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New Generalized Reverse Minkowski Inequality and Related Integral Inequalities via Generalized κ-Fractional Hilfer-Katugampola Derivative

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Abstract: This article aims to present the reverse Minkowski inequality and other related integral inequalities by using the generalized k-fractional Hilfer-Katugampola derivative. We have novelized these inequalities by utilizing the Hölder inequality. Moreover, two new theorems by using this inequality are presented for the generalized κ-fractional Hilfer-Katugampola derivative. The numerical approximations of our consequence have several utilities in applied sciences and fractional integral and differential equations.

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Key Words: Minkowski inequality, Reverse Minkowski inequality, Generalized κ-fractional Hilfer-Katugampola derivative.

1. DESCRIPTION

The calculus of non-integer order pacts derivatives and integral operators' novelization, especially inequalities involving fractional integrals. In the literary text, numerous descriptions of fractional integral operators exist, e.g., Weyl, ErdÃl'lyiâĂŞKober, Hadamard integral, RiemannâĂŞLiouville fractional integral, Hilfer, Katugampola, and Hilfer-Katugampola fractional integral [29, 19, 33, 24]. Abdeljawad [1] and Khalil et al. [27] extend new fractional operators called local fractional conformable derivatives and integral. This individual generalizes such fractional operators via including the new parameters and yield the relevant inequalities like Hermite-Hadamard, Opial, Ostrowski, Hadamard, and others can be seen in [6, 2, 39, 9, 45, 38, 10].
Katugampola [25] proposed a generalized fractional integral summarizing all existing integrals: Weyl, Riemann-Liouville, ErdÃl'lyiâĂŞKober, Hadamard, and Liouville. This

iteration process of fractional calculus yield the generalized fractional integrals and derivative operators by Jarad [23]. Many inequalities are obtained using such generalized operators and motivate the researchers to pioneering concepts to unify the fractional operators [34, 8, 11, 41, 20, 35, 36, 28, 15]. On the other hand, there are numerous approaches to acquiring a generalization of classical fractional integrals inequalities that can be found in various fields of mathematics, science, engineering, physics, impulse equations, [4, 3], the stability of linear transformations, initial value problems, integral differential equations, and boundary value problems. Researchers can find these applications in [39, 17, 46] and various branches of mathematics. Furthermore, Future work, influenced by these advances, will bring innovative thinking to give novelties and create variants concerning these fractional operators. Thus, many applications can be found in [4, 3] by using the integral inequalities. Among them, most known are Hermite-Hadamard, Holder, Minkowski, Jensen, Hardy, and Jensen-Mercer and others [21, 40, 43, 7, 22, 5, 32]. Such generalization motivate us to apply the generalized κ-fractional Hilfer-Katugampola derivative to generalize the reverse Minkowski inequality [42, 26, 16, 12, 31].
Integral inequalities have potential application in several areas of science: technology, mathematics, chemistry, plasma physics, among others; especially, we point out initial value problems, the stability of linear transformation, integral differential equations, and impulse equations. Many researchers have focused on finding the numerous version of the reverse Minkowski inequality for generalized fractional conformable integral by the generalized fractional integral operator and Hadamard fractional integral operators.// The well known Minkowski integral inequality is given for $0 < \int_a^b \psi_1^q(z)dz < \infty$ and $0 < \int_a^b \psi_2^q(z)dz < \infty$ as follows:

$$\left(\int_a^b (\psi_1 + \psi_2)^q (z)\, dz\right)^{\frac{1}{q}} \leq \left(\int_a^b \psi_1^q(z)dz\right)^{\frac{1}{q}} + \left(\int_a^b \psi_2^q(z)dz\right)^{\frac{1}{q}}, \tag{1. 1}$$

where $q \geq 1$.
Similalry, the reverse Minkowski inequality is given as follows:

$$\left(\int_a^b \psi_1^q(z)dz\right)^{\frac{1}{q}} + \left(\int_a^b \psi_2^q(z)dz\right)^{\frac{1}{q}} \leq c\left(\int_a^b (\psi_1 + \psi_2)^q (z)\, dz\right)^{\frac{1}{q}}, \tag{1. 2}$$

where c is a constant and $q > 1$.
The contents of this paper are sorted into different sections. The basic definitions and concept of the generalized k-fractional Hilfer-Katugampola derivative are presented in section 2. We proved the theorem associated with the reverse Minkowski inequality. Our key result is shown in section 3. We advocate essential consequences such as the reverse Minkowski inequality via the generalized κ-fractional Hilfer-Katugampola derivative. Related integral inequalities are proved in section 4. The last section containing the conclusion closed the article.

2. PRELUDE

These basic segment definitions of fractional calculus utilizing the Riemann integral proposed by [40], and the reverse theorem of Minkowski's inequality and its related summary through Riemann-Liouville and Hadamard integration is the motivation of this study. In addition, the fractional conformal integral is discussed, and a theorem is proposed to recover the specific situation.

[29] Let $[a,b]$ be a finite or infinite interval on $\mathbb{R}=(-\infty,\infty)$. The set of Lebesgue complex valued measurable function ψ on $[a,b]$ is defined as

$$M_q\left[a,b\right]=\left\{\psi:\psi_q=\sqrt[q]{\int_a^b\left|\psi\left(z\right)\right|^q dz}<+\infty\right\},\quad 1\leq q<\infty. \tag{2. 3}$$

[7] Let $\psi_1,\psi_2\in M_q\left[a,b\right]$ with $1\leq q<\infty$,$0<\int\limits_a^b\psi_1^q(z)dz<\infty$ and $0<\int\limits_a^b\psi_2^q(z)dz<\infty$. If $0<n\leq\frac{\psi_1(z)}{\psi_2(z)}\leq N$ for $n,N\in\mathbb{R}^+$ and $\forall z\in[a,b]$, then

$$\left(\int\limits_a^b\psi_1^q(z)dz\right)^{\frac{1}{q}}+\left(\int\limits_a^b\psi_2^q(z)dz\right)^{\frac{1}{q}}\leq\frac{N(n+1)+(N+1)}{(n+1)\,(N+1)}\left(\int\limits_a^b\left(\psi_1+\psi_2\right)^q(z)\,dz\right)^{\frac{1}{q}}. \tag{2. 4}$$

[40] Let $\psi_1,\psi_2\in M_q\left[a,b\right]$ with $1\leq q<\infty$,$0<\int\limits_a^b\psi_1^q(z)dz<\infty$ and $0<\int\limits_a^b\psi_2^q(z)dz<\infty$. If $0<n\leq\frac{\psi_1(z)}{\psi_2(z)}\leq N$ for $n,N\in\mathbb{R}^+$ and $\forall z\in[a,b]$, then

$$\begin{aligned}&\left(\int\limits_a^b\psi_1^q(z)dz\right)^{\frac{2}{q}}+\left(\int\limits_a^b\psi_2^q(z)dz\right)^{\frac{2}{q}}\\&\geq\left(\frac{(n+1)\,(N+1)}{N}-2\right)\left(\int\limits_a^b\psi_1^q(z)dz\right)^{\frac{1}{q}}\left(\int\limits_a^b\psi_2^q(z)dz\right)^{\frac{1}{q}}.\end{aligned} \tag{2. 5}$$

[29] A function $\psi(z)$ is said to be in $M_{q,r}\left[a,b\right]$ if

$$M_{q,r}\left[a,b\right]=\left\{\psi:\psi_q=\sqrt[q]{\int_a^b\left|\psi\left(z\right)\right|^q z^r dz}<+\infty\right\},\quad 1\leq q<\infty\quad,\quad r\geq 0. \tag{2. 6}$$

[38] Let $m-1<\omega\leq m$, $m\in\mathbb{N}$, $\rho>0$, $\kappa>0$ and $\psi\in M\left(a,b\right)$ and $a<z<b$, the κ-Riemann Liouville fractional integral of left sided and right sided is defined as

$$\left({}_{\kappa}^{\rho}\Im_{a\pm}^{\omega}\psi\right)(z)=\pm\frac{1}{\kappa\Gamma_{\kappa}(\omega)}\int_{a}^{z}\left(\frac{z^{\rho}-y^{\rho}}{\rho}\right)^{\frac{\omega}{\kappa}-1}y^{\rho-1}\psi(y)\,\mathrm{d}y \quad \omega>0 \quad , \quad x>a. \tag{2. 7}$$

[30] Let $m-1<\omega\leq m$, $0\leq\theta\leq 1$, $m\in\mathbb{N}$, $\rho>0$, $\kappa>0$ and $\psi\in M_{q}(a,b)$, the generalized κ-Hilfer-Katugampola fractional derivative (left sided and right sided) as is defined as

$$\left({}_{\kappa}^{\rho}D_{a\pm}^{\omega,\theta}\psi\right)(z)=\pm\left({}_{\kappa}^{\rho}\Im_{a\pm}^{\theta(\kappa m-\omega)}\left(z^{1-\rho}\frac{d}{dz}\right)^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a\pm}^{(1-\theta)(\kappa m-\omega)}\psi\right)\right)(z) \tag{2. 8}$$

$$=\pm\left({}_{\kappa}^{\rho}\Im_{a\pm}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a\pm}^{(1-\theta)(\kappa m-\omega)}\psi\right)\right)(z), \tag{2. 9}$$

where $\delta_{\rho}^{m}=\left(z^{1-\rho}\frac{d}{dz}\right)^{m}$ and ${}_{\kappa}^{\rho}\Im_{a\pm}^{\omega}$ is the Riemann-Liouville integral defined in equation (2.5).

[16] Let $\psi_1,\psi_2\in M_{1,r}[a,b]$ with $1\leq q<\infty$,$0<\left(\Im_{a+}^{\omega,\theta}\psi_1^{q}\right)(z)<\infty$ and $0<\left(\Im_{a+}^{\omega,\theta}\psi_2^{q}\right)(z)<\infty$. If $0<n\leq\frac{\psi_1(z)}{\psi_2(z)}\leq N$ for $n,N\in\mathbb{R}^{+}$ and $\forall z\in[a,b]$, then

$$\left(\Im_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{2}{q}}+\left(\Im_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{2}{q}}\geq\left(\frac{(n+1)(N+1)}{N}-2\right)\left(\Im_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{1}{q}}\left(\Im_{a+}^{\omega,\theta}\psi_2^{q}(z)\right)^{\frac{1}{q}}. \tag{2. 10}$$

Chinchane et al. [12], and Sabrina et al. [44] developed the following two reverse Minkowski inequality theorems in which Hadamard fractional integral operator is involved.

[12, 44] Let $\psi_1,\psi_2\in M_{1,r}[a,b]$ with $1\leq q<\infty$,$0<\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}\right)(z)<\infty$ and $0<\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_2^{q}\right)(z)<\infty$. If $0<n\leq\frac{\psi_1(z)}{\psi_2(z)}\leq N$ for $n,N\in\mathbb{R}^{+}$ and $\forall z\in[a,b]$, then

$$\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{1}{q}}+\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{1}{q}}\leq\left(\frac{N(n+1)+(N+1)}{(n+1)(N+1)}\right)\left(\mathrm{H}_{a+}^{\omega,\theta}(\psi_1+\psi_2)^{q}(z)\right)^{\frac{1}{q}}. \tag{2. 11}$$

[12, 44] Let $\psi_1,\psi_2\in M_{1,r}[a,b]$ with $1\leq q<\infty$,$0<\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}\right)(z)<\infty$ and $0<\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_2^{q}\right)(z)<\infty$. If $0<n\leq\frac{\psi_1(z)}{\psi_2(z)}\leq N$ for $n,N\in\mathbb{R}^{+}$ and $\forall z\in[a,b]$, then

$$\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{2}{q}}+\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{2}{q}}\geq\left(\frac{(n+1)(N+1)}{N}-2\right)\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_1^{q}(z)\right)^{\frac{1}{q}}\left(\mathrm{H}_{a+}^{\omega,\theta}\psi_2^{q}(z)\right)^{\frac{1}{q}}. \tag{2. 12}$$

Chinchane et al. [13] proposed reverse Minkowski inequality through Saigo's fractional integral, and the same inequality was proved by Chinchane [14] via κ-fractional integral.

3. Reverse Minkowski inequality via generalized κ-fractional Hilfer-Katugampola Derivative

This section has generalized the reverse Minkowski inequality by utilizing the generalized κ-fractional Hilfer-Katugampola derivative defined in Definition 2.6 and the relevant theorems.

Let $\psi_1, \psi_2 \in M_{1,r}\left[a, z\right]$ on $[0, \infty]$ such that $\forall z > a$, ${}_{k}^{\rho}D_{a+}^{\omega,\theta}\psi_1^q(z) < \infty$ with $\kappa > 0$ and $\theta \in R\backslash\{0\}$. $\omega > 0$, $q \geq 1$ and ${}_{k}^{\rho}D_{a+}^{\omega,\theta}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a, z]$, then

$$\left({}_{k}^{\rho}D_{a+}^{\omega,\theta}\psi_1^q(z)\right)^{\frac{1}{q}} + \left({}_{k}^{\rho}D_{a+}^{\omega,\theta}\psi_1^q(z)\right)^{\frac{1}{q}} \leq \left(\frac{N\left(n+1\right)+\left(N+1\right)}{\left(n+1\right)\left(N+1\right)}\right)\left({}_{k}^{\rho}D_{a+}^{\omega,\theta}(\psi_1+\psi_2)^q\left(z\right)\right)^{\frac{1}{q}}. \tag{3. 13}$$

Proof. By the given condition $\frac{\psi_1(z)}{\psi_2(z)} \leq N$, $a \leq y \leq z$, it can be written as

$$\psi_1(z) \leq N\left(\psi_1(z)+\psi_2(z)\right) - N\psi_1(z),$$

which implies that

$$(N+1)^q\psi_1^q(z) \leq N^q(\psi_1(z)+\psi_2(z))^q. \tag{3. 14}$$

Applying the operator ${}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\right)$ to both sides of inequity (3.2), we yield

$$\begin{aligned}&(N+1)^{q}{}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\right)\psi_1^q(z)\\&\leq N^{q}{}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\right)(\psi_1(z)+\psi_2(z))^q.\end{aligned} \tag{3. 15}$$

Accordingly, it can be written as by using equation (2.7)

$$\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\psi_1^q(z)\right)^{\frac{1}{q}} \leq \frac{N}{N+1}\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\left(\psi_1+\psi_2\right)(z)\right)^{\frac{1}{q}}. \tag{3. 16}$$

In contrast, $n \leq \frac{\psi_1(z)}{\psi_2(z)}$, it can be written as

$$\left(1+\frac{1}{n}\right)^q\psi_2(z) \leq \frac{1}{n}(\psi_1(z)+\psi_2(z))^q. \tag{3. 17}$$

Applying the operator ${}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\right)$ on both sides of (3.5), and simplifying the expression , we obtain

$$\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\psi_1^q(z)\right)^{\frac{1}{q}} \leq \frac{1}{n+1}\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\left(\psi_1+\psi_2\right)(z)\right)^{\frac{1}{q}}. \tag{3. 18}$$

The desired result (3.1) stems from (3.4) and (3.6) by adding these inequalities.
Inequality (3.1) is referred to as the reverse Minkowski inequality via generalized κ-fractional Hilfer-Katugampola derivative. □

Let $\psi_1, \psi_2 \in M_{1,r}\,[a,z]$ on $[0,\infty]$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ with $\kappa > 0$ and $\theta \in R\backslash\{0\}$. $\omega > 0$, $q \geq 1$ and ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{2}{q}} + \left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{2}{q}} \leq \left(\frac{(n+1)(N+1)}{N} - 2\right)\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{1}{q}}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{1}{q}}. \tag{3. 19}$$

Proof. The product of inequalities (3.4) and (3.6) yields

$$\frac{(n+1)(N+1)}{N}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{1}{q}}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{1}{q}} \leq \left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_1+\psi_2\right)(z)\right)^{\frac{2}{q}}. \tag{3. 20}$$

Now, utilizing the Minkowski inequality to the right-hand side of (3.8), we yield

$$\frac{(n+1)(N+1)}{N}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{1}{q}}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{1}{q}} \leq \left(\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{1}{q}} + \left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{1}{q}}\right)^2. \tag{3. 21}$$

It can be inferred from (3.9), that

$$\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{2}{q}} + \left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{2}{q}} \geq \left(\frac{(n+1)(N+1)}{N} - 2\right)\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{1}{q}}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{1}{q}}.$$

□

4. Certain Related Inequalities via Generalized κ-fractional Hilfer-Katugampola Derivative

This section is dedicated to the derivation of such related generalized κ-fractional Hilfer-Katugampola derivative operator variants.

Let $\psi_1, \psi_2 \in M_{1,r}\,[a,z]$ on $[0,\infty]$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ with $\kappa > 0$ and $\theta \in R\backslash\{0\}$. $\omega > 0$, $q, r > 1$, $\frac{1}{q} + \frac{1}{r} = 1$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z)\right)^{\frac{1}{q}}\left({}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z)\right)^{\frac{1}{q}} \leq \left(\frac{N}{n}\right)^{\frac{1}{qr}}\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_1^{\frac{1}{q}}(z)\psi_2^{\frac{1}{q}}(z)\right)\right). \tag{4. 22}$$

Proof. Proceeding as in [37] and by the given condition $\frac{\psi_1(z)}{\psi_2(z)} \leq N$, $a \leq y \leq z$, it can be written as

$$\psi_1(z) \leq N\psi_2(z) \quad \Rightarrow \quad \psi_2^{\frac{1}{q}}(z) \geq N^{-\frac{1}{q}}\psi_1^{\frac{1}{q}}(z). \tag{4. 23}$$

We can rewrite it as follows by multiplying both sides of inequality (4.2) by $\psi_1^{\frac{1}{r}}(z)$

$$\psi_1^{\frac{1}{r}}(z)\psi_2^{\frac{1}{q}}(z) \geq N^{-\frac{1}{q}}\psi_1(z). \tag{4. 24}$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$ on both sides, and simplifying the expression , we obtain

$$\begin{aligned}&{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\left(\psi_1^{\frac{1}{r}}(z)\psi_2^{\frac{1}{q}}(z)\right)\\ &\geq N^{-\frac{1}{q}}{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)(\psi_1(z)),\end{aligned} \tag{4. 25}$$

This may also be written as,

$$N^{-\frac{1}{qr}}\left(\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1\right)(z)\right)^{\frac{1}{r}} \leq \left(\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\right)\left(\psi_1^{\frac{1}{q}}(z)\psi_2^{\frac{1}{r}}(z)\right)\right)^{\frac{1}{r}}. \tag{4. 26}$$

In contrast, $n \leq \frac{\psi_1(z)}{\psi_2(z)}$, it can be written as

$$n^{\frac{1}{q}}\psi_2^{\frac{1}{q}}(z) \leq \psi_1^{\frac{1}{q}}(z). \tag{4. 27}$$

Multiply $\psi_2^{\frac{1}{r}}(z)$ to both sides of inequality (4.6) and using the relation $\frac{1}{q}+\frac{1}{r}=1$, we yield

$$n^{\frac{1}{q}}\psi_2(x) \leq \psi_1^{\frac{1}{q}}(x)\psi_2^{\frac{1}{r}}(x). \tag{4. 28}$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$ on both sides, and simplifying the expression , we obtain

$$n^{\frac{1}{qr}}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_2(z))^{\frac{1}{q}} \leq ({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^{\frac{1}{q}}(z)\psi_2^{\frac{1}{r}}(z))^{\frac{1}{r}}. \tag{4. 29}$$

Taking the product between the inequality (4.5) and (4.8) and utilizing $\frac{1}{q}+\frac{1}{r}=1$, the required inequality yields. □

Let $\psi_1, \psi_2 \in M_{1,r}[a,z]$ on $[0,\infty]$ with $\kappa > 0$, $\theta \in R\backslash\{0\}$. $\omega > 0$, $q, r > 1$ and $\frac{1}{q}+\frac{1}{r}=1$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ and ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$ {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1(z)\psi_2(z) \leq \frac{2^{q-1}N^q}{q(N+1)^q}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1^q+\psi_2^q)(z)) + \frac{2^{r-1}}{r(n+1)^r}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^r+\psi_2^r)(z)). \tag{4. 30}$$

Proof. By using the hypothesis, we get the inequality

$$(N+1)^q\psi_1^q(z) \leq N^q(\psi_1+\psi_2)^q(z). \tag{4. 31}$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$on both sides of inequality (4.10) ,we yield

$$\begin{aligned}&(N+1)^q\left({}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\right)(\psi_1^q(z))\\ &\leq N^q\left({}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\right)((\psi_1+\psi_2)^q(z)),\end{aligned} \tag{4. 32}$$

It can be written as

$$ {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^q(z) \leq \frac{N^q}{(N+1)^q}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^q(z). \tag{4. 33} $$

In contrast , $0 < n < \frac{\psi_1(z)}{\psi_2(z)}$, it can be written as

$$ (n+1)^r\psi_2^r(z) \leq (\psi_1+\psi_2)^r(z). \tag{4. 34} $$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_{\rho}\left(\kappa^m{}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$, we yield

$$ {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_2^r(z) \leq \frac{1}{(n+1)^r}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^r(z). \tag{4. 35} $$

Considering the Young's Inequality

$$ \psi_1(z)\ \psi_2(z) \leq \frac{\psi_1^q\,(z)}{q}+\frac{\psi_2^r\,(z)}{r}. \tag{4. 36} $$

Multiplying both sides by ${}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\delta^m_{\rho}\left(\kappa^m{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\right)$, we yield

$$ {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_1\psi_2\right)(z) \leq \frac{1}{q}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_1^q\,(z)\right)+\frac{1}{r}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_2^r\,(z)\right). \tag{4. 37} $$

Invoking inequalities (4.12) and (4.14) into (4.16), we yield

$$ \begin{aligned} {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1\psi_2)(z) &\leq \frac{1}{q}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^q(z))+\frac{1}{r}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_2^q(t)) \\ &\leq \frac{N^q}{q(N+1)^q}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^q(z))+\frac{1}{r(n+1)^r}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^r(z)). \end{aligned} \tag{4. 38} $$

Utilizing the inequality , $(y+z)^r \leq 2^{r-1}(y^r+z^r)$, one yield

$$ {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^q(z) \leq 2^{q-1}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1^q+\psi_2^q)(z), \tag{4. 39} $$

and

$$ {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^r(z) \leq 2^{r-1}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1^q+\psi_2^q)(z). \tag{4. 40} $$

From inequalities (4.17), (4.18), and (4.19) collectively, the proof of inequality (4.9) is done. □

Let $\psi_1, \psi_2 \in M_{1,r}\,[a,z]$ on $[0,\infty]$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ with $\kappa > 0$, $\theta \in R\backslash\{0\}$. $\omega > 0$, $q, r > 1$, $\frac{1}{q}+\frac{1}{r}=1$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$ \begin{aligned} \frac{N+1}{N-c}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1\,(z)-c\psi_2(z))^{\frac{1}{q}} &\leq ({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^q(t))^{\frac{1}{q}}+({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^r(t))^{\frac{1}{r}} \\ &\leq \frac{n+1}{n-c}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1\,(z)-c\psi_2(z))^{\frac{1}{q}}. \end{aligned} \tag{4. 41} $$

Proof. By using $0 < c < n \leq N$, we yield

$$nc \leq Nc \Rightarrow nc + n \leq nc + N \leq Nc + N$$
$$\Rightarrow (N+1)(n-c) \leq (n+1)(N-c).$$

We inferred

$$\frac{(N+1)}{(N-c)} \leq \frac{(n+1)}{(n-c)}$$

Resulting,

$$n - c \leq \frac{\psi_1(x) - c\psi_2(x)}{\psi_2(x)} \leq N - c.$$

which implies that

$$\frac{(\psi_1(x) - c\psi_2(x))^p}{(M-c)^p} \leq \psi_2^p(x) \leq \frac{(J_1(x) - c\psi_2(x))^p}{(m-c)^p}. \quad (4.\ 42)$$

We yield,

$$\frac{1}{N} \leq \frac{\psi_2(z)}{\psi_1(z)} \leq \frac{1}{n} \Rightarrow \frac{n-c}{cn} \leq \frac{\psi_1(z) - c\psi_2(z)}{c\psi_1(x)} \leq \frac{N-c}{cN},$$

Which implies that

$$(\frac{N}{N-c})^q(\psi_1(z) - c\psi_2(z))^q \leq \psi_1^q(z) \leq (\frac{n}{n-c})^q(\psi_1(z) - c\psi_2(z))^q. \quad (4.\ 43)$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$ on both sides of inequality (4.21), we yield

$$\begin{aligned}
&{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\left(\frac{(\psi_1(z) - c\psi_2(z))^q}{(N-c)^q}\right)\\
&\leq^{\rho}_{\kappa} \Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)(\psi_2^q(z))\\
&\leq^{\rho}_{\kappa} \Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\left(\frac{(\psi_1(z) - c\psi_2(z))^q}{(n-c)^q}\right),
\end{aligned}$$

It can be written as accordingly

$$\begin{aligned}
\frac{1}{N-c}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}((\psi_1(z) - c\psi_2(z))^q)^{\frac{1}{q}} &\leq {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_2^q(z))^{\frac{1}{q}}\\
&\leq \frac{1}{n-c}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}((\psi_1(z) - c\psi_2(z))^q)^{\frac{1}{q}}. \quad (4.\ 44)
\end{aligned}$$

Continuing in the same way for the inequality (4.22), we yield

$$\begin{aligned}
\frac{M}{N-c}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}((\psi_1(z) - c\psi_2(z))^q)^{\frac{1}{q}} &\leq {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_2^q(z))^{\frac{1}{q}}\\
&\leq \frac{n}{n-c}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}((\psi_1(z) - c\psi_2(z))^q)^{\frac{1}{q}}. \quad (4.\ 45)
\end{aligned}$$

Now adding the inequality (4.23) and (4.24) , we yield the inequality (4.21). □

Let $\psi_1, \psi_2 \in M_{1,r}[a,z]$ on $[0,\infty]$ with $\kappa > 0$, $\theta \in R\setminus\{0\}$. $\omega > 0$, $q, r > 1$ and $\frac{1}{q} + \frac{1}{r} = 1$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ and ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^q(z))^{\frac{1}{q}} + ({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_2^q(z))^{\frac{1}{q}} \leq \lambda({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^q(z))^{\frac{1}{q}}, \tag{4. 46}$$

where $\lambda = \frac{M(m+N)+N(M+n)}{(M+n)(m+N)}$.

Proof. By the given condition,

$$\frac{1}{N} \leq \frac{1}{\psi_2(z)} \leq \frac{1}{n}. \tag{4. 47}$$

Taking the product of inequality (4.26) and $0 < m \leq \psi_1(z) \leq M$, we obtain

$$\frac{m}{N} \leq \frac{\psi_1(z)}{\psi_2(z)} \leq \frac{M}{n}. \tag{4. 48}$$

From inequality (4.26), we yield

$$\psi_2^q(z) \leq (\frac{N}{m+N})^q(\psi_1(z)+\psi_2(z))^q, \tag{4. 49}$$

and

$$\psi_1^q(z) \leq (\frac{M}{n+M})^q(\psi_1(z)+\psi_2(z))^q. \tag{4. 50}$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_{\rho}\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$ on both sides of inequality (4.28), we obtain

$$\begin{aligned}&{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_{\rho}\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)(\psi_2^q(z))\\ &\leq (\frac{N}{m+N})^q {}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_{\rho}\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)(\psi_1(z)+\psi_2(z))^q,\end{aligned}$$

we can write it as

$$\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_2^q(z))\right)^{\frac{1}{q}} \leq \frac{N}{m+N}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}((\psi_1+\psi_2)^q(z))^{\frac{1}{q}}. \tag{4. 51}$$

Continuing in the same way with the inequality (4.29), we yield

$$\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1^q(z))\right)^{\frac{1}{q}} \leq \frac{M}{n+M}{}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}((\psi_1+\psi_2)^q(z))^{\frac{1}{q}}. \tag{4. 52}$$

Now adding the inequalities (4.30) and (4.31) we get the required inequality (4.25). □

Let $\psi_1, \psi_2 \in M_{1,r}[a,z]$ on $[0,\infty]$with $\kappa > 0$, $\theta \in R\setminus\{0\}$. $\omega > 0$, $q, r > 1$ and $\frac{1}{q} + \frac{1}{r} = 1$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ and ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$\begin{aligned}\frac{1}{M}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1(z)\psi_2(z)) &\leq \frac{1}{(m+1)(M+1)}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^2(z))\\ &\leq \frac{1}{m}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1(z)\psi_2(z)).\end{aligned} \tag{4. 53}$$

Proof. By using $0 < m \leq \frac{\psi_1(z)}{\psi_2(z)} \leq M$, it follows that

$$\psi_2(z)(m+1) \leq \psi_2(z) + \psi_1(z) \leq \psi_2(z)(M+1). \tag{4. 54}$$

Also it can be written as $\frac{1}{M} \leq \frac{\psi_1(z)}{\psi_2(z)} \leq \frac{1}{m}$, we obtain

$$\psi_1(z)(\frac{M+1}{M}) \leq \psi_2(z) + \psi_1(z) \leq \psi_1(z)(\frac{m+1}{m}). \tag{4. 55}$$

Taking product of inequalities (4.33) and (4.34), we yield

$$\frac{\psi_1(z)\psi_2(z)}{M} \leq \frac{(\psi_2(z)+\psi_1(z))^2}{(m+1)(M+1)} \leq \frac{\psi_1(z)\psi_2(z)}{m}. \tag{4. 56}$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\right)$ on both sides of inequality (4.35), we yield

$$\begin{aligned}
&{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\left(\frac{\psi_1(z)\psi_2(z)}{M}\right)\\
&\leq {}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\left(\frac{(\psi_2(z)+\psi_1(z))^2}{(m+1)(M+1)}\right)\\
&\leq {}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho\left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)\left(\frac{\psi_1(z)\psi_2(z)}{m}\right).
\end{aligned}$$

it can be written as

$$\frac{1}{M}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1(z)\psi_2(z)) \leq \frac{1}{(m+1)(M+1)}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}(\psi_1+\psi_2)^2(z)) \leq \frac{1}{m}({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1(z)\psi_2(z)),$$

□

Let $\psi_1, \psi_2 \in M_{1,r}[a,z]$ on $[0,\infty]$ with $\kappa > 0$ and $\theta \in R\backslash\{0\}$. $\omega > 0$, $q, r > 1$ and $\frac{1}{q} + \frac{1}{r} = 1$ such that $\forall z > a$, ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_1^q(z) < \infty$ and ${}^{\rho}_{k}D^{\omega,\theta}_{a+}\psi_2^q(z) < \infty$. If $0 < n \leq \frac{\psi_1(z)}{\psi_2(z)} \leq N$ for $n, N \in \mathbb{R}^+$ and $\forall y \in [a,z]$, then

$$({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_1^q(z))^{\frac{1}{q}} + ({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi_2^q(z))^{\frac{1}{q}} \leq 2({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}h^q(\psi_1(z),\psi_2(z)))^{\frac{1}{q}}, \tag{4. 57}$$

where $h(\psi_1(z),\psi_2(z)) = \max\{M[(\frac{M}{m}+1)\psi_1(t) - M\psi_2(t)]$

Proof. By condition $0 < m \leq \dfrac{\psi_1(z)}{\psi_2(z)} \leq M, a \leq y \leq x$, we can write

$$0 < m \leq M + m - \frac{\psi_1(z)}{\psi_2(z)}. \tag{4. 58}$$

and

$$M + m - \frac{\psi_1(z)}{\psi_2(z)} \leq M. \tag{4. 59}$$

From the inequalities (4.35) and (4.38), we yield

$$\psi_2(z) < \frac{(M+m)\psi_2(z) - \psi_1(z)}{m} \leq h(\psi_1(z),\psi_2(z)). \tag{4. 60}$$

From the given hypothesis , we can write $0 < \frac{1}{M} \leq \frac{\psi_1(z)}{\psi_2(z)} \leq \frac{1}{m}$, which implies

$$\frac{1}{M} \leq \frac{1}{M} + \frac{1}{m} - \frac{\psi_2(z)}{\psi_1(z)}, \tag{4. 61}$$

and

$$\frac{1}{M} + \frac{1}{m} - \frac{\psi_2(z).}{\psi_1(z)} \leq \frac{1}{m} \tag{4. 62}$$

From the inequalities (4.40) and (4.41), we yield

$$\frac{1}{M} \leq \frac{(\frac{1}{M} + \frac{1}{m})\psi_1(z) - \psi_2(z)}{\psi_1(z)} \leq \frac{1}{m}. \tag{4. 63}$$

It can be written as

$$\begin{aligned} \psi_1(z) &\leq M(\frac{1}{M} + \frac{1}{m})\psi_1(z) - M\psi_2(z) \\ &= \frac{M(M+m)\psi_1(z) - M^2 m\psi_2(z)}{mM} \\ &= (\frac{M}{m} + 1)\psi_1(z) - M\psi_2(z) \\ &\leq M[(\frac{M}{m} + 1)\psi_1(z) - M\psi_2(z)] \\ &\leq h(\psi_1(z), \psi_2(z)). \end{aligned} \tag{4. 64}$$

From inequality (4.39) and inequality (4.43)

$$\psi_1^q(z) \leq h^q(\psi_1(z), \psi_2(z)) \tag{4. 65}$$

$$\psi_1^q(z) \leq h^q(\psi_1(z), \psi_2(z)). \tag{4. 66}$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho \left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)$ on both sides of inequality (4.44), we yield

$$\begin{aligned} &{}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho \left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)(\psi_1^q(z)) \\ &\leq {}^{\rho}_{\kappa}\Im^{\theta(\kappa m-\omega)}_{a+}\delta^m_\rho \left(\kappa^m {}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m-\omega)}_{a+}\right)(h^q(\psi_1(z), \psi_2(z))), \end{aligned}$$

It can be written as

$${}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_1^q(z)\right) \leq {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(h^q(\psi_1(z), \psi_2(z))\right). \tag{4. 67}$$

Repeating the process for the inequality (4.45), we yield

$${}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(\psi_1^q(z)\right) \leq {}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\left(h^q(\psi_1(z), \psi_2(z))\right). \tag{4. 68}$$

Adding the inequality (4.46) and (4.47), we yield the required inequality (4.36). □

Theorems (3.1), (3.2), and Theorems (4.1) to (4.6) are proved by using the generalized κ-fractional Hilfer-Katugampola derivative and Riemann-Liouville integral.

5. Concluding remarks

The research paper wrings out, in brief, the newly described fractional integral derivative. We define the novelized strategy for κ-fractional Hilfer-Katugampola derivative for reverse generalization of Minkowski inequality. The related noteworthy variations in regards to generalized derivatives are illustrated. Numerous variants can be set up for the utilization of a few characterized fractional operators. Veritably, the work built up in the given course of action is new and contributes intriguingly to the investigation of integral fractional differential equations.

6. Conflict of Research

The authors do not have any conflict of research.

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