Punjab University Journal of Mathematics (2022),54(11),689-702
https://doi.org/10.52280/pujm.2022.541103

An optimal eighth-order multipoint numerical iterative method to find simple root of scalar nonlinear equations

Malik Zaka Ullah
Department of Mathematics,
King Abdulaziz University, Jeddah 21589, Saudi Arabia,
Email: zmalek@kau.edu.sa

Received: 21 April , 2019 / Accepted: 30 November, 2022 / Published online: 30 November, 2022

Abstract. An optimal eighth-order multipoint numerical iterative method is constructed to find the simple root of scalar nonlinear equations. It is a three-point numerical iterative method that uses three evaluations of func-tion $f(\cdot)$ associated with a scalar nonlinear equation and one of its deriv-atives f' $(\cdot)$. The four functional evaluations are required to achieve the eighth-order convergence. According to Kung-Traub conjecture (KTC), an iterative numerical multipoint method without memory can achieve maximum order of convergence 2^{n-1} where n is the total number of func-tion evaluations in a single instance of the method. Therefore, following the KTC, the proposed method in this article is optimal.

AMS (MOS) Subject Classification Codes: 65H05; 65H20
Key Words: Scalar nonlinear equations; Simple root; Numerical iterative method; Optimal order of convergence.

1. Preliminaries

This section covers some basic definitions and results.

1.1. **Simple root.** Let

$$f(x) = 0, \tag{1. 1}$$

be a nonlinear equation. A scalar α is said to be root of nonlinear equation (1. 1) if $f(\alpha) = 0$. Further, if $f'(\alpha) \neq 0$, then α is called a simple root. Geometrically, when the graph of the nonlinear function intersects the x-axis, that point is called a simple root, and this situation can be visualized in Figure 1. When the graph of the nonlinear function touches the x-axis, or in other words, the graph of the nonlinear function is tangent to the x-axis at the touching point, then the multiplicity of the root is higher than one. A root with multiplicity > 1 is called a non-simple root (multiple roots). Figure 2 shows the situation when the root has a multiplicity higher than one.

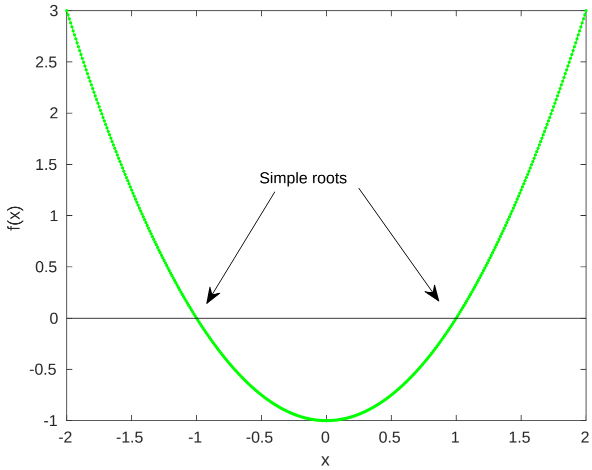

FIGURE 1. Simple roots.

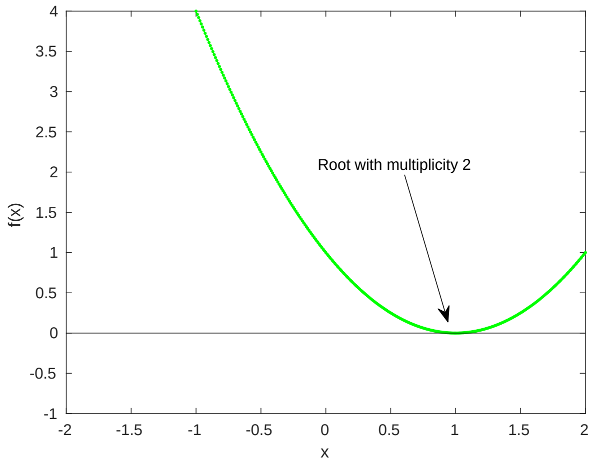

FIGURE 2. Root with multiplicity two.

1.2. **Error at n^{th} step:** Let $x_{n+1} = \phi(x_n)$ be a numerical iterative scheme to solve nonlinear equations. Here x_n is n^{th} approximation to the root. We define the distance of x_n and the root α as the error at n^{th} step by

$$e_n = x_n - \alpha\,.$$

1.3. **Order of convergence:** If the error at $(n+1)^{th}$ step, e_{n+1}, is directly proportional to the m^{th} power of error at n^{th} step e_n, i.e.,

$$e_{n+1} \propto e_n^m$$

then the order of convergence is m.

1.4. **Computational order of convergence:** In Section 1.3, it is observed that the definition of the order of convergence requires the knowledge of the root. We cannot use this definition as we are looking for the root and do not have prior knowledge of it. There are three well-defined computational orders of convergence (COC) as follows:

1- $\mathrm{COC}_1 = \frac{\ln|e_{n+2}/e_{n+1}|}{\ln|e_{n+1}/e_n|}$,
2- $\mathrm{COC}_2 = \frac{\ln|f(x_{n+2})/f(x_{n+1})|}{\ln|f(x_{n+1})/f(x_n)|}$,
3- $\mathrm{COC}_3 = \frac{\ln|(x_{n+3}-x_{n+2})/(x_{n+2}-x_{n+1})|}{\ln|(x_{n+2}-x_{n+1})/(x_{n+1}-x_n)|}$.

Proof of COC$_1$: The expression $e_{n+1} \propto e_n^m$ can be written as $\frac{e_{n+1}}{e_n} \propto e_n^{m-1}$. Similarly, by using easy mathematical manipulation, one can show that

$$\frac{e_{n+2}}{e_{n+1}} \propto e_n^{m(m-1)} .$$

By applying natural logarithm, we get

$$\ln\left|\frac{e_{n+1}}{e_n}\right| \propto (m-1)ln(|e_n|),$$

and

$$\ln\left|\frac{e_{n+2}}{e_{n+1}}\right| \propto m(m-1)ln(|e_n|) .$$

With the help of the above two expressions, we get

$$\mathrm{COC}_1 \propto \frac{m(m-1)ln(|e_n|)}{(m-1)ln(|e_n|)} = m .$$

The above statement proves that the computational order of convergence COC$_1$ is proportional to m.

Proof of COC$_2$: We can expand $f(x_{n+2})$ around α by using Taylor's series as comes next

$$\begin{aligned} f(x_{n+2}) =& f(x_{n+2} - \alpha + \alpha) \\ =& f(e_{n+2} + \alpha) \\ =& f(\alpha) + f'(\alpha)e_{n+2} + O(e_{n+2}^2) \\ f(x_{n+2}) \propto& f'(\alpha)e_{n+2} . \end{aligned} \tag{1. 2}$$

By using the idea of expansion in (1. 2), we can write

$$f(x_n) \propto f'(\alpha)e_n , \tag{1. 3}$$

$$f(x_{n+1}) \propto f'(\alpha)e_{n+1} . \tag{1. 4}$$

Relations (1. 2), (1. 3), and (1. 4) gives

$$\mathrm{COC}_2 \propto \frac{\ln|f(x_{n+2})/f(x_{n+1})|}{\ln|f(x_{n+1})/f(x_n)|},$$

$$\mathrm{COC}_2 \propto \frac{\ln|(f'(\alpha)e_{n+2})/(f'(\alpha)e_{n+1})|}{\ln|(f'(\alpha)e_{n+1})/(f'(\alpha)e_n)|},$$

$$\mathrm{COC}_2 \propto \frac{\ln|e_{n+2}/e_{n+1}|}{\ln|e_{n+1}/e_n|} \propto \mathrm{COC}_1 \propto m .$$

Proof of COC$_3$: We can write

$$\begin{aligned} |x_{n+3}-x_{n+2}| &= |x_{n+3}-\alpha+\alpha-x_{n+2}| \\ &= |e_{n+3}-e_{n+2}| \\ |x_{n+2}-x_{n+1}| &= |e_{n+2}-e_{n+1}| \\ |x_{n+1}-x_n| &= |e_{n+1}-e_n| \ . \end{aligned} \tag{1. 5}$$

We know the relationship

$$\begin{aligned} & e_{n+1} \propto e_n^m \\ & e_{n+2} \propto e_{n+1}^m \propto e_n^{m^2} \\ & e_{n+3} \propto e_{n+2}^m \propto e_n^{m^3} \ . \end{aligned} \tag{1. 6}$$

By using (1. 5) and (1. 6), we get

$$\begin{aligned} \frac{x_{n+3}-x_{n+2}}{x_{n+2}-x_{n+1}} \propto \frac{e_n^{m^3}-e_n^{m^2}}{e_n^{m^2}-e_n^m} & \\ &= \frac{e_n^{m^2-m}-e_n^{m^3-m}}{1-e_n^{m^2-m}} = (1-e_n^{m^2-m})^{-1}(e_n^{m^2-m}-e_n^{m^3-m}) \\ &\propto (1+e_n^{m^2-m})^{-1}(e_n^{m^2-m}-e_n^{m^3-m}) \\ &= e_n^{m^2-m}-e_n^{m^3-m}+e_n^{2m^2-2m}-e_n^{m^2+m^3-2m} \\ &\propto e_n^{m(m-1)} \ . \end{aligned} \tag{1. 7}$$

Similarly, we can write

$$\frac{e_n^{m^2}-e_n^m}{e_n^m-e_n} \propto e_n^{m-1} \ . \tag{1. 8}$$

Relations (1. 7) and (1. 8) give us

$$\text{COC}_3 \propto m \ .$$

Notice that the definitions of computational order COC$_2$ and COC$_3$ do not require the knowledge of the root α.

2. INTRODUCTION

Getting closed-form analytical solutions to a nonlinear problem is only possible in some cases and is also true for systems of nonlinear equations. Polynomials are also nonlinear functions (degree > 1), and it is hard to find the closed form of the analytical solution of the associated nonlinear equation in general. According to Abel-Ruffini theorem [1, 2] (impossibility theorem), "there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients."

Numerical iterative methods are of vital importance in solving nonlinear problems. A large community of researchers is working actively to develop a numerical optimal order method for solving nonlinear equations. According to the KTC [5], an iterative numerical method without memory to solve scalar nonlinear equations can achieve the order of convergence 2^{n-1}, where n is the total number of function evaluations in a single stance of the method. For the two pioneer books related to the paper's subject, we refer to [3, 4]. An optimal order numerical iterative method of order eight is reported in [6].

Several eighth-order optimal methods M$_2$, M$_3$, M$_4$, M$_5$ and M$_6$ that are proposed by Xia et al. [7] as follows:

$$M_1(\phi)=\begin{cases} y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ y_2 = y_1 - \frac{f(y_1)}{f'(x_n)}\, \frac{f(x_n)+bf(y_1)}{f(x_n)+(b-2)f(y_1)} \\ x_{n+1} = y_2 - \frac{f(y_2)}{f'(x_n)} \left(\phi\left(\frac{f(y_1)}{f(x)}\right) + \frac{f(y_2)}{f(y_2)-af(y_2)} + 4\frac{f(y_2)}{f(x_n)}\right). \end{cases}$$

The different value of ϕ's are

$$\begin{aligned} \phi_1(t) &= 12\,t^3 + 5\,t^2 + 2\,t + 1 \\ \phi_2(t) &= \frac{t^2 - 2\,t + 5}{5 - 12\,t} \\ \phi_3(t) &= \left(1 + \frac{t}{1 - 2\,t}\right)^2 \\ \phi_4(t) &= \left(-t^2 - 2\,t + 1\right)^{-1}. \end{aligned}$$

$$M_2=\begin{cases} y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ y_2 = y_1 - \frac{f(x_n)}{f'(x_n)}\, \frac{f(x_n)-f(y_1)}{f(x_n)-2f(y_1)} \\ x_{n+1} = y_2 - \frac{f(y_2)}{f'(x_n)} \left(\frac{1}{2} + \frac{5f(x_n)^2+8f(x_n)f(y_1)+2f(y_1)^2}{5f(x_n)^2-12f(x_n)f(y_1)} \left(\frac{1}{2} + \frac{f(y_2)}{f(y_1)}\right)\right). \end{cases}$$

$$M_3=\begin{cases} y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ y_2 = y_1 - \frac{f(x_n)}{f'(x_n)}\, \frac{f(x_n)-f(y_1)}{f(x_n)-2f(y_1)} \\ x_{n+1} = y_2 - \frac{f(y_2)}{f'(x_n)} \left(\frac{5f(x_n)^2-2f(x_n)f(y_1)+f(y_1)^2}{5f(x_n)^2-12f(x_n)f(y_1)} \left(\frac{1+4\frac{4f(y_1)}{f(x_n)}\, f(y_2)}{f(y_1)}\right)\right). \end{cases}$$

$$M_4=\begin{cases} y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ y_2 = y_1 - \frac{f(x_n)}{f'(x_n)}\, \frac{4f(x_n)^2-5f(x_n)f(y_1)-f(y_1)^2}{4f(x_n)^2-9f(x_n)f(y_1)} \\ x_{n+1} = y_2 - \frac{f(y_2)}{f'(x_n)} \left(1 + 4\frac{f(y_2)}{f(x_n)}\right) \left(\frac{8f(y_1)}{4f(x_n)-11f(y_1)} + 1 + \frac{f(y_2)}{f(y_1)}\right). \end{cases}$$

$$M_5=\begin{cases} y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ y_2 = y_1 - \frac{f(y_1)}{f'(x_n)}\, \frac{-4f(x_n)+f(y_1)}{-4f(x_n)+9f(y_1)} \\ x_{n+1} = y_2 - \frac{f(y_2)}{f'(x_n)}\, \frac{\frac{-f(x_n)+af(y_2)}{f(x_n)+bf(y_2)}}{\frac{4f(x_n)-11f(y_1)}{-4f(x_n)+3f(y_1)} + \frac{f(x_n)+cf(y_1)}{f(x_n)-(a-c+b)f(y_1)}\, \frac{f(y_2)}{f(y_1)}}. \end{cases}$$

$$M_6=\begin{cases} y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ y_2 = y_1 - \frac{f(y_1)}{f'(x_n)}\, \frac{-4f(x_n)+f(y_1)}{-4f(x_n)+9f(y_1)} \\ x_{n+1} = y_2 - \frac{f(y_2)}{f'(x_n)}\, \frac{\frac{4f(x_n)-(3+4a)f(y_2)}{4f(x_n)}}{\frac{-2f(x_n)+(11+2a)f(y_1)}{-4f(x_n)+3f(y_1)} + \frac{2f(x_n)+2af(y_1)}{4f(x_n)-3f(y_1)}\, \frac{f(y_1)-f(y_2)}{f(y_1)+f(y_2)}}. \end{cases}$$

Interested readers can find more details on the related issues in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

3. Proposed multipoint numerical iterative method of optimal order eight

We develop a multipoint numerical iterative method of optimal order eight to find the simple root of nonlinear equations. It is the three-point frozen derivative method. The method MZU_8 utilizes three functional evaluations and one derivative at the initial guess in a single cycle. The total number of function evaluations is four, with the order of convergence $2^{4-1} = 8$. The order of convergence four is optimal according to the Kung-Traub conjecture. We use the idea of weight function to develop the method. A general scheme of weight function can be written as

$$\begin{aligned} x_n &= \text{ initial guess} \\ y_1 &= x_n - f(x_n)/f'(x_n) \\ y_2 &= y_1 - \text{weight function } f(y_1)/f'(x_n) \\ x_{n+1} &= y_2 - \text{weight function } f(y_2)/f'(x_n)\,. \end{aligned}$$

It is a frozen-derivative scheme, as we only evaluate one derivative at the initial point in a single cycle. The weight functions are rational functions in our case that depend on two parameters:

$$\text{MZU}_8 = \begin{cases} x_n = \text{initial guess} \\ y_1 = x_n - \frac{f(x_n)}{f'(x_n)} \\ t_1 = \frac{f(y_1)}{f'(x_n)} \\ p_2 = \frac{{a_1}^3-4\,a_2\,a_1+{a_2}^2+\left(2\,{a_1}^3-{a_1}^2a_2-8\,a_2\,a_1+4\,{a_2}^2\right)t_1+\left({a_1}^4-6\,{a_1}^2a_2+2\,{a_2}^2a_1+4\,{a_2}^2\right){t_1}^2}{{a_1}^3-4\,a_2\,a_1+{a_2}^2+(-{a_1}^2a_2+2\,{a_2}^2)t_1+{a_2}^2a_1\,{t_1}^2-{a_2}^3{t_1}^3} \\ y_2 = y_1 - p_2\,\frac{f(y_1)}{f'(x_n)} \\ t_2 = \frac{f(y_2)}{f(x_n)} \\ t_3 = \frac{f(y_2)}{f(y_1)} \\ p_3 = \frac{q_1}{q_2} + 4\,t_2 + t_3 \\ x_{n+1} = y_2 - p_3\,\frac{f(y_2)}{f'(x_n)}, \end{cases}$$

where

$$\begin{aligned} q_1 =& {a_1}^3 - {a_1}^2 + {a_2}^2 - 5\,a_1 - 12\,a_2 + 33 + \;\; - {a_1}^2a_2 + 6\,{a_1}^2 + 6\,a_2\,a_1 + 4\,{a_2}^2 - 36\,a_1 - 41\,a_2 \\ &+ 102\;\; t_1 + \;\; {a_1}^4 + 2\,{a_1}^2a_2 + 2\,{a_2}^2a_1 - 2\,{a_1}^2 - 4\,a_2\,a_1 + 6\,{a_2}^2 - 24\,a_1 - 54\,a_2 + 121\;\; {t_1}^2, \\ q_2 =& {a_1}^3 - {a_1}^2 + {a_2}^2 - 5\,a_1 - 12\,a_2 + 33 + \;\; - 2\,{a_1}^3 - {a_1}^2a_2 + 8\,{a_1}^2 + 6\,a_2\,a_1 + 2\,{a_2}^2 - 26\,a_1 \\ &- 17\,a_2 + 36\;\; t_1 + \;\; 4\,{a_1}^3 + 4\,{a_1}^2a_2 + {a_2}^2a_1 - 12\,{a_1}^2 - 4\,a_2\,a_1 + {a_2}^2 - 8\,a_2 + 16\;\; {t_1}^2 \\ &+ \;\; - 8\,{a_1}^3 - 12\,{a_1}^2a_2 - 6\,{a_2}^2a_1 - {a_2}^3 + 48\,{a_1}^2 + 48\,a_2\,a_1 + 12\,{a_2}^2 - 96\,a_1 - 48\,a_2 + 64\;\; {t_1}^3\,. \end{aligned}$$

The MZU_8 depends on two parameters, the proper selection of parameters may change the dynamics of the method and provide our higher order of accuracy with the same order of convergence.

Theorem 3.1. *Suppose $f(\cdot)$ has a simple root $\alpha \in D$. If the initial guess x_0 is sufficiently close to α, then the method MZU_8 converges to α with eighth order.*

Proof. Let $e_n = x_n - \alpha$, $c_1 = f'(\alpha)$ and $c_k = \frac{f^{(k)}(\alpha)}{k!\,f'(\alpha)}$ for $k = 2, 3 \cdots$. By expanding $f(x_n)$ around α using Taylor's series, we get
$f(x_n) = f(e_n + \alpha)) = c_1\left(e_n + c_2{e_n}^2 + c_3{e_n}^3 + c_4{e_n}^4 + c_5{e_n}^5 + c_6{e_n}^6 + O\left(e_n^7\right)\right)$.
The derivative of $f(x_n)$ can be computed as

$$f'(x_n) = c_1\left(1 + 2c_2e_n + 3c_3{e_n}^2 + 4c_4{e_n}^3 + 5c_5{e_n}^4 + 6c_6{e_n}^5 + O\left(e_n^6\right)\right)\,.$$

By using $f(x_n)$ and $f'(x_n)$, we can obtain

$y_1 - \alpha = c_2{e_n}^2 + (-2\,{c_2}^2 + 2\,c_3){e_n}^3 + (4\,{c_2}^3 - 7\,c_2c_3 + 3\,c_4){e_n}^4 + (-8\,{c_2}^4 + 20\,{c_2}^2c_3 - 10\,c_2c_4 - 6\,{c_3}^2 + 4\,c_5){e_n}^5 + (16\,{c_2}^5 - 52\,{c_2}^3c_3 + 28\,{c_2}^2c_4 + (33\,{c_3}^2 - 13\,c_5)c_2 - 17\,c_3c_4 +$

$5\,c_6)e_n{}^6+(-32\,c_2{}^6+128\,c_2{}^4c_3-72\,c_2{}^3c_4+(-126\,c_3{}^2+36\,c_5)c_2{}^2+(92\,c_3c_4-16\,c_6)c_2+18\,c_3{}^3-22\,c_3c_5-12\,c_4{}^2)e_n{}^7+(64\,c_2{}^7-304\,c_2{}^5c_3+176\,c_2{}^4c_4+(408\,c_3{}^2-92\,c_5)c_2{}^3+(-348\,c_3c_4+44\,c_6)c_2{}^2+(-135\,c_3{}^3+118\,c_3c_5+64\,c_4{}^2)c_2+75\,c_3{}^2c_4-27\,c_3c_6-31\,c_4c_5)e_n{}^8+O(e_n{}^9)\,.$

The quotient of $f(y_1)$ and $f(x_n)$ gives

$$t_1=c_2e_n+\big(-3\,c_2{}^2+2\,c_3\big)e_n{}^2+\big(8\,c_2{}^3-10\,c_2c_3+3\,c_4\big)e_n{}^3+\big(-20\,c_2{}^4+37\,c_2{}^2c_3-14\,c_2c_4-8\,c_3{}^2+4\,c_5\big)e_n{}^4+\big(48\,c_2{}^5-118\,c_2{}^3c_3+51\,c_2{}^2c_4+\big(55\,c_3{}^2-18\,c_5\big)c_2-22\,c_3c_4+5\,c_6\big)e_n{}^5+\big(-112\,c_2{}^6+344\,c_2{}^4c_3-163\,c_2{}^3c_4+\big(-252\,c_3{}^2+65\,c_5\big)c_2{}^2+\big(150\,c_3c_4-22\,c_6\big)c_2+26\,c_3{}^3-28\,c_3c_5-15\,c_4{}^2\big)e_n{}^6+\big(256\,c_2{}^7-944\,c_2{}^5c_3+480\,c_2{}^4c_4+\big(952\,c_3{}^2-207\,c_5\big)c_2{}^3+\big(-693\,c_3c_4+79\,c_6\big)c_2{}^2+\big(-228\,c_3{}^3+190\,c_3c_5+102\,c_4{}^2\big)c_2+105\,c_3{}^2c_4-34\,c_3c_6-38\,c_4c_5\big)e_n{}^7+O\big(e_n{}^8\big).$$

The expression for p_2 is

$$p_2=1+2\,c_2e_n+\big(\big(a_1-6\big)c_2{}^2+4\,c_3\big)e_n{}^2+\big(\big(-6\,a_1+a_2+16\big)c_2{}^3+4\,c_3\big(a_1-5\big)c_2+6\,c_4\big)e_n{}^3+\big(\big(25\,a_1-9\,a_2-40\big)c_2{}^4-32\,\big(a_1-3/16\,a_2-\tfrac{37}{16}\big)c_3c_2{}^2+\big(6\,a_1\,c_4-28\,c_4\big)c_2+\big(4\,a_1-16\big)c_3{}^2+8\,c_5\big)e_n{}^4+\big(\big(-88\,a_1+51\,a_2+96\big)c_2{}^5+166\,\big(a_1-\tfrac{33\,a_2}{83}-\tfrac{118}{83}\big)c_3c_2{}^3-46\,c_4\big(a_1-\tfrac{9\,a_2}{46}-\tfrac{51}{23}\big)c_2{}^2+\big(\big(-56\,a_1+12\,a_2+110\big)c_3{}^2+8\,a_1\,c_5-36\,c_5\big)c_2+\big(12\,a_1\,c_4-44\,c_4\big)c_3+10\,c_6\big)e_n{}^5+O(e_n^6).$$

Similarly, we evaluated the expression for other terms to get the expansion those are listed below

$$y_2-\alpha=-\big(\big(a_1-5\big)c_2{}^2+c_3\big)c_2e_n{}^4+\big(\big(10\,a_1-a_2-36\big)c_2{}^4+\big(-6\,a_1+32\big)c_3c_2{}^2-2\,c_2c_4-2\,c_3{}^2\big)e_n{}^5+\big(\big(-62\,a_1+13\,a_2+170\big)c_2{}^5+74\,\big(a_1-\tfrac{4\,a_2}{37}-\tfrac{131}{37}\big)c_3c_2{}^3+\big(-9\,a_1\,c_4+48\,c_4\big)c_2{}^2+\big(\big(-12\,a_1+66\big)c_3{}^2-3\,c_5\big)c_2-7\,c_3c_4\big)e_n{}^6+\big(\big(304\,a_1-100\,a_2-660\big)c_2{}^6-544\,\big(a_1-\tfrac{61\,a_2}{272}-\tfrac{45}{17}\big)c_3c_2{}^4+108\,\big(a_1-a_2/9-\tfrac{94}{27}\big)c_4c_2{}^3+\big(\big(204\,a_1-24\,a_2-704\big)c_3{}^2-12\,a_1\,c_5+64\,c_5\big)c_2{}^2+\big(\big(-36\,a_1\,c_4+196\,c_4\big)c_3-4\,c_6\big)c_2+\big(-8\,a_1+44\big)c_3{}^3-10\,c_3c_5-6\,c_4{}^2\big)e_n{}^7+O(e_n^8).$$

$$t_2=-c_2\big(\big(a_2-5\big)c_2{}^2+c_3\big)e_n{}^3+\big(\big(11\,a_2-a_1-41\big)c_2{}^4+\big(-6\,a_2+33\big)c_3c_2{}^2-2\,c_2c_4-2\,c_3{}^2\big)e_n{}^4+\big(\big(-73\,a_2+14\,a_1+211\big)c_2{}^5+81\,\big(a_2-\tfrac{8\,a_1}{81}-\tfrac{100}{27}\big)c_3c_2{}^3+\big(-9\,a_2\,c_4+50\,c_4\big)c_2{}^2+\big(\big(-12\,a_2+69\big)c_3{}^2-3\,c_5\big)c_2-7\,c_3c_4\big)e_n{}^5+\big(\big(377\,a_2-114\,a_1-871\big)c_2{}^6-636\,\big(a_2-\tfrac{131\,a_1}{636}-\tfrac{1781}{636}\big)c_3c_2{}^4+118\,\big(a_2-\tfrac{6\,a_1}{59}-\tfrac{431}{118}\big)c_4c_2{}^3+\big(\big(222\,a_2-24\,a_1-806\big)c_3{}^2-12\,a_2\,c_5+67\,c_5\big)c_2{}^2+\big(\big(-36\,a_2\,c_4+206\,c_4\big)c_3-4\,c_6\big)c_2+\big(-8\,a_2+46\big)c_3{}^3-10\,c_3c_5-6\,c_4{}^2\big)e_n{}^6+O(e_n^7).$$

$$t_3=\big(\big(-a_1+5\big)c_2{}^2-c_3\big)e_n{}^2+\big(\big(8\,a_1-a_2-26\big)c_2{}^3-4\,c_3\big(a_1-5\big)c_2-2\,c_4\big)e_n{}^3+\big(\big(-41\,a_1+11\,a_2+93\big)c_2{}^4+43\,\big(a_1-\tfrac{6\,a_2}{43}-\tfrac{130}{43}\big)c_3c_2{}^2+\big(-6\,a_1\,c_4+29\,c_4\big)c_2+\big(-4\,a_1+19\big)c_3{}^2-3\,c_5\big)e_n{}^4+\big(\big(170\,a_1-73\,a_2-284\big)c_2{}^5-276\,\big(a_1-\tfrac{27\,a_2}{92}-\tfrac{145}{69}\big)c_3c_2{}^3+62\,\big(a_1-\tfrac{9\,a_2}{62}-\tfrac{90}{31}\big)c_4c_2{}^2+\big(\big(76\,a_1-12\,a_2-212\big)c_3{}^2-8\,a_1\,c_5+38\,c_5\big)c_2+\big(-12\,a_1\,c_4+54\,c_4\big)c_3-4\,c_6\big)e_n{}^5+O(e_n^6).$$

$p_3=1+2\,c_2e_n+3\,c_3e_n{}^2+4\,c_4e_n{}^3+\big(\big(10\,a_1-2\,a_2-50\big)c_2{}^4-c_3\big(a_1-20\big)c_2{}^2-c_2c_4-c_3{}^2+5\,c_5\big)e_n{}^4+\big(\big(-108\,a_1+34\,a_2+364\big)c_2{}^5+82\,\big(a_1-\tfrac{17\,a_2}{82}-\tfrac{213}{41}\big)c_3c_2{}^3-2\,c_4\big(a_1-$

$20\big)c_2{}^2+\big(\big(-4\,a_1+70\big)c_3{}^2-2\,c_5\big)c_2-6\,c_3c_4+6\,c_6\big)e_n{}^5+O(e_n^6)$. Summarizing all into the formulation of iterative methods, one obtains the following $e_{n+1}=9\,\big(\big(a_1-5\big)c_2{}^2+c_3\big)c_2\big(\big(a_1-2/9\,a_2-5\big)c_2{}^4-1/9\,c_3\big(a_1-19\big)c_2{}^2-1/9\,c_2c_4-1/9\,c_3{}^2\big)e_n{}^8+O\big(e_n{}^9\big)$ This error equation shows that the order of convergence is eight. □

4. Numerical Simulations

To approximate the computational order of convergence, we use the COC_2 definition in all our numerical computations. To check the validity and numerical accuracy of our proposed numerical iterative method for the computation of simple roots of nonlinear equations, we select five nonlinear functions that are listed in Table 1.

To find the simple roots of five different nonlinear functions, we report five cases of our proposed numerical iterative method MZU_8 namely $\mathrm{MZU}_8(1)$, $\mathrm{MZU}_8(2)$, $\mathrm{MZU}_8(3)$, $\mathrm{MZU}_8(4)$, and $\mathrm{MZU}_8(5)$ for a different selection of parameters a_1 and a_2. To show the numerical accuracy of our method compared to other methods, we present the results in five tables. We select nine methods of optimal order eight namely, $\mathrm{M}_1(\phi_1)$, $\mathrm{M}_1(\phi_2)$, $\mathrm{M}_1(\phi_3)$, $\mathrm{M}_1(\phi_4)$, M_2, M_3, M_4, M_5, and M_6.

These methods are listed in the introduction section. The numerical accuracies that we achieved are 1.12943e-2959 in Table 2, 4.11948e-2405 in Table 3, 2.62365e-9831 in Table 4, 1.34144e-7745 in Table 5, and -9.27570e-10585 in Table 6. In all reported numerical results, our achieved accuracies for five examples are best compared with other reported optimal methods.

Nonlinear functions	Initial guess
$f_1(x)=(t-2)\left(t^{10}+t+1\right)\mathrm{e}^{-t-1}$	2.1
$f_2(x)=\mathrm{e}^{t^2+7\,t-30}-1$	3.1
$f_3(x)=t\mathrm{e}^{t^2}-(\sin(t))^2+3\,\cos(t)+5$	-1.2
$f_4(x)=t^3-10$	2.1
$f_5(x)=(\sin(t))^2-t^2+1$	1.4

TABLE 1. List of nonlinear functions.

$$\mathrm{MZU}_8(1)=\begin{cases} a_1=10,\ a_2=-2.3\\ p_2=\frac{11506960\,{t_1}^2+2435160\,t_1+1097290}{12167\,{t_1}^3+52900\,{t_1}^2+240580\,t_1+1097290}\\ p_3=\frac{-9574740\,{t_1}^2-549460\,t_1-915890}{2571353\,{t_1}^3-2064590\,{t_1}^2+1282320\,t_1-915890}+4\,t_2+t_3,\end{cases}$$

$$\mathrm{MZU}_8(2)=\begin{cases} a_1=10,\ a_2=10\\ p_2=\frac{6400\,{t_1}^2+600\,t_1+700}{-1000\,{t_1}^3+1000\,{t_1}^2-800\,t_1+700}\\ p_3=\frac{13341\,{t_1}^2-68\,t_1+863}{-17576\,{t_1}^3+7436\,{t_1}^2-1794\,t_1+863}+4\,t_2+t_3,\end{cases}$$

$$\mathrm{MZU}_8(4)=\begin{cases} a_1=4,\ a_2=-0.47\\ p_2=\frac{71.7409+151.4436\,t_1+303.7708\,{t_1}^2}{71.7409+7.9618\,t_1+0.8836\,{t_1}^2+0.103823\,{t_1}^3}\\ p_3=\frac{66.8609+70.3936\,t_1+269.9526\,{t_1}^2}{66.8609-63.3282\,t_1+62.3045\,{t_1}^2-43.986977\,{t_1}^3}+4\,t_2+t_3,\end{cases}$$

$$\mathrm{MZU}_8(5)=\begin{cases} a_1=5,\ a_2=1\\ p_2=\frac{489\,{t_1}^2+189\,t_1+106}{-{t_1}^3+5\,{t_1}^2-23\,t_1+106}\\ p_3=\frac{568\,{t_1}^2+40\,t_1+97}{-343\,{t_1}^3+294\,{t_1}^2-154\,t_1+97}+4\,t_2+t_3.\end{cases}$$

Method	i	x_i	$f(x_i)$	COC
$\text{MZU}_8(1)$	0	2.1	7.52812	0
	1	2	-9.46355e-06	0
	2	2	3.24604e-46	6.85769
	3	2	6.21927e-370	8
	4	2	**1.12934e-2959**	8
$\text{M}_1(\phi_1)$	0	2.1	7.52812	0
$a=2$	1	2.00014	0.00734671	0
$b=0$	2	2	2.90483e-24	7.10922
	3	2	1.75272e-195	7.9998
	4	2	**3.07930e-1565**	8
$\text{M}_1(\phi_2)$	0	2.1	7.52812	0
$a=2$	1	2.00006	0.00281328	0
$b=0$	2	2	2.15443e-28	7.32781
	3	2	2.55634e-229	7.99995
	4	2	**1.00442e-1836**	8
	0	2.1	7.52812	0
$\text{M}_1(\phi_3)$	1	2.00006	0.0031786	0
$a=2$	2	2	6.55769e-28	7.31541
$b=0$	3	2	2.15963e-225	7.99994
	4	2	**2.98818e-1805**	
	8			
$\text{M}_1(\phi_4)$	0	2.1	7.52812	0
$a=2$	1	2.00005	0.00275547	0
$b=0$	2	2	1.75798e-28	7.33166
	3	2	4.84013e-230	7.99995
	4	2	**1.59809e-1842**	8
M_2	0	2.1	7.52812	0
	1	2.00002	0.0010339	0
	2	2	4.70842e-33	7.5971
	3	2	8.71195e-268	8
	4	2	**1.19684e-2145**	8
M_3	0	2.1	7.52812	0
	1	2.00006	0.00313171	0
	2	2	5.69684e-28	7.31761
	3	2	6.85395e-226	7.99994
	4	2	**3.00884e-1809**	8
M_4	0	2.1	7.52812	0
	1	2	5.38825e-05	0
	2	2	-5.13931e-44	7.58382
	3	2	-3.52032e-356	8
	4	2	**-1.70610e-2853**	8
M_5	0	2.1	7.52812	0
$a=-1$	1	1.99999	-0.000504053	0
$b=1$	2	2	-6.03183e-36	7.64744
$c=3$	3	2	-2.53612e-291	8
	4	2	**-2.47704e-2334**	8
M_6	0	2.1	7.52812	0
$a=-3$	1	2	4.98230e-05	0
$b=1$	2	2	-3.06204e-44	7.57086
$c=3$	3	2	-6.23273e-358	8
	4	2	**-1.83659e-2867**	8

TABLE 2. Computational order of accuracy of different method for $f_1(x)=0$.

Method	i	x_i	$f(x_i)$	COC
MZU$_8(1)$	0	3.1	2.70617	0
	1	3	-1.90332e-05	0
	2	3	2.50988e-38	6.38092
	3	3	2.29453e-301	8
	4	3	**1.11948e-2405**	8
M$_1(\phi_1)$	0	3.1	2.70617	0
$a=2$	1	3.0017	0.0223765	0
$b=0$	2	3	4.64729e-15	6.0899
	3	3	1.89206e-116	7.99444
	4	3	**1.42830e-927**	8
M$_1(\phi_2)$	0	3.1	2.70617	0
	1	3.00058	0.00763219	0
	2	3	8.14915e-20	6.65627
	3	3	1.43646e-155	7.99891
	4	3	**1.33891e-1241**	8
M$_1(\phi_3)$	0	3.1	2.70617	0
	1	3.00076	0.00999255	0
	2	3	9.02524e-19	6.59529
	3	3	4.22448e-147	7.9985
	4	3	**9.73393e-1174**	8
M$_1(\phi_4)$	0	3.1	2.70617	0
	1	3.00056	0.00732404	0
	2	3	5.43404e-20	6.67144
	3	3	5.19527e-157	7.99898
	4	3	**3.62652e-1253**	8
M$_2$	0	3.1	2.70617	0
	1	3.00018	0.00231844	0
	2	3	-8.42809e-25	6.99001
	3	3	-2.61679e-196	7.99964
	4	3	**-2.25992e-1568**	8
M$_3$	0	3.1	2.70617	0
	1	3.00062	0.00812738	0
	2	3	1.45065e-19	6.63985
	3	3	1.56341e-153	7.99883
	4	3	**2.84551e-1225**	8
M$_4$	0	3.1	2.70617	0
	1	2.99992	-0.000982975	0
	2	3	2.13613e-27	6.87913
	3	3	1.05495e-216	8.00013
	4	3	**3.73316e-1731**	8
M$_5$	0	3.1	2.70617	0
$a=-1$	1	2.99989	-0.00147219	0
$b=1$	2	3	1.50019e-26	7.04322
$c=3$	3	3	1.73081e-210	8.00015
	4	3	**5.43337e-1682**	8
M$_6$	0	3.1	2.70617	0
$a=-3$	1	2.99993	-0.000958999	0
$b=1$	2	3	1.60178e-27	6.89087
$c=3$	3	3	9.63770e-218	8.00012
	4	3	**1.65552e-1739**	8

TABLE 3. Computational order of accuracy of different method for $f_2(x)=0$.

Method	i	x_i	$f(x_i)$	COC
$\mathrm{MZU}_8(3)$	0	-1.2	0.153541	0
	1	-1.20765	9.15747e-19	0
	2	-1.20765	3.49446e-153	7.80393
	3	-1.20765	1.57114e-1228	8
	4	-1.20765	**2.62365e-9831**	8
$\mathrm{M}_1(\phi_1)$	0	-1.2	0.153541	0
$a = 2$	1	-1.20765	-1.38762e-14	0
$b = 0$	2	-1.20765	-5.46154e-119	8.00409
	3	-1.20765	-3.14546e-954	8
	4	-1.20765	**-3.80746e-7636**	8
$\mathrm{M}_1(\phi_2)$	0	-1.2	0.153541	0
	1	-1.20765	-1.39895e-16	0
	2	-1.20765	-6.33946e-137	8.00135
	3	-1.20765	-1.12734e-1099	8
	4	-1.20765	**-1.12741e-8801**	8
$\mathrm{M}_1(\phi_3)$	0	-1.2	0.153541	0
	1	-1.20765	-5.04278e-16	0
	2	-1.20765	-6.37242e-132	8.00207
	3	-1.20765	-4.14365e-1059	8
	4	-1.20765	**-1.32438e-8476**	8
$\mathrm{M}_1(\phi_4)$	0	-1.2	0.153541	0
	1	-1.20765	-4.66641e-17	0
	2	-1.20765	-3.57989e-141	7.99853
	3	-1.20765	-4.29501e-1134	8
	4	-1.20765	**-1.84383e-9077**	8
M_2	0	-1.2	0.153541	0
	1	-1.20765	3.59489e-16	0
	2	-1.20765	2.98503e-133	8.00249
	3	-1.20765	6.74612e-1070	8
	4	-1.20765	**4.59089e-8563**	8
M_3	0	-1.2	0.153541	0
	1	-1.20765	-1.51134e-16	0
	2	-1.20765	-1.27061e-136	8.00136
	3	-1.20765	-3.17107e-1097	8
	4	-1.20765	**-4.77253e-8782**	8
M_4	0	-1.2	0.153541	0
	1	-1.20765	-3.84907e-15	0
	2	-1.20765	-5.37672e-124	8.00352
	3	-1.20765	-7.79496e-995	8
	4	-1.20765	**-1.52120e-7961**	8
M_5	0	-1.2	0.153541	0
$a = -1$	1	-1.20765	3.12409e-16	0
$b = 1$	2	-1.20765	7.82594e-134	8.00471
$c = 3$	3	-1.20765	1.21351e-1074	8
	4	-1.20765	**4.05594e-8601**	8
M_6	0	-1.2	0.153541	0
$a = -3$	1	-1.20765	-3.32418e-15	0
$b = 1$	2	-1.20765	-1.44374e-124	8.00336
$c = 3$	3	-1.20765	-1.82781e-999	8
	4	-1.20765	**-1.20631e-7998**	8

TABLE 4. Computational order of accuracy of different method for $f_3(x) = 0$.

Method	i	x_i	$f(x_i)$	COC
$\text{MZU}_8(4)$	0	2.1	-0.739	0
	1	2.15443	2.00068e-14	0
	2	2.15443	2.48136e-120	7.80591
	3	2.15443	1.38927e-967	8
	4	2.15443	**1.34144e-7745**	8
$\text{M}_1(\phi_1)$	0	2.1	-0.739	0
$a = 2$	1	2.15443	1.53039e-10	0
$b = 0$	2	2.15443	3.23071e-88	8.02115
	3	2.15443	1.27428e-709	8
	4	2.15443	**7.46451e-5681**	8
$\text{M}_1(\phi_2)$	0	2.1	-0.739	0
	1	2.15443	2.53395e-11	0
	2	2.15443	3.32762e-95	8.01557
	3	2.15443	2.94315e-766	8
	4	2.15443	**1.10216e-6134**	8
$\text{M}_1(\phi_3)$	0	2.1	-0.739	0
	1	2.15443	2.85346e-11	0
	2	2.15443	9.67616e-95	8.0157
	3	2.15443	1.69182e-762	8
	4	2.15443	**1.47765e-6104**	8
$\text{M}_1(\phi_4)$	0	2.1	-0.739	0
	1	2.15443	2.44973e-11	0
	2	2.15443	2.46011e-95	8.01545
	3	2.15443	2.54470e-767	8
	4	2.15443	**3.33502e-6143**	8
M_2	0	2.1	-0.739	0
	1	2.15443	1.95413e-12	0
	2	2.15443	4.17716e-105	8.00419
	3	2.15443	1.82092e-846	8
	4	2.15443	**2.37450e-6777**	8
M_3	0	2.1	-0.739	0
	1	2.15443	2.88036e-11	0
	2	2.15443	1.05589e-94	8.01559
	3	2.15443	3.44355e-762	8
	4	2.15443	**4.40650e-6102**	8
M_4	0	2.1	-0.739	0
	1	2.15443	-1.82496e-13	0
	2	2.15443	-1.10692e-114	8.0284
	3	2.15443	-2.02778e-924	8
	4	2.15443	**-2.57190e-7402**	8
M_5	0	2.1	-0.739	0
$a = -1$	1	2.15443	-1.08413e-12	0
$b = 1$	2	2.15443	-1.90881e-107	8.00725
$c = 3$	3	2.15443	-1.76277e-865	8
	4	2.15443	**-9.32552e-6930**	8
M_6	0	2.1	-0.739	0
$a = -3$	1	2.15443	-2.94785e-13	0
$b = 1$	2	2.15443	-1.10147e-112	8.01891
$c = 3$	3	2.15443	-4.18516e-908	8
	4	2.15443	**-1.81813e-7271**	8

TABLE 5. Computational order of accuracy of different method for $f_4(x) = 0$.

Method	i	x_i	$f(x_i)$	COC
$\text{MZU}_8(5)$	0	1.4	0.0111112	0
	1	1.40449	-1.08657e-20	0
	2	1.40449	-2.10275e-165	8.03529
	3	1.40449	-4.13645e-1323	8
	4	1.40449	**-9.27570e-10585**	8
$\text{M}_1(\phi_1)$	0	1.4	0.0111112	0
$a = 2$	1	1.40449	-2.47783e-18	0
$b = 0$	2	1.40449	-1.41221e-143	8.00196
	3	1.40449	-1.57229e-1145	8
	4	1.40449	**-3.71183e-9161**	8
$\text{M}_1(\phi_2)$	0	1.4	0.0111112	0
	1	1.40449	-5.48556e-19	0
	2	1.40449	-1.83070e-149	8.00149
	3	1.40449	-2.81707e-1193	8
	4	1.40449	**-8.85584e-9544**	8
$\text{M}_1(\phi_3)$	0	1.4	0.0111112	0
	1	1.40449	-6.01393e-19	0
	2	1.40449	-4.18675e-149	8.0015
	3	1.40449	-2.31006e-1190	8
	4	1.40449	**-1.98425e-9520**	8
$\text{M}_1(\phi_4)$	0	1.4	0.0111112	0
	1	1.40449	-5.35252e-19	0
	2	1.40449	-1.46820e-149	8.00148
	3	1.40449	-4.70545e-1194	8
	4	1.40449	**-5.23752e-9550**	8
M_2	0	1.4	0.0111112	0
	1	1.40449	-7.94153e-20	0
	2	1.40449	-5.25987e-157	8.0007
	3	1.40449	-1.94778e-1254	8
	4	1.40449	**-6.88741e-10034**	8
M_3	0	1.4	0.0111112	0
	1	1.40449	-6.44813e-19	0
	2	1.40449	-7.84817e-149	8.00148
	3	1.40449	-3.77960e-1188	8
	4	1.40449	**-1.09361e-9502**	8
M_4	0	1.4	0.0111112	0
	1	1.40449	-2.00704e-20	0
	2	1.40449	-2.24817e-162	8.00029
	3	1.40449	-5.57196e-1298	8
	4	1.40449	**-7.93306e-10383**	8
M_5	0	1.5	-0.255004	0
$a = -1$	1	1.40449	1.59377e-09	0
$b = 1$	2	1.40449	8.64289e-75	7.95524
$c = 3$	3	1.40449	6.46435e-597	8
	4	1.40449	**6.33072e-4774**	8
M_6	0	1.4	0.0111112	0
$a = -3$	1	1.40449	-1.22000e-20	0
$b = 1$	2	1.40449	-2.60232e-164	7.99977
$c = 3$	3	1.40449	-1.11527e-1313	8
	4	1.40449	**-1.26919e-10508**	8

TABLE 6. Computational order of accuracy of different method for $f_5(x) = 0$.

5. CONCLUSIONS

This paper retrieves the simple roots of a nonlinear scalar equation with the aid of the weight function and uses the frozen-derivative optimal order numerical iterative method.

The rational function was selected as a weight function that gave the method's accuracy. Numerical results are given to support and uphold the theoretical rate of convergence when solving nonlinear scalar equations.

REFERENCES

[1] R. G. Ayoub, *Paolo Ruffini's Contributions to the Quintic* , Archive for History of Exact Sciences, **22**, No. 3 (1980) 253–277.

[2] A. N. Henrik, *Démonstration de l'impossibilité de la résolution algébrique des èquations générales qui passent le quatrième degré*, Sylow, Ludwig; Lie, Sophus, Œuvres Complètes de Niels Henrik Abel **2**, (1824) 28–33.

[3] H. T. Kung, J. F. Traub, *Optimal order of one-point and multipoint iteration*, Applied Mathematics and Computation **21**, (1974) 643–651.

[4] A. M. Ostrowski, *Solutions of Equations and Systems of Equations*, Academic Press, New York-London, 1966.

[5] J. F. Traub, *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1982.

[6] R. Thukral, M. S. Petković, *A family of three-point methods of optimal order for solving nonlinear equations* , Journal of Computational and Applied Mathematics, **233**, (2010) 2278–2284.

[7] X. Wang, L. Liu, *New eighth-order iterative methods for solving nonlinear equations*, Journal of Computational and Applied Mathematics, **234**, (2010) 1611–1620.

[8] M. Grau, J.L. Diaz-Barrero, *An improvement to Ostrowski root-finding method*, Applied Mathematics and Computation **173**,(2006) 450–456.

[9] R. King, A family of fourth order methods for nonlinear equations, SIAM Journal on Numerical Analysis, **10**, (1973) 876–879.

[10] J. Kou, X. Wang, *Some improvements of Ostrowski's method*, Applied Mathematics Letters, **23**, (2010) 92–96.

[11] L. Liu, X. Wang, *Eighth-order methods with the high-efficiency index for solving nonlinear equations*, Applied Mathematics and Computation, **215**, (2010) 3449–3454.

[12] A. Cordero, J. L. Hueso, E. Martinez, J. R. Torregrosa, *New modifications of Potra-Ptak's method with optimal fourth and eighth-order of convergence*, Journal of Computational and Applied Mathematics, **234**, (2010) 2969–2976.

[13] P. Jarrat, *Some fourth order multipoint iterative methods for solving equations*, Mathematical Computation, **20**, (1966) 434–437.

[14] C. Chun, Y. Ham, *Some sixth-order variants of Ostrowski root-finding methods*, Applied Mathematics and Computation, **193**, (2007) 389–394.

[15] S. Weerakoon, T. G. I. Fernando, *A variant of Newton's method with accelerated third-order convergence*, Applied Mathematics Letters, **13**, No. 8 (2000)87–93.

[16] F. Soleymani, F.W. Khdhr, R.K. Saeed, J. Golzarpoor, *A family of high order iterations for calculating the sign of a matrix*, Mathematical Methods in the Applied Sciences, **43** (2020) 8192-8203.

[17] F.W. Khdhr, F. Soleymani, R.K. Saeed, A. Akgul, *An optimized Steffensen–type iterative method with memory associated with annuity calculation*, The European Physical Journal Plus, Article ID: 146 (2019) 1-12.

[18] A.R. Soheili, M. Amini, F. Soleymani, *A family of Chaplygin–type solvers for Itô stochastic differential equations*, Applied Mathematics and Computation, **340** (2019) 296-304.

[19] A.R. Soheili, F Soleymani, *Iterative methods for nonlinear systems associated with finite difference approach in stochastic differential equations*, Numerical Algorithms, **71** (2016) 89-102.

[20] F. Soleymani, *Two novel classes of two-step optimal methods for all the zeros in an interval*, Afrika Matematika, **25** (2014) 307-321.