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A Note on Paramedial AG-groupoids

Imtiaz Ahmad[1]
Email: iahmaad@hotmail.com

Muhammad Rashad
Email: rashad@uom.edu.pk

Muhammad Iqbal
Email: iqbalmuhammadpk78@yahoo.com

Amanullah
Email: amanswt@hotmail.com
Department of Mathematics,
University of Malakand, Chakdara, Pakistan.

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Abstract.: An AG-groupoid satisfying the property: $(ub)(cd) = (db)(cu)$ is known as paramedial AG-groupoid [22]. In this note, we study some characteristics and constructions of paramedial AG-groupoids. Latest computational techniques of Mace-4 and GAP are used for generating various examples and counterexamples to strengthen the study of this subclass. Various relations of this subclass with other known algebraic structures are established. Furthermore, paramedial AG-groupoid is decomposed with the help of some congruences.

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1. Introduction

A groupoid G that satisfies the left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in G$ is known as an Abel-Grassmann's groupoid (AG-groupoid) [16] or left-almost semigroup (LA-semigroup) [13]. Generally, AG-groupoid is a non-associative and non-commutative structure lying midway between a magma and a commutative semigroup. Every AG-groupoid is medial, i.e. it satisfies the medial law: $(ub)\,(cd) = (uc)\,(bd)$. A groupoid G is called paramedial if $ub \cdot cd = db \cdot cu$, $\forall u, b, c, d \in G$. We use juxtaposition and the notation "$\cdot$" to avoid frequent use of parenthesis, e.g. $((u \cdot b) \cdot c)d$ will denote the same as $(ub \cdot c)d$.

AG-groupoid has a variety of applications in various areas such as: algebra, finite mathematics, flock theory, geometry and topology [13, 3, 23, 18, 2]. Various subclasses of AG-groupoids have been introduced and investigated by different researchers [9, 19, 24, 17, 1, 20, 8]. The concept of paramedial groupoid [5] is extended to paramedial AG-groupoid by Shah et al. [22] and some fundamental characteristics are proved such as: (i) a Bol*-AG-groupoid is paramedial, (ii) an AG-groupoid semigroup is paramedial, (iii) a paramedial AG-groupoid is left nuclear square. An AG-groupoid with left identity is called AG-monoid. It is easy to show that every AG-monoid is paramedial and that a commutative AG-groupoid is a semigroup[9, Proposition 1], i.e. it satisfies the associative law: $ub \cdot c = u \cdot bc$.

[1]Corresponding author

The class of paramedial AG-groupoid is investigated in detail and various results are proved in this note. Furthermore, a variety of constructions are established for this class and the concept of inverse paramedial AG-groupoid is introduced and various congruences on the same class are established.

2. PRELIMINARIES

The following is a list of fundamentals [18, 9, 24, 1, 11, 10, 15] that shall frequently be used throughout this note. An AG-groupoid G is called —

(a) — T^1 if $ub = cd \Rightarrow bu = dc$, $\forall u, b, c, d \in G$,
(b) — T^3_r if it satisfies the identity $bu = cu \Rightarrow ub = uc \forall u, b, c \in G$,
(c) — T^3_l if the identity $ub = uc \Rightarrow bu = cu$ holds $\forall u, b, c \in G$,
(d) — T^3 if it is T^3_l and T^3_r,
(e) — T^4_f if the identity $ub = cd \Rightarrow ud = cb$ holds $\forall u, b, c, d \in G$,
(f) — T^4_b if $ub = cd$ implies $du = bc \forall u, b, c, d \in G$,
(g) — T^4 if G is T^4_f and T^4_b,
(h) — bi-commutative (BC) if G is left commutative (LC), (i.e. if the identity: $ub \cdot c = bu \cdot c$ is true in G) and is right commutative (RC) (i.e. if G satisfies the identity $u \cdot bc = u \cdot cb$),
(i) — right permutable (RP) if G satisfies the identity: $ub \cdot c = uc \cdot b$,
(j) — AG-band if $uu = u \forall u \in G$,
(k) — semi-lattice if G is a commutative AG-band,
(l) — AG* if $ub \cdot c = b \cdot uc \forall u, b, c \in G$,
(m) — AG** if $u \cdot bc = b \cdot uc \forall u, b, c \in G$,
(n) — right nuclear square if $ub \cdot c^2 = u \cdot bc^2 \forall u, b, c \in G$,
(o) — self-dual if $u \cdot bc = c \cdot bu \forall u, b, c \in G$,
(p) — Bol* if $u(bc \cdot d) = (ub \cdot c)d \forall u, b, c \in G$.

3. CHARACTERIZATION OF PARAMEDIAL AG-GROUPOIDS

Definition 1. [22] *An AG-groupoid G is called paramedial if*

$$ub \cdot cd = db \cdot cu \forall u, b, c, d \in G \tag{3. 1}$$

Note that if G is paramedial AG-groupoid, then by medial law: $ub \cdot cd = db \cdot cu = dc \cdot bu$, i.e.

$$ub \cdot cd = dc \cdot bu \tag{3. 2}$$

The following example depicts the existence and the fact that paramedial AG-groupoid is non-associative in nature.

Example 1. *Let $G = \{1, 2, 3, 4\}$, then it is easy to show that $(G, *)$ and $(G, \cdot)$ with the following tables are non-associative paramedial AG-groupoid of size 4.*

$*$	1	2	3	4
1	1	2	2	2
2	2	1	1	1
3	2	1	1	1
4	3	1	1	1

$\cdot$	1	2	3	4
1	1	1	3	3
2	1	1	4	4
3	3	3	1	1
4	3	3	1	1

The following example shows that paramedial and self-dual are two different subclasses of AG-groupoids. Furthermore, neither of these is a subclass of the right nuclear square. However, the subclass satisfying the properties of both these subclasses is a right nuclear square as proved in the next theorem.

Example 2. *Table (i) represents a paramedial AG-groupoid of size 3, which is not self-dual as* $1(1 \cdot 2) \neq 2(1 \cdot 1)$. *The AG-groupoid in Table (ii) is self-dual of size 4, but it is not paramedial as* $(1 \cdot 2)(3 \cdot 4) \neq (4 \cdot 2)(3 \cdot 1)$.

(i)

·	1	2	3
1	1	1	1
2	2	1	1
3	1	1	1

(ii)

·	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Now, we provide a counterexample to show that in general, neither the self-dual nor the paramedial AG-groupoid is a right nuclear square.

Example 3. *(i) Let $G = \{1, 2, 3\}$, then $(G, \cdot)$ a paramedial AG-groupoid of size 3, which is not right nuclear square.*

(ii) Let $H = \{1, 2, 3, 4\}$, then $(H, *)$ is a self-dual AG-groupoid of size 4, which is not a right nuclear square.

(i)

·	1	2	3
1	1	1	1
2	1	1	1
3	1	2	2

(ii)

∗	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Theorem 1. *Every self-dual paramedial AG-groupoid is right nuclear square.*

Proof. Let G be a self-dual paramedial AG-groupoid and $u, b, c \in G$. Then by self-duality and (3. 1) we have,

$$u \cdot bc^2 = c^2 \cdot bu = cb \cdot cu = ub \cdot cc = ub \cdot c^2.$$

Therefore, $u \cdot bc^2 = ub \cdot c^2$. Hence, G is right nuclear square. □

Theorem 2. *A paramedial AG-band is a commutative semigroup.*

Proof. Let G be a paramedial AG-band and $u, b \in G$. Then, by the assumption and (3. 2), we have

$$ub = uu \cdot bb = bb \cdot uu = bu.$$

Thus $ub = bu$ for all u, b in G. Equivalently G is commutative and hence a commutative semigroup, as a commutative AG-groupoid is always associativie [9, Proposition 1]. □

The following counterexample shows that T^3-AG-groupoid may not be paramedial in general.

Example 4. *A T^3-AG-groupoid that is not paramedial as,* $(1 \cdot 2)(3 \cdot 4) \neq (4 \cdot 2)(3 \cdot 1)$.

∗	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Theorem 3. *Each of the following is a subclass of paramedial AG-groupoid.*

(i) T^1-AG-groupoid,
(ii) T^4-AG-groupoid,
(iii) BC-AG-groupoid,
(iv) RP-AG-groupoid,
*(v) AG**-groupoid,*
(vi) AG-groupoid.*

Proof. We prove in each case that the identity (3. 1) holds.

(i). Let G be a T^1-AG-groupoid and $u,b,c,t \in G$. Then by the definition of T^1 and the medial law,

$$\begin{aligned} ub\cdot ct = uc\cdot bt &\quad\Rightarrow\quad ct\cdot ub = bt\cdot uc \\ \Rightarrow cu\cdot tb = bt\cdot uc &\quad\Rightarrow\quad tb\cdot cu = uc\cdot bt \end{aligned}$$

Thus $tb\cdot cu = ub\cdot ct$. Hence G is paramedial.

(ii). Let G be a T^4-AG-groupoid and $u,b,c,t \in G$. Then by definition of T^4_f, T^4_b and the medial law,

$$\begin{aligned} ub\cdot ct = uc\cdot bt &\quad\Rightarrow\quad ub\cdot bt = uc\cdot ct \\ \Rightarrow ct\cdot ub = bt\cdot uc &\quad\Rightarrow\quad cu\cdot tb = bu\cdot tc \\ \Rightarrow tc\cdot cu = tb\cdot bu &\quad\Rightarrow\quad bu\cdot tc = cu\cdot tb \\ \Rightarrow bt\cdot uc = cu\cdot tb &\quad\Rightarrow\quad bt\cdot tb = cu\cdot uc \\ \Rightarrow uc\cdot bt = tb\cdot cu &\quad\Rightarrow\quad ub\cdot ct = tb\cdot cu. \end{aligned}$$

Hence G is paramedial.

(iii). Let G be a BC-AG-groupoid and $u,b,c,t \in G$. Then using the BC property and the medial law,

$$\begin{aligned} ub\cdot ct = bu\cdot ct = bu\cdot tc &\quad=\quad bt\cdot uc = tb\cdot uc \\ = tb\cdot cu = tc\cdot bu &\quad=\quad tb\cdot cu. \end{aligned}$$

Thus $ub\cdot ct = tb\cdot cu$. Hence G is paramedial.

(iv). Let G be an RP-AG-groupoid and $u,b,c,t \in G$. To prove that G is paramedial, use left invertive law, medial law and definition of right permutablity

$$\begin{aligned} ub\cdot ct &= (ct\cdot b)u = (cb\cdot t)u = (tb\cdot c)u = uc\cdot tb \\ &= ut\cdot cb = (u\cdot cb)t = (t\cdot cb)u = tu\cdot cb \\ &= tc\cdot ub = (ub\cdot c)t = (uc\cdot b)t = (bc\cdot u)t \\ &= (bu\cdot c)t = tc\cdot bu = tb\cdot cu. \end{aligned}$$

Thus $ub\cdot ct = tb\cdot cu$. Hence paramedial law holds in G.

(v). Let G be an AG**-groupoid and $u,b,c,t \in G$. Then by the medial law and the property of AG** ,

$$ub\cdot ct = uc\cdot bt \quad=\quad b(uc\cdot t) = b(tc\cdot u) = tc\cdot bu = tb\cdot cu.$$

Thus $ub\cdot ct = tb\cdot cu$. Hence G posses the law of paramedial.

(vi). Let $u,b,c,t \in G$, such that G is an AG*-groupoid. Then by the alternative repeated use of left invertive law and the identity of AG*,

$$\begin{aligned} ub\cdot ct = (ct\cdot b)u = (t\cdot cb)u &\quad=\quad (u\cdot cb)t \\ = (cu\cdot b)t = (bu\cdot c)t &\quad=\quad tc\cdot bu = tb\cdot cu. \end{aligned}$$

Thus, $ub\cdot ct = tb\cdot cu$. Equivalently G is paramedial.

Hence the theorem is proved. □

T^4 AG-groupoid is parmedial as proved in Theorem 3 *(ii)*, however it is depicted in the following example that neither T^4_f nor T^4_b-AG-groupoid is paramedial.

Example 5. *(i) Example 3 (ii) is a T^4_f-AG-groupoid, which is not a paramedial.*
(ii) A T^4_b-AG-groupoid of size 5 *is given below is not a paramedial as* $2 = (1\cdot 3)(4\cdot 5) \neq (5\cdot 3)(4\cdot 1) = 1$.

$\cdot$	1	2	3	4	5
1	1	3	2	5	4
2	4	2	5	1	3
3	5	4	3	2	1
4	3	5	1	4	2
5	2	1	4	3	5

Proposition 3.1. [1]*Every T^2-AG-groupoid is T^1.*

Using Theorem 3 and Proposition 3.1, we have the following obvious corollary;

Corollary 1. *Every T^2-AG-groupoid is paramedial.*

4. CONSTRUCTION OF PARAMEIDAL AG-GROUPOID

Construction of algebraic structures is an important tool for their development, wherein one structure is modified to achieve the desired one while using some simple procedures. Sometime the examples so achieved by this method is not even possible through computers. The constructions are even sometimes used effectively to answer a conjecture or to solve an open problem. A variety of constructions are available for quasigroups, loops, semigroups and other algebraic structures. Some constructions for AG-groupoids [21] have been done by the authors. In the following we discuss some constructions of various other structures from paramedial groupoid and vice versa via a series of theorems. For a fixed element of a paramedial groupoid $(G, \cdot)$ we define an operation and implement an additional condition to get an AG**. On the other hand we construct AG* and AG**-groupoid from paramedial groupoid under some specific suitable conditions.

Theorem 4. *Let $(G, \cdot)$ be a paramedial groupoid and let s be a fixed element of G. Define a binary operation " $\circ$ " on G as $x \circ y = x(sy)$ for all $x, y \in G$. Then $(G, \circ)$ is an AG-groupoid. In addition, if $(G, \cdot)$ satisfies $x(yz) = y(xz)$, for all $x, y \in G$, then $(G, \circ)$ is an AG**-groupoid.*

Proof. Let $x, y, z \in G$. Then by the definition of $\circ$, (3. 1) and the medial law;

$$(x \circ y) \circ z = (x(sy))(sz) = (z(sy))(sx) = (z \circ y) \circ x$$
$$\Rightarrow (x \circ y) \circ z = (z \circ y) \circ x.$$

Thus $(G, \circ)$ satisfy the left invertive law and hence is an AG-groupoid. Now let $(G, \cdot)$ satisfy the identity $x(yz) = y(xz)$. Then by repeated use of this identity

$$\begin{aligned} x \circ (y \circ z) = x(s(y(sz))) &= x(y(s(sz))) = y(x(s(sz))) \\ = y(s(x(sz))) = y \circ (x \circ z) &\Rightarrow x \circ (y \circ z) = y \circ (x \circ z). \end{aligned}$$

Hence $(G, \circ)$ is an AG**-groupoid. □

Theorem 5. *Let $(G, \triangleright)$ be an AG-groupoid. Define a binary operation "$\cdot$" on $(G, \triangleright)$ as $x \cdot y = \varphi(x) \triangleright \psi(y)$ for all $x, y \in G$, where $\varphi, \psi \in End(G)$. Then*

(i) $(G, \cdot)$ is paramedial groupoid if $\varphi^2 = \psi^2, \varphi\psi = \psi\varphi$ and any of the following holds:
   *(a) $(G, \triangleright)$ is an AG**-groupoid,*
   (b) $(G, \triangleright)$ is an AG-groupoid.*
(ii) $(G, \cdot)$ is paramedial, if $\varphi^2, \psi^2, \varphi$ and ψ are constants.

Proof. (i) Let $(G, \triangleright)$ be an AG-groupoid and
(a). Let $(G, \triangleright)$ be an AG**-groupoid and $u, b, c, t \in G$. Then by definition of " $\cdot$ "

$$\begin{aligned} ub \cdot ct &= \varphi(\varphi(u) \triangleright \psi(b)) \triangleright \psi(\varphi(c) \triangleright \psi(t)) \\ &= (\varphi^2(u) \triangleright \varphi\psi(b)) \triangleright (\psi\varphi(c) \triangleright \psi^2(t)) \end{aligned} \quad (4.\ 1)$$

Again, by the definition of " $\cdot$ ", AG** and the left invertive law

$$\begin{aligned} tb \cdot cu &= \varphi(\varphi(t) \triangleright \psi(b)) \triangleright \psi(\varphi(c) \triangleright \psi(u)) \\ &= (\varphi^2(t) \triangleright \varphi\psi(b)) \triangleright (\psi\varphi(c) \triangleright \psi^2(u)) \\ &= [(\psi\varphi(c) \triangleright \psi^2(u)) \triangleright \varphi\psi(b)] \triangleright \varphi^2(t) \quad \text{[by AG**]} \\ &= [(\varphi\psi(b) \triangleright \psi^2(u)) \triangleright \psi\varphi(c)] \triangleright \varphi^2(t)] \quad \text{[by left invertive law]} \\ &= (\varphi^2(t) \triangleright \psi\varphi(c)) \triangleright (\varphi\psi(b) \triangleright \psi^2(u)) \quad \text{[by left invertive law]} \end{aligned} \quad (4.\ 2)$$

From (4. 1) and (4. 2), $(G, \cdot)$ is paramedial if $\varphi^2 = \psi^2$ and $\varphi\psi = \psi\varphi$.

(b). Let $(G, \triangleright)$ be an AG*-groupoid and $u, b, c, t \in G$. By definition of " $\cdot$ ", left invertive law and AG*-groupoid

$$\begin{aligned} ub \cdot ct &= \varphi(\varphi(u) \triangleright \psi(b)) \triangleright \psi(\varphi(c) \triangleright \psi(t)) \\ &= (\varphi^2(u) \triangleright \varphi\psi(b)) \triangleright (\psi\varphi(c) \triangleright \psi^2(t)) \\ &= [(\psi\varphi(c) \triangleright \psi^2(t)) \triangleright (\varphi\psi(b)] \triangleright \varphi^2(u) && \text{[by left invertive law]} \\ &= [(\varphi\psi(b) \triangleright \psi^2(t)) \triangleright \psi\varphi(c)] \triangleright \varphi^2(u) && \text{[by left invertive law]} \\ &= [\psi^2(t) \triangleright (\varphi\psi(b) \triangleright \psi\varphi(c))] \triangleright \varphi^2(u) && \text{[by AG*]} \\ &= [\varphi^2(u) \triangleright (\varphi\psi(b) \triangleright \psi\varphi(c))] \triangleright \psi^2(t) && \text{[by left invertive law]} \\ &= (\varphi\psi(b) \triangleright \psi\varphi(c)) \triangleright (\varphi^2(u) \triangleright \psi^2(t)). && \text{[by AG*]} \end{aligned} \tag{4. 3}$$

Again, applying the left invertive law, medial law and the definition of AG*-groupoid

$$\begin{aligned} tb \cdot cu &= \varphi(\varphi(t) \triangleright \psi(b)) \triangleright \psi(\varphi(c) \triangleright \psi(u)) \\ &= (\varphi^2(t) \triangleright \varphi\psi(b)) \triangleright (\psi\varphi(c) \triangleright \psi^2(u)) \\ &= [(\psi\varphi(c) \triangleright \psi^2(u)) \triangleright \varphi\psi(b)] \triangleright \varphi^2(t) && \text{[by left invertive law]} \\ &= \varphi\psi(b) \triangleright [(\psi\varphi(c)) \triangleright \psi^2(u)) \triangleright \varphi^2(t)] && \text{[by AG*]} \\ &= \varphi\psi(b) \triangleright [(\varphi^2(t)) \triangleright \psi^2(u)) \triangleright \psi\varphi(c)] && \text{[by left invertive law]} \\ &= [(\varphi^2(t)) \triangleright \psi^2(u)) \triangleright \varphi\psi(b)] \triangleright \psi\varphi(c) && \text{[by AG*]} \\ &= (\psi\varphi(c) \triangleright \varphi\psi(b)) \triangleright (\varphi^2(t) \triangleright \psi^2(u)). && \text{[by left invertive law]} \end{aligned} \tag{4. 4}$$

From (4. 3) and (4. 4), $(G, \cdot)$ is paramedial if $\varphi^2 = \psi^2$ and $\varphi\psi = \psi\varphi$.

(ii) Let $(G, \triangleright)$ be an AG-groupoid and $u, b, c, t \in G$. Then by definition of " $\cdot$ " and left invertive law

$$\begin{aligned} ub \cdot ct &= \varphi(\varphi(u) \triangleright \psi(b)) \triangleright \psi(\varphi(c) \triangleright \psi(t)) \\ &= (\varphi^2(u) \triangleright \varphi\psi(b)) \triangleright (\psi\varphi(c) \triangleright \psi^2(t)) \\ &= [(\psi\varphi(c) \triangleright \psi^2(t)) \triangleright \varphi\psi(b)] \triangleright \varphi^2(u). \end{aligned} \tag{4. 5}$$

Again, by definition of " $\cdot$ " and left invertive law

$$\begin{aligned} tb \cdot cu &= \varphi(\varphi(t) \triangleright \psi(b)) \triangleright \psi(\varphi(c) \triangleright \psi(u)) \\ &= (\varphi^2(t) \triangleright \varphi\psi(b)) \triangleright (\psi\varphi(c) \triangleright \psi^2(u)) \\ &= [(\psi\varphi(c) \triangleright \psi^2(u)) \triangleright \varphi\psi(b)] \triangleright \varphi^2(t). \end{aligned} \tag{4. 6}$$

From (4. 5) and (4. 6), $(G, \cdot)$ is paramedial if $\varphi^2(u) = \varphi^2(t)$, $\psi^2(t) = \psi^2(u)$ and $\psi\varphi(b) = \psi\varphi(c)$, $\varphi\psi(b) = \varphi\psi(c)$, i.e. φ^2, ψ^2 and $\varphi\psi$ and $\psi\varphi$ are constants.

Hence the theorem is proved. □

Theorem 6. *Let $(G, \cdot)$ be a paramedial AG-groupoid and q be fixed element in G. Define a binary operation $\oplus$ on $(G, \cdot)$ as $u \oplus b = (uq)b$, $\forall u, b \in G$. Then $(G, \oplus)$ is a commutative semigroup.*

Proof. Let $u, b, t \in G$. Then by definition of $\oplus$ and left invertive law

$$u \oplus b = (uq)b = (bq)u = b \oplus u.$$

Now by medial law, left invertive law, (3. 1) and (3. 2)

$$\begin{aligned} (u \oplus b) \oplus t = (((uq)b)q)t &= (tq)(uq \cdot b) = (tq)(bq \cdot u) = (u(bq))(qt) \\ = (uq)((bq)t) &= u \oplus (b \oplus t) \Rightarrow (u \oplus b) \oplus t = u \oplus (b \oplus t). \end{aligned}$$

Hence $(G, \oplus)$ is commutative semigroup. □

5. ENUMERATION OF PARAMEDIAL AG-GROUPOIDS

Enumeration and classification of associative and non associative structures play an important role in their characterizations. Using GAP, AG-groupoids are enumerated by Distler et al.[6] up to order 6. We use the same techniques and tools with different codes in GAP for enumeration of the paramedial AG-groupoids. We further categorize these AG-groupoids into non-commutative, associative, and non-associative as given in the following table. Note that all AG-groupoids of size 2 or less are commutative and hence associative. Furthermore, it is pertinent to mention that, since GAP only counts the non-isomorphic tables so as these enumerations. The presented data in the table shows that most of the AG-groupoids are paramedial, this fact can also be verified by Theorem (3), Corollary 1 and the investigated results of [22] wherein it is proved that Bol*-AG-groupoid, AG-monoid and AG-groupoid semigroup are paramedial and that every AG** and T^1-AG-groupoid is Bol* hence are paramedial. Further, it is investigated that every AG-monoid is AG** and that a T^4-AG-groupoid is T^2 and inductively are paramedial. In order to effectively visualize these facts, the Venn diagram is presented in Fig. 1.

Size	3	4	5	6
Total AG-groupoids	20	331	31913	40104513
Non-associative	8	269	31467	40097003
Paramedial	18	313	31294	39960206
Non-associative paramedial	6	251	30848	39952696
Non-commutative & associative paramedial	0	4	121	5367
Commutative & associative paramedial	12	58	325	2143

Table 1. Enumeration and classification of paramedial AG-groupoids

6. CONGRUENCES ON PARAMEDIAL AG-GROUPOIDS

In this section, some equivalence relations and congruences on paramedial AG-groupoid are defined and investigated. Moreover, some partial and compatible partial orders on an inverse paramedial AG-groupoid is provided. Various examples are provided to illustrate the relative concept of parmedial and inverse paramedial AG-groupoid.

Theorem 7. *Let u, b be elements of a paramedial AG-groupoid G. Define a relation σ on G as, $u\sigma b \Leftrightarrow xu = xb$, for all $x \in G$. Then σ is an equivalence relation on G.*

Proof. Clearly σ is reflexive, as for any $u \in G$ and for all $x \in G$, $xu = xu \Rightarrow u\sigma u$. Again for any $u, b \in G$ and for all $x \in G$, let $u\sigma b$ then by definition of σ, $u\sigma b \Leftrightarrow xu = xb \Leftrightarrow xb = xu \Leftrightarrow b\sigma u$. Hence σ is symmetric. Now, for transitivity, let $u\sigma b$ and $b\sigma c$. Then $u\sigma b \Leftrightarrow xu = xb$ and $b\sigma c \Leftrightarrow xb = xc$ for all $x \in G$. This implies $xu = xb = xc \Leftrightarrow xu = xc \Leftrightarrow u\sigma c$. Thus σ is transitive. Hence σ is an equivalence relation on G. □

The following example illustrate the above result.

Example 6. *Let $G = \{1, 2, 3, 4, 5\}$. Then $(G, \cdot)$ in the following table is a paramedial AG-groupoid.*

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	2	2	1	2	2
5	2	2	3	2	3

The equivalence relation σ on G is given as:

$$\sigma = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (1,4), (4,1), (2,4), (4,2)\}.$$

Theorem 8. *Let G be a paramedial AG-groupoid with $E(G) \neq \phi$ where $E(G)$ is the set of idempotents of G and ϱ be defined on G as,*

$$\varrho = \{(u,b) \in G \times G,\ eu = eb,\ \text{for some } e \in E(G)\}.$$

Then ϱ is a congruence on G.

Proof. First we show that ϱ is an equivalence relation on G. As $eu = eu \Rightarrow u\varrho u$ for any $u \in G$ and $e \in E(G)$. Hence ϱ is reflexive. Again, let $u\varrho b$ then, $u\varrho b \Leftrightarrow eu = eb \Leftrightarrow eb = eu \Leftrightarrow b\varrho u$. Hence ϱ is symmetric. Now for transitivity, let $u\varrho b$ and $b\varrho t$, then $eu = eb,\ fb = ft$ for some $e, f \in E(G)$. Now,

$$\begin{aligned} (ef)\,u &= (ee \cdot f)\,u = uf \cdot ee = ef \cdot eu = ef \cdot eb \\ &= ee \cdot fb = ee \cdot ft = te \cdot fe = tf \cdot ee = (ef)\,t \\ \Rightarrow (ef)\,u &= (ef)\,t. \end{aligned}$$

Since $ef \in E(G)$ we conclude that $u\varrho t$ and thus ϱ is transitive. Hence ϱ is an equivalence relation on G.

ϱ is right compatible: Let $u\varrho b$, then for some $e \in E(G)$, $eu = eb$.

$$\begin{aligned} u\varrho b &\Rightarrow eu = eb \\ &\Rightarrow eu \cdot t = eb \cdot t,\ \forall t \in G \\ &\Rightarrow tu \cdot e = tb \cdot e \quad \text{[by Left invertive law]} \\ &\Rightarrow tu \cdot ee = tb \cdot ee \quad [e \in E(G)] \\ &\Rightarrow eu \cdot et = eb \cdot et \quad \text{[by Eqn. 3.1]} \\ &\Rightarrow ee \cdot ut = ee \cdot bt \quad \text{[by medial law]} \\ &\Rightarrow e \cdot ut = e \cdot bt \quad [e \in E(G)] \\ &\Rightarrow ut\varrho bt,\ \forall t \in G \end{aligned}$$

ϱ is left compatible: Let $u\varrho b$, then for some $e \in E(G)$, $eu = eb$. Since ϱ is reflexive so $et = et$,$\forall t \in G$. Thus,

$$\begin{aligned} u\varrho b &\Rightarrow eu = eb \\ &\Rightarrow et \cdot eu = et \cdot eb,\ \forall t \in G \\ &\Rightarrow ee \cdot tu = ee \cdot tb \quad \text{[by medial law]} \\ &\Rightarrow e \cdot tu = e \cdot tb \quad [e \in E(G)] \\ &\Rightarrow tu\varrho tb. \end{aligned}$$

Thus ϱ is compatible. Consequently ϱ is a congruence on G. □

The following example illustrate the above result.

Example 7. *Let $G = \{1,2,3,4\}$. Then $(G,\cdot)$ with the following table is a paramedial AG-groupoid. It is easy to verify that the relation ϱ given below is a congruence for $e = 1$,*

·	1	2	3	4
1	1	1	3	3
2	1	1	4	4
3	3	3	1	1
4	3	3	1	1

$$\varrho = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}.$$

6.1. **Inverse Paramedial AG-groupoid.** Now, we introduce the following notions to investigate inverse paramedial AG-groupoid. An inverse AG-groupoid G is an AG-groupoid in which for all $u \in G$ there exists some $u' \in G$ such that $uu' \cdot u = u$ and $u'u \cdot u' = u'$ [14, 7]. G is called completely inverse AG-groupoid if $uu' = u'u$ for every $u \in G$, where u' is called the inverse of u. By $V(u)$ we shall mean the set of all inverses of u [15].

Remark 1. [14] *Let $a' \in V(a)$ and $b' \in V(b)$ in an AG-groupoid then $a'b' \in V(ab)$ and $(ab)' = a'b'$. Moreover, aa' and $a'a$ are not necessarily idempotents.*

Remark 2. *Let G be a paramedial AG-groupoid and $e, h \in E(G)$. Then by medial and pramedial laws,*

$$eh = ee \cdot hh = he \cdot he = hh \cdot ee = he \Rightarrow eh = he.$$

Consequently, $E(G)$ is a semi-lattice.

Remark 3. [4] *Let G be an inverse AG-groupoid, such that $v' \in V(v)$ for $v \in G$ and $vv' = v'v$. Then*

$$(vv')^2 = vv' \cdot vv' = vv' \cdot v'v = (v'v \cdot v')v = v'v = vv',$$

implies that $vv' \in E(G)$. However, vv' and $v'v$ are not necessarily idempotent.

Lemma 6.2. *Let G be an inverse parmedial AG-groupoid and $v \in G$. Then $vv', v'v \in E(G)$ if and only if $vv' = v'v$.*

Proof. Let vv', $v'v \in E(G)$. Then by left invertive law

$$vv' \cdot v'v = (v'v \cdot v')v = v'v \tag{6. 1}$$

$$v'v \cdot vv' = (vv' \cdot v)v' = vv' \tag{6. 2}$$

.

Since $vv', v'v \in E(G)$, thus by Remark 2, (6. 1) and (6. 2) $vv' \cdot v'v = v'v \cdot vv'$. Hence $vv' = v'v$. The converse follows by Remark 3. □

Example 8. *An inverse AG-groupoid G of size 4, is given below.*

∗	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

$E(G) = \{2, 3\}$ is a semi-lattice, elements 1 and 4 are mutually inverses and $1 * 4 \neq 4 * 1$.

Remark 4. [18]*Let G be an inverse paramedial AG-groupoid, and $x, y \in V(u)$. Then $ux = (uy \cdot u)x = xu \cdot uy = yu \cdot ux = (ux \cdot u)y = uy$. Hence*

$$\begin{aligned} x &= xu \cdot x = (xu(xu \cdot x)) = (x \cdot xu)(ux) = (x \cdot xu)(uy) \\ &= (y \cdot xu)(ux) = (y \cdot xu)(uy) = (yu)(xu \cdot y) \\ &= (yu)(yu \cdot x) = (y \cdot yu)(ux) = (y \cdot yu)(uy) \\ &= (yu)(yu \cdot y) = yu \cdot y = y \\ &\Rightarrow x = y. \end{aligned}$$

It follows that $|V(u)| = 1$, and the inverse of $u \in G$ is unique. We shall denote it by u^{-1}.

6.3. **Partial Order on Inverse Paramedial AG-groupoid.** A relation $\leq$ is called a partial order on AG-groupoid G, if it satisfies the conditions:

(i) $\leq$ is reflexive that is $u \leq u \, \forall u \in G$,
(ii) $\leq$ is antisymmetric that is $u \leq w$ and $w \leq u \Rightarrow u = w \, \forall u, w \in G$,
(iii) $\leq$ is transitive that is $u \leq w, w \leq v \Rightarrow u \leq v \, \forall u, w, v \in G$.

$\leq$ is called right (left) compatible if $u \leq w$ implies $uc \leq wc$ $(cu \leq cw)$ $\forall u, w \in G$. A compatible equivalence relation is called a congruence.

Theorem 9. *Let G be an inverse paramedial AG-groupoid. Then for any $v, w \in G$, the relation $\leq$ defined as,*

$$v \leq w \quad \Leftrightarrow \quad v = vv^{-1} \cdot w \tag{6. 3}$$

is a partial order and is compatible.

Proof. The relation $\leq$ is clearly reflexive.
$\leq$ is antisymmetric: Assume that $v \leq w$ and $w \leq v$, then $v = vv^{-1} \cdot w$ and $w = ww^{-1} \cdot v$. Now, by assumption (Assump), medial law (ML), left invertive law (LIL), (3. 1),(3. 2)

$$\begin{aligned} v &= vv^{-1} \cdot w = (vv^{-1})(ww^{-1} \cdot v) \overset{LIL}{=} (vv^{-1})(vw^{-1} \cdot w) \\ &\overset{3.2}{=} (w \cdot vw^{-1})(v^{-1}v) \\ &\overset{3.1}{=} (v \cdot vw^{-1})(v^{-1}w) \overset{LIL}{=} (v^{-1}w \cdot vw^{-1})v \\ &\overset{3.1}{=} (w^{-1}w \cdot vv^{-1})v \\ &\overset{LIL}{=} (v \cdot vv^{-1})(w^{-1}w) \overset{3.2}{=} (ww^{-1})(vv^{-1} \cdot v) \\ &\overset{Assump}{=} ww^{-1} \cdot v = w. \end{aligned}$$

Thus $v = w$. Therefore G is antisymmetric.
$\leq$ is transitivitve: Assume that $v \leq w$ and $w \leq c$, then $v = vv^{-1} \cdot w$ and $w = ww^{-1} \cdot c$. Now, by assumption (Assump), medial law (ML), left invertive law (LIL), (3. 1),(3. 2) and Remark (1) we have,

$$\begin{aligned} v &\overset{Assump}{=} vv^{-1} \cdot w \overset{Assump}{=} (vv^{-1})(ww^{-1} \cdot c) \overset{Assump}{=} ((vv^{-1} \cdot v)v^{-1})(ww^{-1} \cdot c) \\ &\overset{LIL}{=} (v^{-1}v \cdot vv^{-1})(ww^{-1} \cdot c) \overset{ML}{=} (v^{-1}v \cdot ww^{-1})(vv^{-1} \cdot c) \\ &\overset{3.2}{=} (c \cdot vv^{-1})(ww^{-1} \cdot v^{-1}v) \overset{3.2}{=} (c \cdot vv^{-1})(vv^{-1} \cdot w^{-1}w) \\ &\overset{3.1}{=} (c \cdot vv^{-1})(wv^{-1} \cdot w^{-1}v) \overset{LIL}{=} (c \cdot vv^{-1})((w^{-1}v \cdot v^{-1})w) \\ &\overset{Remark\,(1)}{=} (c \cdot vv^{-1})((wv^{-1} \cdot v)^{-1})w) \overset{LIL}{=} (c \cdot vv^{-1})((vv^{-1} \cdot w)^{-1}w) \\ &\overset{Assump}{=} (c \cdot vv^{-1})(v^{-1}w) \overset{3.2}{=} (wv^{-1})(vv^{-1} \cdot c) \overset{LIL}{=} (wv^{-1})(cv^{-1} \cdot v) \\ &\overset{ML}{=} (w \cdot cv^{-1})(v^{-1}v) \overset{3.1}{=} (v \cdot cv^{-1})(v^{-1}w) \overset{ML}{=} (vv^{-1})(cv^{-1} \cdot w) \\ &\overset{LIL}{=} (vv^{-1})(wv^{-1} \cdot c) \overset{3.2}{=} (c \cdot wv^{-1})(v^{-1}v) \overset{ML}{=} (cv^{-1})(wv^{-1} \cdot v) \\ &\overset{LIL}{=} (cv^{-1})(vv^{-1} \cdot w) \overset{Assump}{=} cv^{-1} \cdot v \overset{LIL}{=} vv^{-1} \cdot c. \end{aligned}$$

Thus $v \leq c$. Therefore $\leq$ is transitive. Hence $\leq$ is a partial order on G.
$\leq$ is compatible: Assume that $v \leq w$ and $t \in G$. Then by assumption (Assump), medial law (ML), left invertive law (LIL), (3. 1),(3. 2) and Remark (1) we have,

$$\begin{aligned} tv &\overset{Assump}{=} t(vv^{-1} \cdot w) \overset{\leq}{\equiv} (tt^{-1} \cdot t)(vv^{-1} \cdot w) \overset{3.2}{=} (w \cdot vv^{-1})(t \cdot tt^{-1}) \\ &\overset{ML}{=} (wt)(vv^{-1} \cdot tt^{-1}) \overset{ML}{=} (wt)(vt \cdot v^{-1}t^{-1}) \overset{3.2}{=} (v^{-1}t^{-1} \cdot vt)(tw) \\ &\overset{3.2}{=} (tv \cdot t^{-1}v^{-1})(tw) \overset{Remark\,(1)}{=} (tv \cdot (tv)^{-1})(tw) \end{aligned}$$

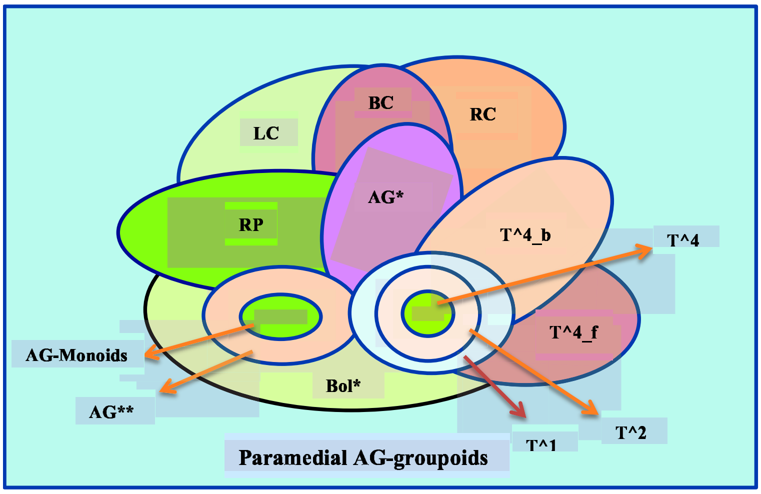

FIGURE 1. Relations of various subclasses of AG-groupoids with paramedial

Thus $tv \leq tw$. Hence $\leq$ is left compatible. Again

$$\begin{aligned} vt &\overset{Assump}{=} (vv^{-1} \cdot w)t \overset{Assump}{=} (vv^{-1} \cdot w)(tt^{-1} \cdot t) \\ &\overset{ML}{=} (vv^{-1} \cdot tt^{-1})(wt) \\ &\overset{ML}{=} (vt \cdot v^{-1}t^{-1})(wt) \overset{Remark\,(1)}{=} (vt \cdot (vt)^{-1})(wt) \\ \Rightarrow vt &\leq wt. \end{aligned}$$

Thus $\leq$ is also right compatible and hence compatible. □

7. CONCLUSION

In this note, we studied some characteristics and constructions for the paramedial AG-groupoids as a subclass and established various results. The modern computational techniques of Mace-4 and GAP are used for enumeration and producing various examples and counterexamples to strengthen this studies up to the mark. Various relations of this subclass with other known algebraic structures are established as depicted in Figure 1. Furthermore, paramedial AG-groupoid is decomposed with the help of some congruences and a partial order is defined and investigated for inverse paramedial AG-groupoid.

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