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On the Structural Properties and Some Topological Indices of Young-Fibonacci Graphs

Iqra Zaman, FM Bhatti
Department of Mathematics,
Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan.
Email: iqrazamankh@gmail.com, fmbhatti@riphah.edu.pk

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Abstract.: In this paper, we study Young Fibonacci graphs G_n, a special family of graphs that are constructed with the help of integer partitions. Young diagrams are also used in the construction of graphs. The family of graphs is rich in structure. Thus, we investigate various properties of the family of graphs which include degree based structure and topological in-dices. Topological indices like Zagreb Index, Wiener Index, Randic Index and Connective Eccentricity Index of these graphs are computed. We also study the eigenvalues and energy of the graph.

AMS (MOS) Subject Classification Codes: 68R10; 05C30; 05C75
Key Words: Young Fibonacci graphs; Integer Partitions; Young Diagram; Zagreb Index; Randic Index; Eigenvalues; Energy.

1. Introduction

The term **graph** $G(V, E)$ is typically defined as the set of vertices $V = \{v_1, v_2, v_3, ..., v_n\}$ and edges $E = \{e_1, e_2, e_3, ..., e_n\}$. The total number of vertices denotes as n and total number of edges denotes as m. The vertex set V comprises all the vertices $v_i's$ of the graph G and degree of the vertex v_i is simply the total edges which are incident to v_i. Degree of the graph G or $d_{eg}(G)$ is total degrees of all vertices in V, maximum degree denotes as $\Delta(G)$ and minimum degree as $\delta(G)$.The distance between v_i and v_j vertices is measured by the length between the vertices whereas, the path is shortest. Maximum distance between vertex v and all other vertices of the graph is called eccentricity denoted by ε of the vertex v. The minimum eccentricity from all the vertices is called radius r of the graph and the maximum eccentricity from all the vertices is called diameter of the graph and denoted by ω. The adjacency matrix $\mathcal{A}(G)$ is obtained by $\mathcal{A}(G)_{i,j} = \begin{cases} 1, & if\ v_i \sim v_j\ and\ i \neq j \\ 0, & otherwise \end{cases}$

The degree matrix $\mathcal{D}(G)$ is defined as $\mathcal{D}(G)_{i,j} = \begin{cases} d_{eg}(v_{ij}), & i = j \\ 0, & otherwise \end{cases}$, the Laplacian matrix L is defined as $L(G) = \mathcal{D}(G) - \mathcal{A}(G)$. The eigenvalues of any matrix M are the roots λ_i of the characteristic polynomial of $det(\lambda I - M)$. The energy of a graph G with respect to the adjacency matrix with λ_i eigenvalues, is defined in [27] as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The Laplacian Energy $E_l(G)$ with respect to the Laplacian matrix with μ_i eigenvalues [28] is defined as

$$E_l(G) = \sum_{j=1}^{n} |\mu_j - \frac{2m}{n}|$$

.

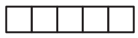
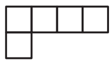
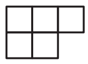
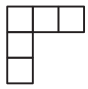
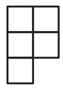
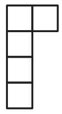
TABLE 1. Some Topological indices where $d_{eg}(v_i)$ is degree of vertex v_i, $\varepsilon(v_i)$ is the eccentricity of vertex v_i

Topological Index	**Formulation**
First Zagreb Index [14]	$M_1(G) = \sum_{v_i} d_{eg}(v_i)^2$
Second Zagreb Index [14]	$M_2(G) = \sum_{v_i \sim v_j} d_{eg}(v_i) d_{eg}(v_j)$
Wiener Index [15]	$W(G) = \frac{\sum_{a=1}^{k} \sum_{b=1}^{k} d(v_a, v_b)}{2}$
Randić Index [17]	$R(G) = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_{eg}(v_i) d_{eg}(v_j)}}$
Connective Eccentricity Index [16]	$C^{\zeta}(G) = \sum_{v_i} \frac{d_{eg}(v_i)}{\varepsilon(v_i)}$

Topological index or molecular descriptor is a numerical number which gives the molecular properties of the graph like boiling point, stability, similarity, chirality and melting point of chemical species [33–35]. There are several topological indices which have been discussed in multiple research articles and successfully applicable in different fields like pharmaceutical sciences, chemical studies. Some of them are discussed in this article. Estrada index was introduced by Ernesto Estrada [22]. For a graph $G(V, E)$, the Estrada index is denoted by $\mathcal{EE}(G)$, defined as $\mathcal{EE}(G) = \sum_{i=1}^{n} e^{\lambda_i}$ where λ_i are the eigenvalues of adjacency matrix of the graph G. The Laplacian Estrada index [23] is denoted by $\mathcal{LEE}(G)$ and defined as $\mathcal{LEE}(G) = \sum_{j=1}^{n} e^{\mu_j}$ where μ_j are the eigenvalues of Laplacian matrix of graph G.

Integer Partitions is an interesting field in Combinatorics, which gives a number of applications in different fields like genetics, statistical mechanics, and modern algebra [3,4,31,32]. A Partition Function P_n [19], is the function counting the number of partitions for a given natural number. A partition is a non increasing sequence $(n_1, n_2, n_3, ..., n_k)$, such that $|P_n| = \sum_{j=1}^{k} n_j = n$ where n_j are the parts of the partition P_n and $P_{-n} = 0$, $P_0 = 1$ (by default). A Young diagram [26] of P_n, of length h, is a collection of empty cells of h rows of left-justified order where row j containing n_j cells for $1 \leq j \leq h$. The Young-Fibonacci graphs G_n [18] are finite ranked graphs with one level for each positive integer *n* constructed using integer partitions and Young diagrams. For $n \in \mathbb{Z}^+$, the vertices at level *n* are precisely the partition of integer *n* and edges are drawn by connecting a partition of integer *n* and a partition of integer $n + 1$ if the Young diagram for the partition of integer $n + 1$ can be obtained by inserting one block into the Young diagram of the partition of integer *n*. The connection between the Young diagram of n and $n - 1$ is defined by the rule that Young diagram of integer n can be obtained by adding a Young tab in the inner corners or at the bottom of the Young diagram of $n - 1$ integer. The levels are drawn as horizontal

.

Partitions	5	4+1	3+2	3+1+1	2+2+1	2+1+1+1	1+1+1+1+1
Young Diagram							

rows with levels increasing vertically upward and each Young diagram is replaced by a vertex in the graph, see Figure 1.

Definition 1.1. *The Young Fibonacci graph G_n is a simple undirected graph consists of total number of partitions up to n as a set of vertices and the set of edges as the connection between the Young diagrams of integer n and integer $n-1$.*

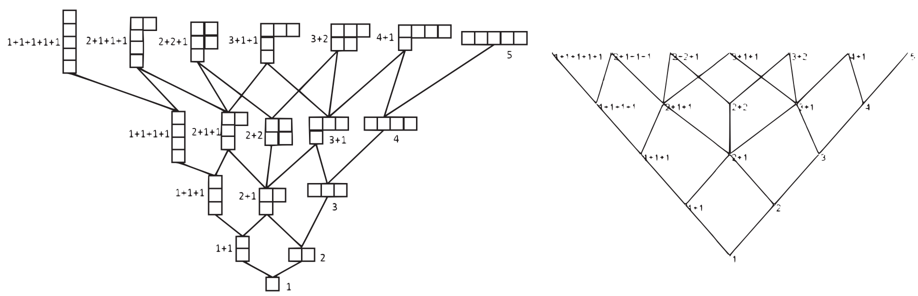


FIGURE 1. Young Diagram and corresponding Young-Fibonacci graph G_5

In this paper, we discuss further properties of Young Fibonacci graphs G_n [18]. We need the following results which have been given in [**?**, 24], for our later use in this paper.

Theorem 1.2. *For the graph G_n, the total number of vertices $N(G_n)$ is given by*

$$N(G_n) = \sum_{i=1}^{n} P_i \ , \ n \in \mathbb{Z}^+ \tag{1. 1}$$

where P_i is the number of partitions of integer i.

Theorem 1.3. *Let $m(G_n)$ is the total number of edges of the graph G_n, then*

$$m(G_n) = (n-1)P_0 + \sum_{i=1}^{n-1}(n-i)P_i \tag{1. 2}$$

where $n \geq 1$ and $P_0 = 1$.

We need the following results which have been given in the literature [6–11], [13, 20] for our later use.

$$\mathcal{E}\mathcal{E}(G) \geq a_n + pe^{\frac{E(G)}{2p}} + qe^{\frac{-E(G)}{2p}} \tag{1. 3}$$

where p, q, a_n and $E(G)$ are the positive, negative, zero eigenvalues and energy respectively with respect to $\mathcal{A}(G)$.

$$(n-1)^2 \leq W(T) \leq \frac{1}{6}n(n-1)(n+1) \tag{1. 4}$$

where $W(T)$ is the Wiener Index of tree T .

$$W(G) \geq n^2 - n - m + 1 \tag{1. 5}$$

with $\omega \geq 3$. Further, the equality holds if G has exactly two vertices with eccentricity 3 and rest are of eccentricity 2.

$$W(G) \geq \frac{\omega(\omega-1)(\omega-2)}{6} + n(n-1) - m \tag{1. 6}$$

$$and \quad W(G) \leq \frac{\omega(n-1)n}{2} - \frac{\omega(\omega-1)(\omega-2)}{3} - m(\omega-1) \tag{1. 7}$$

$$W(G) \leq r(n-r)\left[\frac{r(r-1)}{n-1} + n - r\right] \tag{1. 8}$$

equality holds if and only if $G \cong K_{1,n-1}$.

$$\sum_{i=0}^{k} d_i^2 \leq m\left(\frac{2m}{n-1} + n - 2\right) \tag{1. 9}$$

$$\frac{2m}{\omega} \leq C^{\zeta} \leq \frac{2m}{r} \tag{1. 10}$$

with equality if and only if the eccentricities of all the vertices are same.

2. Main Results

Theorem 2.1. *Let G_n be the Young Fibonacci graph then*
(a) the maximum degree of the graph G_{n+1}, denoted by $\Delta(G_{n+1})$ for $n \geq 3$ appears on the vertex having the part of non-increasing sequence of consecutive integers i-e $(n, n-1, n-2, ..., 2, 1)$.
(b) Let $a(n)$ denotes the n^{th} term of maximum degree's sequence for $n \geq 2$ then

$$a(n) = 2n - 3 + \left\lfloor \frac{n}{2} \right\rfloor$$

Proof of (a): To prove this we use direct combinatorial proof. The Young diagram for any consecutive decreasing integer part is given in Figure 2(a). One of the Young diagrams of the integer parts of *n* other than the consecutive non-increasing integer part is given in Figure 2(b). In the diagram in Figure 2(a), exactly *l+1* cells can be added to make a Young diagram of integer parts of *(n+1)* integer in the graph and exactly *l* cells can be deleted to connect the integer parts of previous *(n-1)* integer in the graph. But in Figure 2(b) the number of cells that can be added is less than *(l+1)* and the number of cells that can be

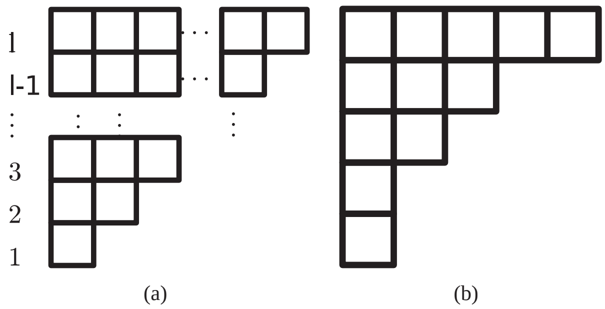


FIGURE 2. (a) Young Diagram of integer part *l+(l-1)+(l-2)+...+4+3+2+1*, (b) Young Diagram of other integer part

deleted is also less than *l*. It can be easily seen that the number of added and deleted cells in Figure 2(a) are more than the diagram in Figure 2(b). So that Δ will be at the vertex where the integer part of integer *n* is consecutive and non-increasing. Now, it is evident that the diagram of the maximum degree vertex will connect to the diagrams of integers $n+1$ to complete the graph if it is on the integer part of n. The graph of maximum degree will be G_{n+1}.
(b) This statement is proved by mathematical induction.
For $n = 2$; $\Delta(2) = 2(2) - 3 + \left\lfloor \frac{2}{2} \right\rfloor = 2$ which is trivial to see in G_2, the maximum degree is 2. Now we recall the result's proof of (a), where it has been shown that maximum degree vertex has *(l+1)* edges, which are connecting to the next integer parts and *l* edges are connecting to previous integer parts. The degree of a vertex of any graph is the number of edges incident to it. So the total number of edges are $(l+1)+l = 2l+1$, where $l \geq 1$. For $n \geq 3$, we have to show that $2l+1 = 2n-3$. $\because$ As $\left\lfloor \frac{2}{n} \right\rfloor = 0$, for $n \geq 3$.
For $n = l+2$, the right hand side becomes $2l+1$, which is sufficient to show. □
Furthermore, it is readily seen that the radius $r(G_n)$ is calculated as $r(G_n) = n-1$ and the diameter $\omega(G_n)$ is calculated as

$$\omega(G_n) = 2(n-1) \tag{2. 11}$$

$$\omega(G_n) = 2r(G_n) \tag{2. 12}$$

In the following, we investigate the topological indices for the graph G_n by using the formulas given in the Table 1, we have converted all the results in the form of P_n.

Theorem 2.2. *Let G_n be the Young Fibonacci graph and $W(G_n)$ be the Wiener Index of G_n then*

$$W(G_n) \leq \frac{\sum_{i=1}^{n} P_i \left(\left(\sum_{i=1}^{n} P_i\right)^2 - 1 \right)}{6}$$

and

$$W(G_n) \geq \left(\sum_{i=1}^{n} P_i \right)^2 - \sum_{i=1}^{n} (n-i+1) P_i - n + 2$$

Equality holds for G_1 and G_2.

Proof This can be easily observe that for any tree T of connected graph G, we have $W(G) \leq W(T)$ and using equations (1. 1) and (1. 4) for graph G_n, we obtain

$$W(G_n) \leq \frac{\left(\sum_{i=1}^{n} P_i\right)\left(\sum_{i=1}^{n} P_i - 1\right)\left(\sum_{i=1}^{n} P_i + 1\right)}{6}$$

$$W(G_n) \leq \frac{\sum_{i=1}^{n} P_i \left(\left(\sum_{i=1}^{n} P_i\right)^2 - 1\right)}{6}$$

Now for lower bound, using equation (1. 5) for graph G_n, we get

$$W(G_n) \geq (N(G_n))^2 - N(G_n) - m(G_n) + 1$$

Using equations (1. 1) and (1. 2),we get

$$W(G_n) \geq \left(\sum_{i=1}^{n} P_i\right)^2 - \sum_{i=1}^{n} P_i - (n-1)P_0 - \sum_{i=1}^{n-1}(n-i)P_i + 1$$

$$W(G_n) \geq \left(\sum_{i=1}^{n} P_i\right)^2 - \sum_{i=1}^{n}(n+1-i)P_i - n + 2$$

This completes the proof. □

2.3. **Bounds for Wiener Index in the form of radius and sum of partitions of integer P_i.**

Theorem 2.4. *Let G_n be a graph and $W(G_n)$ be the wiener index of G_n then*

$$W(G_n) \leq r\left(\frac{r(r-1)}{\sum_{i=1}^{n} P_i - 1} + \sum_{i=1}^{n} P_i - r\right)\left(\sum_{i=1}^{n} P_i - r\right)$$

Equality holds for G_1 and G_2.

$$W(G_n) \geq \frac{2r}{3}(r-1)(2r-1) + \left(\sum_{i=1}^{n} P_i\right)^2 - \sum_{i=1}^{n}(n+1-i)P_i - n + 1$$

where r is the radius of graph G_n.

Proof Using equations (1. 1) and (1. 8), we have

$$W(G_n) \leq r\left(\sum_{i=1}^{n} P_i - r\right)\left[\sum_{i=1}^{n} P_i - r + \frac{r(r-1)}{\sum_{i=1}^{n} P_i - 1}\right]$$

Now for lower bound we use equation (1. 6) for graph G_n, as here $\omega = \omega(G_n)$ and $r = r(G_n)$ are diameter and radius of G_n respectively and for G_n, $\omega(G_n) = 2r(G_n)$, we have

$$W(G_n) \geq \frac{1}{6}(2r-2)(2r-1)2r + n(n-1) - m$$

by using equations (1. 1) and (1. 2), we get

$$W(G_n) \geq \frac{2}{3}(r-1)(2r-1)r + \sum_{i=1}^{n} P_i \left(\sum_{i=1}^{n} P_i - 1\right) - (n-1)P_0 - \sum_{i=1}^{n-1}(n-i)P_i$$

$$W(G_n) \geq \frac{2}{3}(r-1)(2r-1)r + \left(\sum_{i=1}^{n} P_i\right)^2 - \sum_{i=1}^{n} P_i - n + 1 - \sum_{i=1}^{n}(n-i)P_i$$

$$W(G_n) \geq \frac{2r}{3}(r-1)(2r-1) + \left(\sum_{i=1}^{n} P_i\right)^2 - \sum_{i=1}^{n}(n+1-i)P_i - n + 1$$

This completes the proof. □

Theorem 2.5. *Let G_n be the graph, m is number of edges and d_i is degree of vertex v_i of the graph G_n then*

$$M_1(G_n) = (2m)^2 - 2\sum_{i<j} d_i d_j$$

Proof Let $M_1(G_n) = \sum_{i=1}^{n} F(d_i)$, where $F(d_i)$ is a function of d_i. Let

$$F(d_i) = F(x_i) \tag{2. 13}$$

$$M_1(G_n) = \sum_{i=1}^{n} x_i^2 \tag{2. 14}$$

It is readily seen that $\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\sum_{i<j} x_i x_j$, equation (2. 14) implies

$$M_1(G_n) = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\sum_{i<j} x_i x_j$$

By using equation (2. 13), we get $M_1(G_n) = (\sum_{i=1}^{n} d_i)^2 - 2\sum_{i<j} d_i d_j$, for graph G_n, $\sum_{i=1}^{n} d_i = 2m$, implies

$$M_1(G_n) = (2m)^2 - 2\sum_{i<j} d_i d_j$$

This completes the proof. □

In the following, we discuss the bounds for first Zagreb Index $M_1(G_n)$.

Theorem 2.6. *Let G_n be the graph then*

$$0 \leq M_1(G_n) \leq (n+\alpha-1)\left(2(n+\alpha) + \left(\sum_{i=1}^{n} P_i - 3\right)\sum_{i=1}^{n} P_i\right)$$

where $\alpha = \sum_{i=1}^{n-1}(n-i)P_i$ and equality holds for G_1 and G_2.

Proof As $M_1(G_n) = \sum_{i=0}^{n} d_i^2$, it is clear that $M_1(G_n) \geq 0$. For upper bound we use equation (1. 9) for G_n then by using equations (1. 1) and (1. 2), we get

$$M_1(G_n) \leq \left((n-1) + \sum_{i=1}^{n-1}(n-i)P_i\right)\left(2(n-1) + 2\sum_{i=1}^{n-1}(n-i)P_i + \left(\sum_{i=1}^{n} P_i\right)^2 - 3\sum_{i=1}^{n} P_i + 2\right)$$

Put $\alpha = \sum_{i=1}^{n-1}(n-i)P_i$, after simplify we get,

$$M_1(G_n) \leq (n+\alpha-1)\left(2(n+\alpha) + \left(\sum_{i=1}^{n} P_i - 3\right)\sum_{i=1}^{n} P_i\right)$$

□

In the following, we give the bounds for the Connective Eccentricity Index C^{ζ}.

Theorem 2.7. *Let G_n be the graph then*

$$1+\frac{\alpha}{n-1} \leq C^{\zeta}(G_n) \leq 2\left(1+\frac{\alpha}{n-1}\right)$$

where $\alpha = \sum_{i=1}^{n-1}(n-i)P_i$, and the equality holds for G_1 only.

Proof Using equations (1. 2) and (1. 10), we have

$$\frac{2\left((n-1)P_0 - \sum_{i=1}^{n-1}(n-i)P_i\right)}{d} \leq C^{\zeta}(G_n) \leq \frac{2\left((n-1)P_0 - \sum_{i=1}^{n-1}(n-i)P_i\right)}{r}$$

Now by equations (2. 11) and (2. 12), we obtain

$$\frac{2\left((n-1) - \sum_{i=1}^{n-1}(n-i)P_i\right)}{2(n-1)} \leq C^{\zeta}(G_n) \leq \frac{2\left((n-1) - \sum_{i=1}^{n-1}(n-i)P_i\right)}{n-1}$$

by putting $\alpha = \sum_{i=1}^{n-1}(n-i)P_i$, we get

$$1+\frac{\alpha}{n-1} \leq C^{\zeta}(G_n) \leq 2\left(1+\frac{\alpha}{n-1}\right)$$

This completes the proof. □

Here we give a conjucture for Randic Index, $R(G_n)$.

Conjecture 2.8. *Let G_n be the graph, $R(G_n)$ be the Randic Index then we have*

$$\left\lfloor \frac{R(G_n)}{2} \right\rfloor + 1 = \sum_{i=1}^{\infty} \frac{x^i\left(1-x^{\left(i\lfloor \frac{i}{2}\rfloor\right)}\right)}{\prod_{j=1}^{i}(1-x^j)}$$

Here, we give an example for the above conjecture. For G_5 given in Figure 1, we have Randic index $R(G_5) = 8.5258846$ and

$$\left\lfloor \frac{R(G_5)}{2} \right\rfloor + 1 = 5$$

Now if we see the generating function

$$\sum_{i=1}^{\infty} \frac{x^i\left(1-x^{\left(i\lfloor \frac{i}{2}\rfloor\right)}\right)}{\prod_{j=1}^{i}(1-x^j)}$$

gives the series

$$= 1x^2 + 2x^3 + 3x^4 + 5x^5 + 7x^6 + 11x^7 + 16x^8 + 23x^9 + ...$$

here the coefficient of x^5 gives the value of required result for G_5, which is 5.

Young Diagrams of integer parts of 4	**Number of 1's in integer parts**	**Conjugates of integer parts of 4**	**Number of 1's in integer parts**
4	0	1+1+1+1	4
3+1	1	2+1+1	2
2+2	0	2+2	0
2+1+1	2	3+1	1
1+1+1+1	4	4	0

TABLE 2. Number of 1's, integer parts 4 and $3+1$ have fewer 1's than their conjugates $1+1+1+1$ and $2+1+1$ respectively.

2.9. **Spectral Properties of Young Fibonacci graphs.** In the following, we have some results related to the spectral properties of G_n. We compute some eigenvalues based results which are discussed in the form of theorems.
Before moving to the results, let us discuss some terminologies first. Let we have integer $n = 4$ then we have integer partition of 4:$\{4, 3+1, 2+2, 2+1+1, 1+1+1+1\}$. There is a generating function to find number of parts of integer partition P_{n+1} for $n \geq 1$, such that P_{n+1} contains fewer 1's than its conjugate that is

$$\left(-1 + \frac{1}{\prod_{k>0}(1-x^k)}\right) \cdot \frac{x}{1+x} \tag{2. 15}$$

In which the coefficient terms of x^n indicate the value of number of parts of integer partition P_{n+1} for $n \geq 1$ such that P_{n+1} contains fewer 1's than its conjugate. We determine some results related to the zero eigenvalues of G_n.

Remark 2.10. *Let a_n denotes the number of zero eigenvalues of G_n then the recurrence relation to obtain a_n is*

$$a_n = P_n - a_{n-1} \tag{2. 16}$$

for $a_1 = 1$, and P_n is number of integer partition of integer n for G_n.

We give the following result to find a_n for graph G_n with the help of integer partitions P_n.

Theorem 2.11. *Let a_n denotes the number of zero eigenvalues of $G_n, n \geq 1$ then*

$$a_n = \begin{cases} \sum_{i=1}^{n} (-1)^i P_i; & when\ n\ is\ even \\ \sum_{i=1}^{n} (-1)^{i+1} P_i; & otherwise \end{cases}$$

Proof
We use induction to prove this theorem.
Case1: When n is even, $n \geq 2$: for $n = 2$,

$$a_2 = \sum_{i=1}^{2} (-1)^i P_i$$

we have $a_2 = 1$. Now for induction step, let us suppose that the result is true for $n = k$.

$$a_k = \sum_{i=1}^{k} (-1)^i P_i$$

Now we have to show that the result is true for $n = k + 2$, for this we need to add to both sides

$$(-1)^{k+1} P_{k+1} + (-1)^{k+2} P_{k+2}$$

we get

$$a_k + (-1)^{k+1} P_{k+1} + (-1)^{k+2} P_{k+2} = \sum_{i=1}^{k} (-1)^i P_i + (-1)^{k+1} P_{k+1} + (-1)^{k+2} P_{k+2}$$

Implies

$$a_k - P_{k+1} + P_{k+2} = \sum_{i=1}^{k+2} (-1)^i P_i \tag{2. 17}$$

Now using equation (2. 16), we have

$$P_{k+1} = a_{k+1} + a_k \tag{2. 18}$$
$$P_{k+2} = a_{k+2} + a_{k+1} \tag{2. 19}$$

Now using Equations (2. 18) and (2. 19) in Equation (2. 17) , we get

$$a_{k+2} = \sum_{i=1}^{k+2} (-1)^i P_i$$

Case2: When n is odd, $n \geq 1$: let for $k + 1$ being an odd integer, Let us assume that the result is true for $n = k + 1$, that is

$$a_{k+1} = \sum_{i=1}^{k+1} (-1)^{i+1} P_i$$

Now we have to show that the result is true for $n = k + 3$, for this add to both sides

$$(-1)^{k+2+1} P_{k+2} + (-1)^{k+3+1} P_{k+3}$$

we get

$$a_{k+1} + (-1)^{k+2+1} P_{k+2} + (-1)^{k+3+1} P_{k+3} = \sum_{i=1}^{k+1} (-1)^{i+1} P_i + (-1)^{k+2+1} P_{k+2} + (-1)^{k+3+1} P_{k+3}$$

implies

$$a_k - P_{k+2} + P_{k+3} = \sum_{i=1}^{k+3} (-1)^{i+1} P_i \tag{2. 20}$$

Now using equation (2. 16), We have

$$P_{k+2} = a_{k+2} + a_{k+1} \tag{2. 21}$$
$$P_{k+3} = a_{k+3} + a_{k+2} \tag{2. 22}$$

Now using equation (2. 21) and (2. 22) in equation (2. 20) , we get

$$a_{k+3} = \sum_{i=1}^{k+3} (-1)^{i+1} P_i$$

which completes the proof. □

Theorem 2.12. *The number of zero eigenvalues of graph G_n is exactly same as the number of partitions P_{n+1} that contains fewer $1's$ than its conjugate.*

Proof First recall the result which is given in equation (2. 15). The generating function for the number of partition P_{n+1}, that contains fewer $1's$ than its conjugate is

$$\left(-1 + \frac{1}{\prod_{k>0}(1-x^k)}\right) . \frac{x}{1+x}$$

expanding this

$$= \frac{-x}{1+x} + \frac{x}{1+x}\left(\frac{1}{\prod_{k>0}(1-x^k)}\right)$$

$= -x(1+x)^{-1} + x(1+x)^{-1}(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}...$
$= -x(1-x+x^2-x^3+x^4-...) + x(1-x+x^2-x^3+x^4-...)(1+x+x^2+x^4+x^6+...)(1+x^3+x^6+x^9+x^{12}+...)...$
$= (-x+x^2-x^3+x^4-x^5+...) + (x-x^2+x^3-x^4+...)(1+x+x^2+x^4+x^6+...)(1+x^3+x^6+x^9+x^{12}+...)...$
If we simplify and calculate the coefficients i.e for the coefficient of x^2, we will find $1.x^2 + 1.x^2 - 1.x^2$. The coefficient of x^2 is 1 exactly same as the number of zero eigenvalues of graph G_1. In the same way if we check the coefficient of x^5 that is 3 and the number of zero eigenvalues of graph G_4 is 3, and the term that has the $(n+1)^{th}$ power in the expansion of equation is obtained by selecting x^{1a_1} from the first factor, x^{2a_2} from the second factor and so on, where

$$1a_1 + 2a_2 + 3a_3 + ... = n + 1$$

Since there is one to one correspondence between the number of times the term x^{n+1} is obtained in the sequence and the number of zero eigenvalues of graph G_n. So the coefficient of x^{n+1} will represent the number of zero eigenvalues of graph G_n. □
In the following, we obtain bounds for the energy of graph G_n in terms of λ_1 which is the largest eigenvalue of the graph and integer partitions function P_n.

Theorem 2.13. *Let G_n be the graph then*

$$E(G_n) \leq \lambda_1 + \sqrt{\left(\sum_{i=2}^{n} P_i\right)\left(2n - 2 + 2\sum_{i=1}^{n-1}(n-i)P_i - \lambda_1^2\right)}$$

the equality holds only for G_1.

Proof As in [29], the bounds for energy of bipartite graphs is discussed in detail, for any bipartite graph G with m edges and n vertices

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left(2m - \left(\frac{2m}{n}\right)^2\right)} \tag{2. 23}$$

Since G_n is also a bipartite graph and $\lambda_1 \geq \frac{2m}{n}$, then

$$E(G_n) \leq \lambda_1 + \sqrt{(n-1)(2m - \lambda_1^2)}$$

using (1. 1) and (1. 2) in above inequality,

$$E(G_n) \leq \lambda_1 + \sqrt{\left(\sum_{i=2}^{n} P_i\right)\left(2n - 2 + 2\sum_{i=1}^{n-1}(n-i)P_i - \lambda_1^2\right)}$$

□

Furthermore, we obtain bounds for the energy of graph G_n in terms of λ_1 which is the largest eigenvalue of the graph, integer partition function P_n and and Energy with respect to Laplacian matrix.

Theorem 2.14. *Let G_n be the graph then*

$$E(G_n) \leq \lambda_1 + \sqrt{\left(\sum_{i=2}^{n} P_i\right)(E_l(G_n) - \lambda_1^2)}$$

Equality holds for G_1 and $E_l(G_n)$ is the energy with respect to Laplacian matrix of G_n.

Proof To prove this inequality we will use the inequality

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)(2m - (\frac{2m}{n})^2)} \tag{2. 24}$$

As $\lambda_1 \geq \frac{2m}{n}$ and $n = \sum_{i=1}^{n} P_i$ implies $n - 1 = \sum_{i=2}^{n} P_i$ and for G_n, the energy with respect to Laplacian matrix E_l is twice the number of edges of G_n, that is

$$E_l(G_n) = 2m(G_n)$$

using these facts we get

$$E(G_n) \leq \lambda_1 + \sqrt{(\sum_{i=2}^{n} P_i))(E_l(G_n) - \lambda_1)^2}$$

this proves the inequality. □

We will discuss some topological indices related results of G_n.

Theorem 2.15. *Let G_n be the graph then*

$$\mathcal{E}\mathcal{E}(G_n) \geq a_n + \left(\sum_{i=1}^{n} P_i - a_n\right) Cosh\left(\frac{E(G_n)}{\sum_{i=1}^{n} P_i - a_n}\right)$$

where a_n is number of zero eigenvalues, $E(G_n)$ is energy of G_n.

Proof As G_n is bipartite graph so number of positive eigenvalues p and negative eigenvalues q are equal, so that $p = q$, using (1. 3) for G_n, we have

$$\mathcal{E}\mathcal{E}(G_n) \geq a_n + p\left(e^{\frac{E(G)}{2p}} + e^{\frac{-E(G)}{2p}}\right)$$

$$\geq a_n + 2p\left(cosh\left(\frac{E(G)}{2p}\right)\right)$$

as$N(G_n) = a_n + p + q$ so that $N(G_n) = a_n + 2p$, this implies

$$\mathcal{E}\mathcal{E}(G_n) \geq a_n + \left(\sum_{i=1}^{n} P_i - a_n\right) cosh\left(\frac{E(G)}{\sum_{i=1}^{n} P_i - a_n}\right)$$

□

Theorem 2.16. *For graph G_n, the Laplacian Estrada index is*

$$\mathcal{L}\mathcal{E}\mathcal{E}(G_n) = \sum_{i=0}^{n}(1 - n + i)P_i + e^2\mathcal{E}\mathcal{E}(L(G_n))$$

where $\mathcal{E}\mathcal{E}(L(G_n))$ =Estrada index of line graph.

Proof By using bound of Laplacian Estrada index for any graph from the article [30] for G_n, we get

$$\mathcal{L}\mathcal{E}\mathcal{E}(G_n) = N(G_n) - m(G_n) + e^2\mathcal{E}\mathcal{E}(L(G_n))$$

$$\mathcal{L}\mathcal{E}\mathcal{E}(G_n) = \sum_{i=1}^{n} P_i - (n-1)P_0 - \sum_{i=1}^{n-1}(n-i)P_i + e^2\mathcal{E}\mathcal{E}(L(G_n))$$

On simplification, we get

$$\mathcal{L}\mathcal{E}\mathcal{E}(G_n) = \sum_{i=0}^{n}(1 - n + i)P_i + e^2\mathcal{E}\mathcal{E}(L(G_n))$$

□

3. Conclusion

In this paper, we perceive results to construct Young Fibonacci graphs G_n, using concepts of Integer partitions. We also identify the vertices of the graphs which have the maximum degree. The radius and diameter of the graph G_n are calculated. Topological indices are also calculated for G_n and particularly Zagreb Index. Some results for the energy of the graphs are constructed. Moreover, results on eigenvalues and Laplacian Estrada index are also given.

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