+3}, r is odd

i.e., delete one edge $v_{r-1}v_r$ from P_{2r-1} and add 2 edges, $v_{r-1}v_{2r+1}$ from C_3 and $v_{2r+1}v_r$ from C_{2r+1}.

$$C_{2r+3}^{'} : (P_{2r-2} \cup P_3 \backslash v_r v_0) \cup [v_r, v_{2r}, v_{2r+1}, v_0]$$

i.e., delete one edge $v_r v_0$ from P_3 and add 3 edges, $v_r v_{2r}$ from C_4, $v_{2r}v_{2r+1}$ from the triangle and $v_{2r+1}v_0$ from C_{2r+1}.

The remaining paths and edges are $v_{2r-1}v_{2r+1}$, $v_{2r}v_0$, $[v_{r+1}, v_{2r+1}, v_1]$, $[v_{2r-1}, v_{2r}, v_{r-1}]$, $[v_{r+1}, v_{2r}, v_1]$, ∞v_{2r}, ∞v_{2r+1}, $v_r v_0$ and $v_{r-1}v_r$. These paths and edges are used to construct $P_3^{'}, P_4^{'}$ and $P_5^{'}$, see Figure 3.

$$P_3^{'} \quad :[v_{r+1}, v_{2r}, v_{2r-1}, v_{2r+1}]$$
$$P_4^{'} \quad :[v_{r-1}, v_{2r}, v_1, v_{2r+1}, v_{r+1}]$$
$$P_5^{'} \quad :[v_{r-1}, v_r, v_0, v_{2r}, \infty, v_{2r+1}]$$

Case II: $r \geq 4$ is even.
We construct $C_3, C_4, C_5, C_{2r}, C_{2r+1}, C_{2r+2}$ in K_{2r+3} as in Case II of Theorem 3.1. By using these cycles and the triangle (v_r, v_{2r}, v_{2r+1}), we construct the paths $P_3^{'}, P_4^{'}, P_5^{'}$ and cycles $C_{2r+1}^{'}, C_{2r+2}^{'}$ and $C_{2r+3}^{'}$ as follows:

$$C_{2r+1}^{'} : P_2 \cup P_{2r-1}$$

$$C_{2r+2}^{'} : (P_1 \cup P_{2r} \backslash v_{r-1}v_r) \cup [v_{r-1}, v_{2r+1}, v_r]$$

i.e., delete one edge $v_{r-1}v_r$ from P_{2r} and add 2 edges, $v_{r-1}v_{2r+1}$ from C_5 and $v_{2r+1}v_r$ from the triangle.

$$C_{2r+3}^{'} : (P_{2r-2} \cup P_3 \backslash v_{r-1}v_0) \cup [v_{r-1}, v_{2r}, v_{2r+1}, v_0]$$

i.e., delete one edge $v_{r-1}v_0$ from P_3 and add 3 edges, $v_{r-1}v_{2r}$ from C_{2r}, $v_{2r}v_{2r+1}$ from the triangle and $v_{2r+1}v_0$ from C_3.

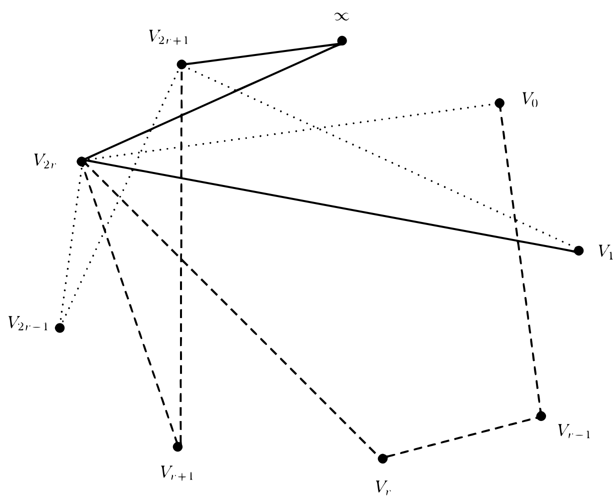

FIGURE 4. $P_3^{'}, P_4^{'}$ and $P_5^{'}$ in K_{2r+3}, r is even

The remaining paths and edges are ∞v_{2r+1}, $[v_{r+1}, v_{2r}, v_1]$, $v_{2r-1}v_{2r+1}$, $[\infty, v_{2r}, v_0]$,$[v_{r+1}, v_{2r+1}, v_1]$, $v_{2r-1}v_{2r}$, $v_{r-1}v_0$, v_rv_{2r} and $v_{r-1}v_r$. These paths and edges are used to construct $P_3^{'}, P_4^{'}$ and $P_5^{'}$, see Figure 4.

$$P_3^{'} \quad :[v_{2r+1}, \infty, v_{2r}, v_1]$$
$$P_4^{'} \quad :[v_1, v_{2r+1}, v_{2r-1}, v_{2r}, v_0]$$
$$P_5^{'} \quad :[v_0, v_{r-1}, v_r, v_{2r}, v_{r+1}, v_{2r+1}]$$

Hence we get the required decomposition as in the previous theorem. □

4. ACKNOWLEDGMENTS

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