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New Quartic B-spline Approximations for Numerical Solution of Fourth Order Singular Boundary Value Problems

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Abstract. In this work, we have proposed a new quartic B-spline (QBS) approximation technique for numerical treatment of fourth order singular boundary value problems. The typical QBS functions in association with new approximations for third and fourth order derivatives are employed to interpolate the solution in spatial domain. An error analysis is presented and the proposed numerical technique is proved to be uniformly convergent. We have corroborated this work by considering some test examples containing singularity at one of their boundaries. The comparison of approximate results affirms the superiority of our new approximation method over current methods on the topic.

AMS (MOS) Subject Classification Codes: 34B15; 34B16; 74H15; 65L10; 65L11.
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1. INTRODUCTION

Singular boundary value problems (SBVP's) arise in Mathematical modeling of several real life phenomena such as chemical reactions, electrohydrodynamics, aerodynamics, thermal explosions, elastic stability, gravity assisted flows, inelastic flows, atomic nuclear reactions and electrically charged fluid flows. In this work, we have considered the following class of 4^{th} order SBVP's

$$\alpha y^{(4)}(x) + \frac{m}{x} y^{(3)}(x) + p(x)y''(x) + q(x)y'(x) = f(x, y),\ 0 \leq x \leq 1, \tag{1. 1}$$

subject to one of the following sets of initial/boundary conditions

$$\begin{cases} y(0) = \alpha_1,\ y'(0) = \alpha_2,\ y''(0) = \alpha_3,\ y'''(0) = 0, \\ y(0) = \alpha_1,\ y'(0) = \alpha_2,\ y(1) = \alpha_3,\ \ y'(1) = \alpha_4, \\ y(0) = \alpha_1,\ y''(0) = \alpha_2,\ y(1) = \alpha_3,\ \ y''(1) = \alpha_4, \end{cases} \tag{1. 2}$$

where m, α, α_i's are constants and $p(x)$, $q(x)$, $f(x, y)$ are known functions. To ensure the existence of unique solution to (1. 1)–(1. 2), we presume that f and f_u are sufficiently smooth and f_u is non-negative in the entire domain. In recent years, many numerical and analytical techniques have been proposed for solving SBVP's. A new decomposition method based on Adomian polynomials has been developed by Khuri [15] for numerical treatment of generalized Lane-Emden type equations. Kim and Chun [17] proposed modified adomian decomposition method for series solution of higher order SBVP's. Aruna and Kanth [5] employed differential transformation method to examine the power series solution of higher order SBVP's. Wazwaz [24] investigated the approximate solution of fourth order initial value problems by means of variational iteration method. Taiwo and Hassan [22] presented a new iterative decomposition method for solving higher order initial and boundary value problems. The authors in [23] studied the series solution of a class of fourth order singular initial value problems using Adomian decomposition method. Hasan and Zhu [10, 11] used modified Adomian decomposition method for solving singular boundary value problems of higher-order ordinary differential equations.

The spline approximation techniques have been extensively applied for numerical simulation of initial and boundary value problems (BVP's). The cubic spline (CS) functions have been utilized for solving second order SBVP's in [2, 7, 8, 9, 12]. Khuri and Sayfy [16] developed a new adaptive cubic B-spline (CBS) collocation approach for numerical solution of second order SBVP's. The quartic polynomial spline functions were used by Akram [3] for solving a class of 3^{rd} order singularly perturbed BVP's. Mishra and Saini [20] explored the approximate solution of 3^{rd} order self adjoint singularly perturbed BVP's using typical QBS collocation method. Akram and Amin [4] employed fifth degree polynomial spline functions for solving fourth order singularly perturbed BVP's. Lodhi and Mishra

[19] utilized quintic B-spline (QnBS) functions for numerical treatment of fourth order singularly perturbed SBVP's.

In this paper, we have explored the approximate solution of fourth order SBVP's using fourth degree basis spline functions powered by new QBS approximations for 3^{rd} and 4^{th} order derivatives. In recent years, several numerical methods have been proposed to delve into numerical solution of SBVP's but as far as we know, this technique is novel and has not been employed for solving BVP's before.

This paper is arranged as: In section 2, we have presented some basic ideas of typical QBS interpolation. The new QBS approximations for third and fourth order derivatives have been formulated in section 3 and 4. The numerical scheme is described in section 5. The derivation of uniform convergence is given in section 6. The approximate results and discussion are presented in section 7.

2. Quartic B-spline Functions

We uniformly partition the unit interval $[a, b]$ by $n + 1$ equally spaced knots $x_i = a + ih,\ i = 0 : 1 : n$, where $n \in \mathbb{Z}^+$, $a = x_0$, $b = x_n$ and $h = \frac{1}{n}(b - a)$. We extend $[a, b]$ to $[a - 4h, b + 4h]$ with equally spaced knots $x_i = a + ih,\ i = -4, -3, -2, \cdots, n + 4$, and define the typical QBS functions as [9, 18]

$$B_j(x) = \frac{1}{24h^4} \begin{cases} (x - x_{j-2})^4, & x \in [x_{j-2}, x_{j-1}] \\ (x - x_{j-2})^4 - 5(x - x_{j-1})^4, & x \in [x_{j-1}, x_j] \\ (x - x_{j-2})^4 - 5(x - x_{j-1})^4 + 10(x - x_j)^4, & x \in [x_j, x_{j+1}] \\ (x_{j+3} - x)^4 - 5(x_{j-2} - x)^4, & x \in [x_{j+1}, x_{j+2}] \\ (x_{j+3} - x)^4, & x \in [x_{j+2}, x_{j+3}] \\ 0 & \text{otherwise.} \end{cases} \tag{2. 3}$$

For an adequately smooth function $y(x)$, there always exists a unique quartic spline $Y(x)$ which satisfies the prescribed interpolating conditions such that

$$Y(x) = \sum_{j=-2}^{n+1} c_j B_j(x), \tag{2. 4}$$

where, the real constants, c_j's are to be determined. Let Y_i, m_i, M_i, T_i and F_i denote the QBS approximations for $y(x)$ and its first four derivatives at the i^{th} knot respectively.

Using (2. 3) and (2. 4), we have

$$Y_i = Y(x_i) = \sum_{j=i-2}^{i+1} c_j B_j(x_i) = \frac{1}{24}\big(c_{i-2} + 11c_{i-1} + 11c_i + c_{i+1}\big), \quad (2.\ 5)$$

$$m_i = Y'(x_i) = \sum_{j=i-2}^{i+1} c_j B'_j(x_i) = \frac{1}{6h}\big(- c_{i-2} - 3c_{i-1} + 3c_i + c_{i+1}\big), \quad (2.\ 6)$$

$$M_i = Y''(x_i) = \sum_{j=i-2}^{i+1} c_j B''_j(x_i) = \frac{1}{2h^2}\big(c_{i-2} - c_{i-1} - c_i + c_{i+1}\big), \quad (2.\ 7)$$

$$T_i = Y'''(x_i) = \sum_{j=i-2}^{i+1} c_j B'''_j(x_i) = \frac{1}{h^3}\big(- c_{i-2} + 3c_{i-1} - 3c_i + c_{i+1}\big). \quad (2.\ 8)$$

Moreover, from (2. 5)-(2. 8), we can established the following relations [9]

$$m_i = y'(x_i) + \frac{h^4}{720} y^{(5)}(x_i) + \cdots, \quad (2.\ 9)$$

$$M_i = y''(x_i) - \frac{h^4}{240} y^{(6)}(x_i) + \cdots, \quad (2.\ 10)$$

$$T_i = y'''(x_i) - \frac{h^2}{12} y^{(5)}(x_i) + \frac{h^4}{240} y^{(7)}(x_i) + \cdots. \quad (2.\ 11)$$

From (2. 9)-(2. 11), we have

$$\|m_i - y'(x_i)\|_\infty = \max_{0\le j\le n} \|m_i - y'(x_i)\| = O(h^4),$$
$$\|M_i - y''(x_i)\|_\infty = O(h^4),$$
$$\|T_i - y^{(3)}(x_i)\|_\infty = O(h^2).$$

To improve the truncation error in T_i, we formulate a new approximation for $y^{(3)}(x)$. Moreover, a new QBS approximation for $y^{(4)}(x)$ is also required.

3. THE NEW APPROXIMATION FOR $y^{(3)}(x)$

Using (2. 11), the following expression can be established for T_{i-2} at the knot x_i, ($i = 2, 3, 4, \cdots, n-2$)[14]

$$\begin{aligned} T_{i-2} &= y^{(3)}(x_{i-2}) - \frac{h^2}{12} y^{(5)}(x_{i-2}) + \frac{h^4}{240} y^{(7)}(x_{i-2}) + \cdots \\ &= y^{(3)}(x_i) - 2h y^{(4)}(x_i) + \frac{23h^2}{12} y^{(5)}(x_i) - \frac{7h^3}{6} y^{(6)}(x_i) + \frac{121h^4}{240} y^{(7)}(x_i) + \cdots. \end{aligned}$$

We can derive similar relations for T_{i-1}, T_{i+1} and T_{i+2} at i^{th} knot as

$$\begin{aligned} T_{i-1} &= y^{(3)}(x_i) - hy^{(4)}(x_i) + \frac{5h^2}{12}y^{(5)}(x_i) - \frac{h^3}{12}y^{(6)}(x_i) + \frac{h^4}{240}y^{(7)}(x_i) + \cdots , \\ T_{i+1} &= y^{(3)}(x_i) + hy^{(4)}(x_i) + \frac{5h^2}{12}y^{(5)}(x_i) + \frac{h^3}{12}y^{(6)}(x_i) + \frac{h^4}{240}y^{(7)}(x_i) + \cdots , \\ T_{i+2} &= y^{(3)}(x_i) + 2hy^{(4)}(x_i) + \frac{23h^2}{12}y^{(5)}(x_i) + \frac{7h^3}{6}y^{(6)}(x_i) + \frac{121h^4}{240}y^{(7)}(x_i) + \cdots . \end{aligned}$$

Let, $\widetilde{T}_i$ be the new approximation for $y^{(3)}(x_i)$, such that

$$\widetilde{T}_i = a_1 T_{i-2} + a_2 T_{i-1} + a_3 T_i + a_4 T_{i+1} + a_5 T_{i+2}. \tag{3. 12}$$

The relation (3. 12) returns five equations involving a_1, a_2, a_3, a_4 and a_5 as

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 &= 1, \\ -2a_2 - a_3 + a_4 + 2a_5 &= 0, \\ 23a_1 + 5a_2 - a_3 + 5a_4 + 23a_5 &= 0, \\ -14a_1 - a_2 + a_4 + 14a_5 &= 0, \\ 121a_1 + a_2 + a_3 + a_4 + 121a_5 &= 0. \end{aligned}$$

Hence, $a_1 = -\frac{1}{240}$, $a_2 = \frac{1}{10}$, $a_3 = \frac{97}{120}$, $a_4 = \frac{1}{10}$, and $a_5 = -\frac{1}{240}$. Substituting a_i's into (3. 12), we get

$$\widetilde{T}_i = \frac{1}{240h^3}\big(c_{i-4} - 27c_{i-3} - 119c_{i-2} + 485c_{i-1} - 485c_i + 119c_{i+1} + 27c_{i+2} - c_{i+3}\big). \tag{3. 13}$$

Now we approximate $y'''(x)$ at the knot x_0 using four neighbouring values, such that

$$\widetilde{T}_0 = a_1 T_0 + a_2 T_1 + a_3 T_2 + a_4 T_3, \tag{3. 14}$$

where

$$\begin{aligned} T_0 &= y^{(3)}(x_0) - \frac{h^2}{12}y^{(5)}(x_0) + \frac{h^4}{240}y^{(7)}(x_0) + \cdots , \\ T_1 &= y^{(3)}(x_0) + hy^{(4)}(x_0) + \frac{5h^2}{12}y^{(5)}(x_0) + \frac{h^3}{12}y^{(6)}(x_0) + \frac{1h^4}{240}y^{(7)}(x_0) + \cdots , \\ T_2 &= y^{(3)}(x_0) + 2hy^{(4)}(x_0) + \frac{23h^2}{12}y^{(5)}(x_0) + \frac{7h^3}{6}y^{(6)}(x_0) + \frac{121h^4}{240}y^{(7)}(x_0) + \cdots , \\ T_3 &= y^{(3)}(x_0) + 3hy^{(4)}(x_0) + \frac{53h^2}{12}y^{(5)}(x_0) + \frac{17h^3}{4}y^{(6)}(x_0) + \frac{721h^4}{240}y^{(7)}(x_0) + \cdots . \end{aligned}$$

The expression (3. 14) yields the following four equations

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 1, \\ a_2 + 2a_3 + 3a_4 &= 0, \\ -a_1 + 5a_2 + 23a_3 + 53a_4 &= 0, \\ a_2 + 14a_3 + 51a_4 &= 0. \end{aligned}$$

Hence, $a_1 = \frac{7}{6}$, $a_2 = -\frac{5}{12}$, $a_3 = \frac{1}{3}$ and $a_4 = -\frac{1}{12}$. Using these values in (3. 14), we obtain

$$\widetilde{T}_0 = \frac{1}{12h^3}\big(- 14c_{-2} + 47c_{-1} - 61c_0 + 42c_1 - 20c_2 + 7c_3 - c_4\big). \tag{3. 15}$$

Similarly, for the knots x_1, x_{n-1} and x_n, we can derive the following approximations respectively

$$\widetilde{T}_1 = \frac{1}{12h^3}\big(- c_{-2} - 7c_{-1} + 26c_0 - 26c_1 + 7c_2 + c_3\big), \tag{3. 16}$$

$$\widetilde{T}_{n-1} = \frac{1}{12h^3}\big(- c_{n-4} - 7c_{n-3} + 26c_{n-2} - 26c_{n-1} + 7c_n + c_{n+1}\big), \tag{3. 17}$$

$$\widetilde{T}_n = \frac{1}{12h^3}\big(c_{n-5} - 7c_{n-4} + 20c_{n-3} - 42c_{n-2} + 61c_{n-1} - 47c_n + 14c_{n+1}\big). \tag{3. 18}$$

4. The New Approximation for $y^{(4)}(x)$

Let, $\widetilde{F}_i$ be the new QBS approximation to $y^{(4)}(x_i)$ such that

$$\widetilde{F}_i = a_1T_{i-2} + a_2T_{i-1} + a_3T_i + a_4T_{i+1} + a_5T_{i+2}. \tag{4. 19}$$

The above expression returns five equations involving a_i's as

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 &= 0, \\ -2ha_2 - ha_3 + ha_4 + 2ha_5 &= 1, \\ 23a_1 + 5a_2 - a_3 + 5a_4 + 23a_5 &= 0, \\ -14a_1 - a_2 + a_4 + 14a_5 &= 0, \\ 121a_1 + a_2 + a_3 + a_4 + 121a_5 &= 0. \end{aligned}$$

Hence, $a_1 = \frac{1}{24h}$, $a_2 = -\frac{7}{12h}$, $a_3 = 0$, $a_4 = -\frac{1}{24h}$ and $a_5 = -\frac{1}{24h}$. Substituting a_i's back into (4. 19), we obtain

$$\widetilde{F}_i = \frac{1}{24h^4}\big(-c_{i-4}+17c_{i-3}-45c_{i-2}+29c_{i-1}+29c_i-45c_{i+1}+17c_{i+2}-c_{i+3}\big). \tag{4. 20}$$

Now we approximate $y^{(4)}(x)$ at x_0 using four neighbouring values, as

$$\widetilde{F}_0 = a_1T_0 + a_2T_1 + a_3T_2 + a_4T_3. \tag{4. 21}$$

The relation (4. 21) yields the following four equations

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0, \\ ha_2 + 2ha_3 + 3ha_4 &= 1, \\ -a_1 + 5a_2 + 23a_3 + 53a_4 &= 0, \\ a_2 + 14a_3 + 51a_4 &= 0. \end{aligned}$$

Hence, $a_1 = -\frac{23}{12h}$, $a_2 = \frac{13}{4h}$, $a_3 = -\frac{7}{4h}$ and $a_4 = \frac{5}{12h}$. Using these values in (4. 21), we get

$$\widetilde{F}_0 = \frac{1}{12h^4}\big(23c_{-2} - 108c_{-1} + 207c_0 - 208c_1 + 117c_2 - 36c_3 + 5c_4\big). \tag{4. 22}$$

Similarly, using four neighbouring values at the knots x_1, x_{n-1} and x_n, we can obtain the following approximations respectively

$$\widetilde{F}_1 = \frac{1}{12h^4}\big(5c_{-2} - 12c_{-1} - 3c_0 + 32c_1 - 33c_2 + 12c_3 + c_4\big), \quad (4.\ 23)$$

$$\widetilde{F}_{n-1} = \frac{1}{12h^4}\big(- c_{n-4} + 12c_{n-3} - 33c_{n-2} + 32c_{n-1} - 3c_n - 12c_{n+1} + 5c_{n+2}\big), \quad (4.\ 24)$$

$$\widetilde{F}_n = \frac{1}{12h^4}\big(5c_{n-5} - 36c_{n-4} + 117c_{n-3} - 208c_{n-2} + 207c_{n-1} - 108c_n + 23c_{n+1}\big). \quad (4.\ 25)$$

5. Numerical Method

The problem (1. 1) is linearized by applying Quasi-linearization technique [13] as

$$\alpha y^{(4)}_{k+1}(x) + \frac{m}{x} y'''_{k+1}(x) + p(x)y''_{k+1}(x) + q(x)y'_{k+1}(x) + V_k(x)y_{k+1}(x) = W_k(x), \ \ 0 \le x \le 1, \quad (5.\ 26)$$

where $V_k(x) = -\big(\frac{\partial f}{\partial y}\big)_{(x,y_k)}$ and $W_k(x) = f(x, y_k) - \big(\frac{\partial f}{\partial y}\big)_{(x,y_k)}$, $k = 0, 1, 2, \cdots$.
Similarly, transforming the boundary conditions (1. 2), we get

$$\begin{cases} y_{k+1}(a) = \alpha_1, \ y'_{k+1}(a) = \alpha_2, \\ y_{k+1}(b) = \alpha_3, \ y'_{k+1}(b) = \alpha_4, \end{cases} \quad (5.\ 27)$$

Let the QBS solution to (5. 26) be given by

$$Y(x) = \sum_{j=-2}^{n+1} c_j B_j(x). \quad (5.\ 28)$$

Discretizing (5. 26) at the i^{th} knot, we obtain

$$\alpha Y^{(4)}_{k+1}(x_i) + \frac{m}{x_i} Y'''_{k+1}(x_i) + p(x_i)Y''_{k+1}(x_i) + q(x_i)Y'_{k+1}(x_i) + V_k(x_i)Y_{k+1}(x_i) = W_k(x_i). \quad (5.\ 29)$$

For $i = 2, 3, 4, \cdots, n-2$, using (2. 5)–(2. 7), (3. 13) and (4. 20) in equation (5. 29), we obtain

$$\begin{aligned} &\alpha\left(\frac{-c_{i-4} + 17c_{i-3} - 45c_{i-2} + 29c_{i-1} + 29c_i - 45c_{i+1} + 17c_{i+2} - c_{i+3}}{24h^4}\right) \\ &+ \frac{m}{x_i}\left(\frac{c_{i-4} - 27c_{i-3} - 119c_{i-2} + 485c_{i-1} - 485c_i + 119c_{i+1} + 27c_{i+2} - c_{i+3}}{240h^3}\right) \\ &+ p(x_i)\left(\frac{c_{i-2} - c_{i-1} - c_i + c_{i+1}}{2h^2}\right) + q(x_i)\left(\frac{-c_{i-2} - 3c_{i-1} + 3c_i + c_{i+1}}{6h}\right) \\ &+ V_k(x_i)\left(\frac{c_{i-2} + 11c_{i-1} + 11c_i + c_{i+1}}{24}\right) = W_k(x_i). \quad (5.\ 30) \end{aligned}$$

Similarly, at the knots $x_1,\ x_{n-1}$ and x_n, relation (5. 29) produces the following equations respectively

$$\alpha\Big(\frac{5c_{-2}-12c_{-1}-3c_0+32c_1-33c_2+12c_3+c_4}{12h^4}\Big) + \frac{m}{x_1}\Big(\frac{-c_{-2}-7c_{-1}+26c_0-26c_1+7c_2+c_3}{12h^3}\Big)+p(x_1)\Big(\frac{c_{-1}-c_0-c_1+c_2}{2h^2}\Big) + q(x_1)\Big(\frac{-c_{-1}-3c_0+3c_1+c_2}{6h}\Big)+V_k(x_1)\Big(\frac{c_{-1}+11c_0+11c_1+c_2}{24}\Big) = W_k(x_1), \quad (5.\ 31)$$

$$\alpha\Big(\frac{-c_{n-4}+12c_{n-3}-33c_{n-2}+32c_{n-1}-3c_n-12c_{n+1}+5c_{n+2}}{12h^4}\Big) + \frac{m}{x_{n-1}}\Big(\frac{-c_{n-4}-7c_{n-3}+26c_{n-2}-26c_{n-1}+7c_n+c_{n+1}}{12h^3}\Big) + p(x_{n-1})\Big(\frac{c_{n-3}-c_{n-2}-c_{n-1}+c_n}{2h^2}\Big)+q(x_{n-1})\Big(\frac{-c_{n-3}-3c_{n-2}+3c_{n-1}+c_n}{6h}\Big) + V_k(x_{n-1})\Big(\frac{c_{n-3}+11c_{n-2}+11c_{n-1}+c_n}{24}\Big)=W_k(x_{n-1}), \quad (5.\ 32)$$

$$\alpha\Big(\frac{5c_{n-5}-36c_{n-4}+117c_{n-3}-208c_{n-2}+207c_{n-1}-108c_n+23c_{n+1}}{12h^4}\Big) + \frac{m}{x_n}\Big(\frac{c_{n-5}-7c_{n-4}+20c_{n-3}-42c_{n-2}+61c_{n-1}-47c_n+14c_{n+1}}{12h^3}\Big) + p(x_n)\Big(\frac{c_{n-2}-c_{n-1}-c_n+c_{n+1}}{2h^2}\Big)+q(x_n)\Big(\frac{-c_{n-2}-3c_{n-1}+3c_n+c_{n+1}}{6h}\Big) + V_k(x_n)\Big(\frac{c_{n-2}+11c_{n-1}+11c_n+c_{n+1}}{24}\Big)=W_k(x_n). \quad (5.\ 33)$$

The set of boundary conditions (5. 27) as well give the following four equations

$$\frac{c_{-2}+11c_{-1}+11c_0+c_1}{24} = \alpha_1, \quad (5.\ 34)$$

$$\frac{-c_{-2}-3c_{-1}+3c_0+c_1}{6h} = \alpha_2, \quad (5.\ 35)$$

$$\frac{c_{n-2}+11c_{n-1}+11c_n+c_{n+1}}{24} = \alpha_3, \quad (5.\ 36)$$

$$\frac{-c_{n-2}-3c_{n-1}+3c_n+c_{n+1}}{6h} = \alpha_4. \quad (5.\ 37)$$

In this way, we get a system of $n+4$ equations (5. 30)–(5. 37) with unknowns c_i's, $i=-2,-1,0,\cdots,n+1$. We start from $k=0$ with an initial guess $Y_0(x)$ and solve it for c using well known Thomas algorithm. The values of c_i's are plugged into (5. 28) to get $Y_{k+1}(x)$. The procedure is replayed for $k = 1, 2, 3, \cdots$ until $max|Y_{k+1}(x_i) - Y_k(x_i)| \le 10^{-8}$. The numerical simulation is run in $Mathematica$ 9.

6. Error Analysis

Using QBS approximations (2. 5)–(2. 7), we can establish the following relations [9]

$$h\big[Y'(x_{i-2}) + 11Y'(x_{i-1}) + 11Y'(x_i) + Y'(x_{i+1})\big] = 4\big[Y(x_{i+1}) + 3Y(x_i) - 3Y(x_{i-1}) - Y(x_{i-2})\big], \quad (6.\ 38)$$

$$h^2Y''(x_i) = 2\big[Y(x_{i+1}) - 2Y(x_i) + Y(x_{i-1})\big] - \frac{h}{2}\big[Y'(x_{i+1}) - Y'(x_{i-1})\big]. \quad (6.\ 39)$$

Similarly, using (3. 13) and (4. 20), we have

$$h^3Y^{(3)}(x_i) = \frac{h}{20}\big[59Y'(x_{i-1}) - 72Y'(x_i) + 13Y(x_{i+1})\big] + \frac{h^2}{120}[Y''(x_{i-2}) + 92Y''(x_{i-1}) + 184Y''(x_i) - Y''(x_{i+2})], \quad (6.\ 40)$$

$$h^4Y^{(4)}(x_i) = \frac{h^2}{12}[-Y''(x_{i-2}) + 16Y''(x_{i-1}) - 30Y''(x_i) + 16Y''(x_{i+1}) - Y''(x_{i+2})]. \quad (6.\ 41)$$

Using the operator notation $E^{\lambda}(Y'(x_i)) = Y'(x_{i+\lambda})$, $\lambda \in \mathbb{Z}$, equation (6. 38) is expressed as

$$h[E^{-2} + 11E^{-1} + 11E^0 + E^1]Y'(x_i) = 4[E^1 + 3E^0 - 3E^{-1} - E^{-2}]y(x_i).$$

Hence,

$$hY'(x_i) = 4\left[\frac{E^1 + 3E^0 - 3E^{-1} - E^{-2}}{E^{-2} + 11E^{-1} + 11E^0 + E^1}\right]y(x_i). \quad (6.\ 42)$$

Using $E = \mathrm{e}^{hD}$, $D \equiv d/dx$, we obtain

$$\begin{aligned} E^1 + 3E^0 - 3E^{-1} - E^{-2} &= \mathrm{e}^{hD} + 3 - 3\mathrm{e}^{-hD} - \mathrm{e}^{-2hD} \\ &= 24hD - 12h^2D^2 + 8h^3D^3 - 3h^4D^4 + \cdots, \end{aligned}$$

$$\begin{aligned} E^{-2} + 11E^{-1} + 11E^0 + E^1 &= \mathrm{e}^{-2hD} + 11\mathrm{e}^{-hD} + 11 + \mathrm{e}^{hD} \\ &= 24 - 12hD + 8h^2D^2 - 3h^3D^3 + \frac{7}{6}h^4D^4 + \cdots. \end{aligned}$$

Therefore, equation (6. 42) can be expressed as

$$\begin{aligned} Y'(x_i) &= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 + \cdots\right)\left(1 - \frac{1}{2}hD + \frac{1}{3}h^2D^2 + \cdots\right)^{-1} y(x_i) \\ &= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 + \cdots\right)\left[1 + \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 + \cdots\right)\right]^{-1} y(x_i) \\ &= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 + \cdots\right)\left[1 + \frac{1}{2}hD - \frac{1}{12}h^2D^2 + \cdots\right] y(x_i) \\ &= \left(D + \frac{1}{720}h^4D^5 - \frac{1}{2016}h^6D^7 + \cdots\right) y(x_i). \end{aligned}$$

Simplifying, we obtain

$$Y'(x_i) = y'(x_i) + \frac{1}{720}h^4 y^{(5)}(x_i) - \frac{1}{2016}h^6 y^{(7)}(x_i) + \cdots . \tag{6. 43}$$

Similarly, writing (6. 39) in operator form, we obtain

$$\begin{aligned} h^2Y''(x_i) &= 2\left[E^{-1} - 2 + E\right] y(x_i) - \frac{h}{2}\left[E - E^{-1}\right] y'(x_i) \\ &= \left(2h^2D^2 + \frac{h^4D^4}{6} + \frac{h^6D^6}{180} + \frac{h^8D^8}{10080} + \cdots\right) y(x_i) \\ &\quad - \left(h^2D + \frac{h^4D^3}{6} + \frac{h^6D^5}{120} + \frac{h^8D^7}{5040} + \cdots\right) y'(x_i). \end{aligned}$$

Simplifying the above relation, we have

$$Y''(x_i) = y''(x_i) - \frac{h^4}{240} y^{(6)}(x_i) + \frac{h^6}{6048} y^{(8)}(x_i) + \cdots . \tag{6. 44}$$

In the same way, (6. 40)-(6. 41) give the following relations

$$Y^{(3)}(x_i) = y^{(3)}(x_i) + \frac{23h^3}{1800} y^{(6)}(x_i) - \frac{7h^4}{1200} y^{(7)}(x_i) + \frac{23h^5}{37800} y^{(8)}(x_i) \cdots , \tag{6. 45}$$

$$Y^{(4)}(x_i) = y^{(4)}(x_i) - \frac{h^4}{90} y^{(8)}(x_i) - \frac{h^6}{1008} y^{(10)}(x_i) + \cdots . \tag{6. 46}$$

We define the error term at i^{th} knot as $e(x_i) = Y(x_i) - y(x_i)$. Using the relations (5. 30)-(5. 32) in Taylor series expansion of error term, we have

$$e(x_i + \theta h) = \frac{\theta}{720} h^5 y^{(5)}(x_i) + \frac{\theta^2(-45 + 46\theta)}{21600} h^6 y^{(6)}(x_i) + \cdots , \tag{6. 47}$$

where $\theta \in [0, 1]$. From (6. 47), it is clear that the new QBS approximation is uniformly convergent.

7. Numerical Results

The experimental outcomes of new QBS approximation method are presented in this section. In order to validate the accuracy of numerical scheme, the error norm L_∞, L_2 and are calculated as [1, 21]

$$L_\infty = \max_i |Y_i - y_i|$$

$$L_2 = \sqrt{h \sum_{i=0}^{n} (y_i - Y_i)^2}$$

where Y_i and y_i are the numerical and analytical solutions at the i^{th} knot respectively.

Example 7.1. *Consider the following fourth order singularly perturbed SBVP* [19]

$$\epsilon y^{(4)}(x) + \frac{1}{x} y(x) = \mathrm{e}^x [1 - x - \epsilon(8 + 7x + x^2)] - \frac{2}{3}\mathrm{e}(1 - x^2), \quad 0 \le x \le 1,$$

$$y(0) = 0,\ y''(0) = 0,\ y(1) = 0,\ y''(1) = 0.$$

The analytical exact solution is $x\mathrm{e}^x(1 - x) - \frac{2}{3}\mathrm{e}x(1 - x^2)$. The maximum absolute error, L_∞, for different choices of h and ϵ is reported in Table 1. It can be observed that the current experimental outcomes exhibit a better accordance with the analytical results as compared to those obtained by the typical quintic B-spline collocation method (QnBSM) [19]. In Figure 1, the approximate and analytical exact solutions are portrayed, when $h = 1/16$ and $\epsilon = 10^{-4}$. In order to show the slope rate of convergence, the $Log(L_\infty)$ and $Log(L_2)$ versus $Log(h)$ plots are displayed in Figure 2.

Table 1. Numerical error norm for Example 7.1.

n	$\epsilon = 10^{-k}$	$k = 0$	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$
8	QnBSM [19]	8.51×10^{-4}	2.62×10^{-4}	7.52×10^{-6}	1.30×10^{-7}	1.35×10^{-9}	1.35×10^{-11}
	Proposed method	1.03×10^{-5}	2.86×10^{-6}	3.20×10^{-7}	8.81×10^{-9}	9.10×10^{-11}	9.10×10^{-13}
16	QnBSM [19]	2.12×10^{-4}	6.62×10^{-5}	1.86×10^{-6}	2.87×10^{-8}	3.82×10^{-10}	3.84×10^{-12}
	Proposed method	4.63×10^{-7}	1.37×10^{-7}	1.86×10^{-8}	1.61×10^{-9}	2.33×10^{-11}	2.34×10^{-13}
32	QnBSM [19]	5.30×10^{-5}	1.65×10^{-5}	4.65×10^{-7}	6.92×10^{-9}	9.40×10^{-11}	1.02×10^{-12}
	Proposed method	2.12×10^{-8}	7.55×10^{-9}	1.12×10^{-9}	1.27×10^{-10}	5.35×10^{-12}	5.83×10^{-14}
64	QnBSM [19]	1.33×10^{-5}	4.13×10^{-6}	1.16×10^{-7}	1.72×10^{-9}	2.04×10^{-11}	2.60×10^{-13}
	Proposed method	1.07×10^{-9}	3.90×10^{-10}	6.76×10^{-11}	7.78×10^{-12}	7.33×10^{-13}	1.43×10^{-14}
128	QnBSM [19]	3.31×10^{-6}	1.03×10^{-6}	2.90×10^{-8}	4.30×10^{-10}	4.98×10^{-12}	5.70×10^{-14}
	Proposed method	4.86×10^{-11}	2.30×10^{-11}	4.12×10^{-12}	4.79×10^{-13}	4.63×10^{-14}	2.96×10^{-15}

Example 7.2. *Consider the fourth order Emden-Flower type equation* [17, 22]

$$y^{(4)}(x) + \frac{3}{x} y^{(3)}(x) + y^2(x) - y^3(x) = g(x), \quad 0 \le x \le 1,$$

$$y(0) = 0,\ y'(0) = 0,\ y(1) = \mathrm{e},\ y'(1) = 3\mathrm{e},$$

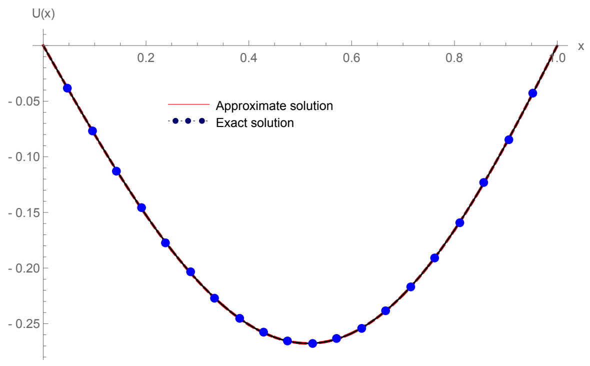


FIGURE 1. Exact and approximate solution for Example 7.1 when $h = 1/16$ and $\epsilon = 10^{-4}$.

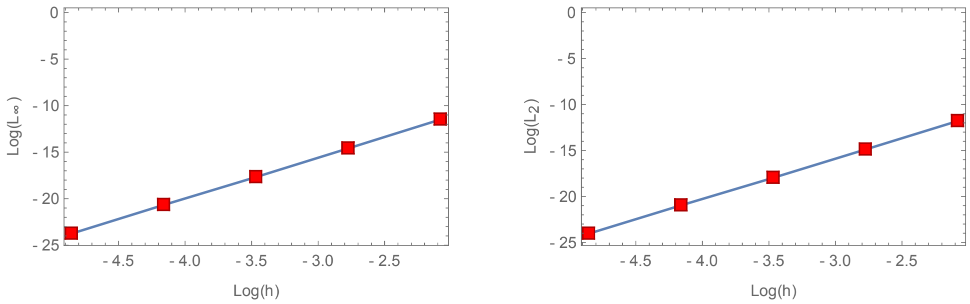


FIGURE 2. The slope rate of convergence for Example 7.1

where

$$g(x) = \frac{18}{x}\mathrm{e}^{x} + 30\mathrm{e}^{x} + 11x\mathrm{e}^{x} + x^2\mathrm{e}^{x} + x^4\mathrm{e}^{2x} - x^6\mathrm{e}^{3x}$$

The analytical exact solution is $x^2\mathrm{e}^{x}$. Table 2 depicts that our computed results are better than modified form of Adomian decomposition method (MADM) [17] and Iterative decomposition method (IDM) proposed in [22]. In Figure 3, the absolute computational error corresponding to four different choices of step size is displayed. It is obvious that as we decrease the grid spacing, the approximate solution converges to the analytical exact solution. In order to show the slope rate of convergence, the $Log(L_\infty)$ and $Log(L_2)$ versus $Log(h)$ plots are displayed in Figure 4.

Example 7.3. *Consider the fourth order Emden-Flower type equation* [23]

$$y^{(4)}(x) + \frac{3}{x}y^{(3)}(x) = 96(1 - 10x^4 + 5x^8)\mathrm{e}^{-4y(x)}, \ \ 0 \le x \le 1,$$
$$y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

TABLE 2. Approximate results for Example 7.2 when $h = 0.05$.

x	MADM [17]	IDM [22]	Proposed method	Exact solution
0.0	0	0	0.0000000000	0
0.1	0.0110383122	0.0110383121	0.0110517825	0.0110517092
0.2	0.0488025223	0.0488025152	0.0488562518	0.0488561103
0.3	0.1213667199	0.1213665374	0.1214874789	0.1214872927
0.4	0.2384776008	0.2384757785	0.2386921617	0.2386919516
0.5	0.4118453983	0.4118345370	0.4121805322	0.4121803177
0.6	0.6554804967	0.6554337960	0.6559629690	0.6559627681
0.7	0.9860824332	0.9859221477	0.9867389971	0.9867388267
0.8	1.42348892400	1.4230224464	1.4243463196	1.4243461942
0.9	1.99119360394	1.9899967121	1.9922785876	1.9922785200
1.0	2.71694235913	2.7141618691	2.7182818285	2.7182818285
L_∞	1.34×10^{-3}	4.12×10^{-3}	2.15×10^{-7}	$\cdots$

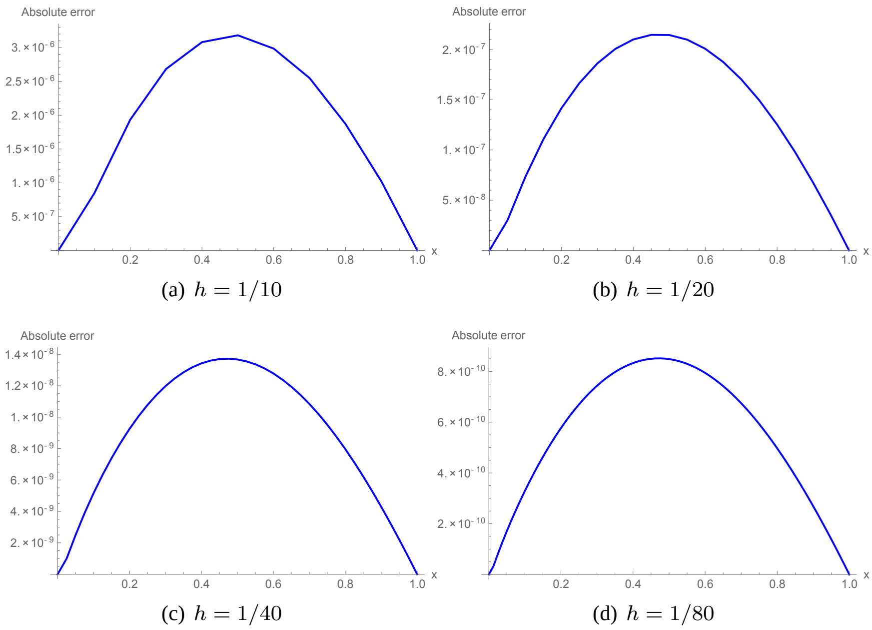


FIGURE 3. Absolute computational error for Example 7.2.

The true solution is $\log(1 + x^4)$. The approximate results are listed in Table 3. It is obvious that the obtained results are well balanced as compared to Adomian decomposition method (ADM) [23]. Figure 5 portrays absolute computational error for $n = 10, 20, 40, 80$.

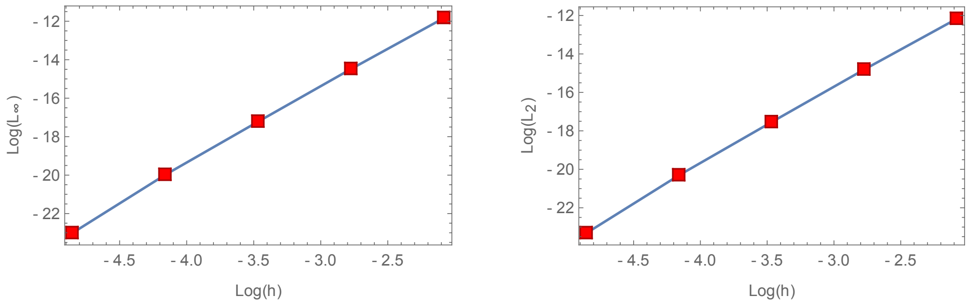


FIGURE 4. The slope rate of convergence for Example 7.2

TABLE 3. Numerical results for Example 7.3 when $h = 0.0125$.

x	ADM [23]	Proposed method	Exact solution
0.0	0	0.0000000000	0
0.1	0.0000999950	0.0000999939	0.0000999950
0.2	0.0015987214	0.0015987150	0.0015987214
0.3	0.0080673721	0.0080673514	0.0080673711
0.4	0.0252779124	0.0252777651	0.0252778072
0.5	0.0606282552	0.0606245617	0.0606246218
0.6	0.1219275141	0.1218635524	0.1218635878
0.7	0.2158897574	0.2151920981	0.2151920215
0.8	0.3486204122	0.3433062524	0.3433059762
0.9	0.5350095738	0.5044659432	0.5044654406
1.0	0.8333333333	0.6931478683	0.6931471806
L_∞	1.40×10^{-1}	6.88×10^{-7}	$\cdots$

Example 7.4. *Consider the fourth order Emden-Flower type equation* [23]

$$y^{(4)}(x) + \frac{1}{x}y^{(3)}(x) = -24(1 - 27x^4 + 38x^8 - 4x^{12})y^9(x), \quad 0 \le x \le 1,$$

$$y(0) = 1, \ y'(0) = y''(0) = y^{(3)}(0) = 0.$$

The true solution is $\frac{1}{\sqrt{1+x^4}}$. Table 4 displays a comparison of experimental outcomes with ADM [23]. The computational results using new QBS approximation technique displays a better accordance with the analytical results as compared to ADM, especially when $x \to 1$. The absolute numerical error corresponding to $n = 10, 20, 40, 80$ is shown in Figure 6. One can see that the computational error decreases as the mesh size is decreased.

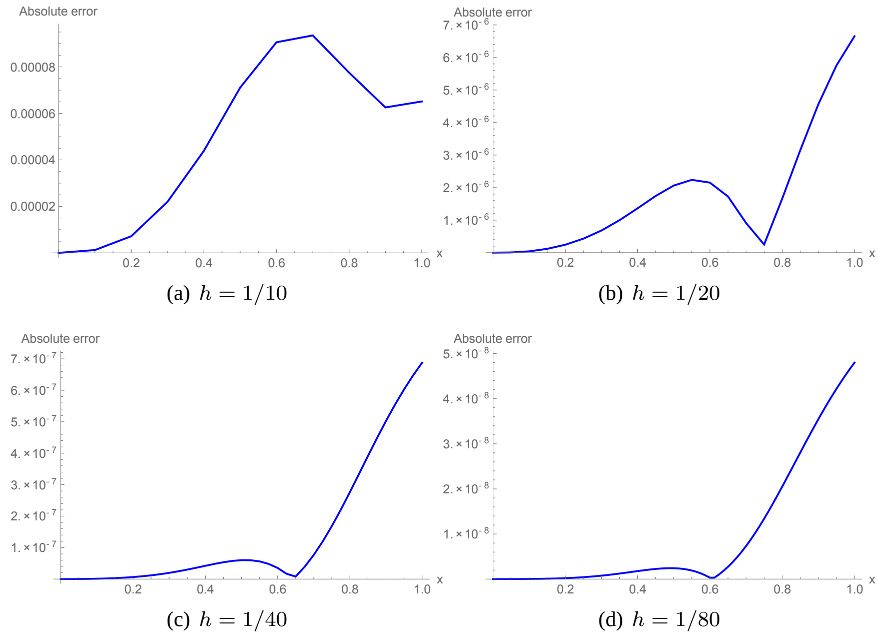


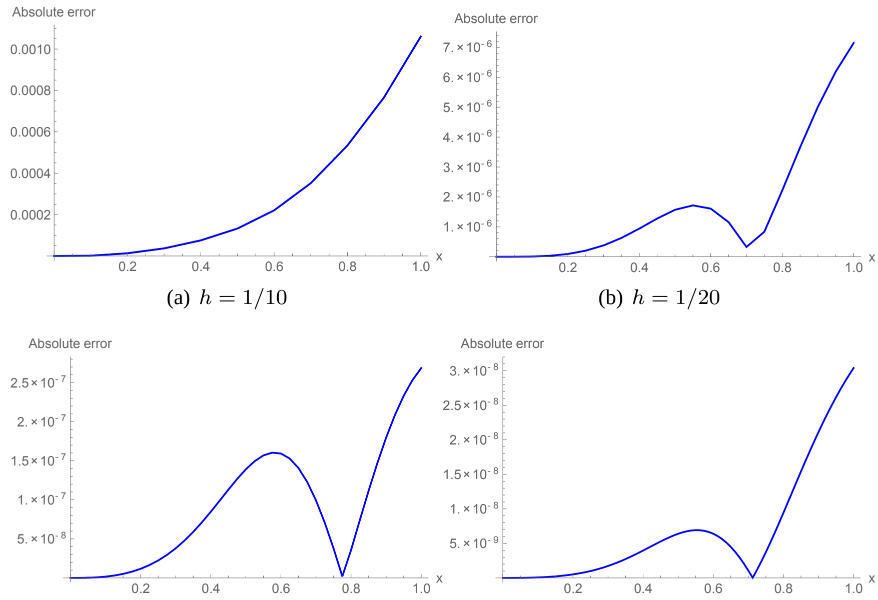
FIGURE 5. Absolute computational error for Example 7.3.

TABLE 4. Approximate results for Example 7.4 when $h = 0.0125$.

x	ADM [23]	Proposed method	Exact solution
0.0	1	1.0000000000	1
0.1	0.9999500037	0.9999500054	0.9999500037
0.2	0.9992009587	0.9992009704	0.9992009587
0.3	0.9959744388	0.9959744767	0.9959744388
0.4	0.9874406319	0.9874407166	0.9874406319
0.5	0.9701424874	0.9701426391	0.9701425001
0.6	0.9408864577	0.9408875711	0.9408874119
0.7	0.8979549507	0.8979904008	0.8979903016
0.8	0.8414990371	0.8422713649	0.8422714007
0.9	0.7658652395	0.7770636989	0.7770638785
1.0	0.58984375	0.7071065126	0.7071067812
L_∞	1.17×10^{-1}	2.69×10^{-7}	$\cdots$

8. CONCLUSION

In this work, a new QBS approximation technique has been presented for numerical treatment of fourth order SBVP's. We conclude this work as:



(a) $h = 1/10$ (b) $h = 1/20$

(c) $h = 1/40$ (d) $h = 1/80$

FIGURE 6. Absolute computational error for Example 7.4.

(1) The presented numerical algorithm is based on typical QBS collocation method strengthened by new QBS approximations for third and fourth order derivatives.
(2) The proposed technique is novel for fourth order singular boundary value problems.
(3) This scheme is proved to be uniformly convergent.
(4) As the grid spacing is decreased, the approximate solution approaches to the exact solution, which affirms the convergence of proposed scheme.
(5) By virtue of straightforward and simple implementation, it returns better outcomes as compared to MADM [17], IDM [22] and ADM [23].

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