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Core Fundamental Groupoid Bundle, Its Sections and Relatedness of Homeomorphisms and Sections

Chidanand Badiger and T.Venkatesh
[1,2]Department of Mathematics,
Rani Channamma University, Belagavi-591 156, Karnataka, INDIA
Email: [1]chidanandbadiger1@gmail.com and [2]tmathvenky@yahoo.co.in

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Abstract.: In this paper, we introduce bundle, fibre bundle and principal G-bundle structures on the Core fundamental groupoid keeping its standard projections and quotient topology intact. We give an explicit description of Core fundamental groupoids as such bundles including for the uniquely geodesic spaces and formulate some results on bundle maps. Further, we introduce sections on the Core fundamental groupoid bundle, and also, present some basic properties including composition and inverse with the help of the induced groupoid homomorphisms on the Core fundamental groupoids. With a group structure on the set of all continuous sections of the Core fundamental groupoid bundle, a group action has been built on the Core fundamental groupoid. A notion of relatedness of homeomorphism and section are defined and discussed the pushforward, pullback of sections and their properties. Finally, we investigate more about relatedness notions $rel_{\Gamma^0(\bar{\pi}_1 M)}(f), rel_{Homeo(M)}(X)$ and same on the subsets based on both section related homeomorphisms and homeomorphism related sections. Further, some consequences based on an algebraic structure on the new class of $rel_{\Gamma^0(\bar{\pi}_1 M)}(f), rel_{Homeo(M)}(X)$, etc. have been placed. We present an interrelationship between subsets of $Homeo(M)$ and $\Gamma^0(\bar{\pi}_1 M)$, which have nice applications in the left-invariant sections and topological groups.

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1. Introduction

The theory of bundles as a branch of algebraic topology and geometry has a prominent role in multi-purpose objectives of both geometry and topology. The fibre bundles and vector bundles are heavily used in quantum mechanics, the theory of relativity under the

Riemannian, sub-Riemannian setting. Bloch vector bundle $E(T)$ over the Brillouin zone T can be addressed for instance, where the fibres are the spaces of states with the same Bloch momentum k [17]. The theory of fibre bundle has similar applications as vector bundles, it plays a big role in geometry and physics. Developments in the theory of bundles have taken a nice part in the classification theory, as classification of spaces developed due to a lot of invariants like Euler characteristics, homotopy, the fundamental group, homology, cohomology groups, etc.

In the decade of 1930, bundle theory grew tremendously and remarkably influenced all branches of topology and geometry. For the first time in 1933, the terms fibre and fibre space appeared in the paper of Herbert Seifert [8]. In 1935, Hassler Whitney [9] gave the first definition of fibre space under the name sphere space, but later in 1940, he changed it to sphere bundle [10]. W. S. Massey mentioned in [23] about the conference on fibre bundles and differential geometry, which was held at Cornell University from May 3 to May 7, 1953. In the same paper, he discussed the developments and various definitions of different mathematicians, as well as the profound research work of participants on the theory of bundles existing at that time. Besides, in the paper, one can find different definitions of fibre bundle and they are: fibre bundle in the American sense, fibre space in the sense of Ehresmann and Feldbau, fibre space in the sense of Hurewicz and Steenrod, fibre space in the sense of Serre, and also mentioned about the locally trivial fibre space as defined by the French school. Sophistication and necessity objectives of the study have been taken into the part of the existing recent definitions of bundle, fibre bundle, principal G-bundle and vector bundle. A more general definition of bundle indeed appears in category theory. Covering projection is a kind of fibre bundle with discrete fibre, and fibrations are also the same.

Like the Hairy ball theorem on the non-existence of nowhere vanishing continuous vector field on the even-dimensional sphere, there are many issues in the theory of fibre bundles, principal bundles in topology about homotopy, fibre and existence of sections. For every topological group M, J. Milnor in [12, 13] has shown that there exist contractible fibre spaces having M as fibre. Relating to the similar result there is a question in [23] - what should be the conditions to be put on an H-space M such that there could exist a contractible fibre space having M as its fibre? For the same question, a well-known result is that such a space always exists if M is a compact Lie group [11]. Similar to this kind of problem, Robert Herman [20] gave a sufficient condition for a mapping of Riemannian manifolds to be a fibre bundle. The triviality of bundles and the existence of a global section of bundles are the most common problems in this theory. Norman Steenrod [19] discussed the homotopy of maps of bundles explicitly whenever the base space is the same. One can extend this definition to the bundle maps with two different base spaces. He mentioned that every fibre bundle whose base space is contractible is trivial and included the theorem of first covering homotopy. The homotopy of bundle maps is essential to the study of the fundamental topological structures of bundles. We are discussing a part of the homotopy theory of bundle maps and their consequences here.

Concepts like the relatedness of a map on sections, vector fields, tensor fields and differential forms lead to important theories like the theory of invariant vector fields, invariant metrics and invariant sections. This has a dominant responsibility in the theory of the Lie groups, One can see such a discussion of invariant vector fields in the articles [4, 6, 14].

Besides, it is essential to study related morphisms of given sections of a bundle. Such an essential study has been conducted in this paper. Sections of a bundle is general notion to the vector field concept, because vector fields, differential forms are certain smooth sections of vector bundles. Vector bundles are crucial in the theory of connections and metrics in Riemannian manifolds. Concerning the existence of a regular group action on space and admittance of a topological group structure, a Lie group structure on a space requires some basic quality on space. Such a need can be realised by the theory of bundles. In fact, in this paper, we construct some classes of bundles on topological spaces, and their applications will be discussed in our next paper, while a hint of applications is mentioned in the conclusion part.

There are a lot of studies that have taken place on an algebraic structure groupoid. One may see that they have different arms like groupoid, topological groupoid, Lie groupoid, and also we may see similar names with different ideas, i.e. hypergroupoid and topological hypergroupoid [21, 22]. In [2], we have introduced a sufficient topological invariant, namely the Core fundamental groupoid which contains the path homotopy equivalence class of loop-based at all points of the space. In section 3 of [2], we have proved that the standard projection on the Core fundamental groupoid to its base space is a quotient map. Here, we have constructed some bundle structures on the Core fundamental groupoid and discussed some canonical bundle maps including principal G-bundle maps. We have studied the sections of such bundles including algebraic structure associated with the set of all sections of the Core fundamental groupoid bundle, and discussed relatedness on both sections and functions (mainly on the homeomorphisms), including pullback and push-forward of sections. The notions of rel for sections, maps, homeomorphisms, subsets of $Homeo(M)$ and $\Gamma^0(\bar{\pi}_1 M)$ have been broadly studied for their algebraic structure on respective outcomes.

2. Preliminaries

In the entire paper, we denote by $(M, \mathfrak{I}_M)$ or whenever there is no confusion, simply M for a topological space. Generally, $\pi_1(M, x)$ is the fundamental group for topological space M and with a base point $x \in M$ [1, 15, 16]. Throughout this paper, we denote γ_x for a loop-based at x and $\overline{\gamma}$ for the reverse of a path γ, and also c_x to the constant loop based at x. The $\simeq_p$ denotes path homotopy, the $*$ denotes the concatenation of two paths/loops as defined in the Core fundamental groupoid or fundamental groupoid or in the fundamental group. In the fundamental group $\pi_1(M, x)$, indeed, the path homotopy equivalence class $[c_x]$ is the identity element.

A description of Groupoid structure (algebraic sense) is available in [7, 18]. A non-empty set G associated with $\star^{-1} : G \to G$ a unary operation and $\star : G \times G \to G$ a partial function, but not a binary operation satisfying i) Associativity: If $a \star b$ and $b \star a$ defined then $a \star (b \star c)$ and $(a \star b) \star c$ are defined and $a \star (b \star c) = (a \star b) \star c$, ii) Inverse: $a^{-1} \star a$ and $a \star a^{-1}$ are always defined. iii) Identity: If $a \star b$ defined, then $a \star b \star b^{-1} = a$ and $a^{-1} \star a \star b = b$ are always defined, is called a groupoid. Generally, one can see that $(a^{-1})^{-1} = a$ and $(a \star b)^{-1} = b^{-1} \star a^{-1}$ for defined $a \star b$, are often using properties in groupoid. Commonly, G_0 denotes the set of all identities of groupoid G, it is called the identity set of G. Here, there are some important definitions and results that will be used later.

Definition 2.1. [7] *Let G, G' be groupoids under partial functions $\star$ and $\star'$ respectively, then a map $T : G \to G'$ is called a groupoid homomorphism if $\forall a, b \in G$ and $a \star b$ defined implies $T(a) \star' T(b)$ is defined in such case $T(a \star b) = T(a) \star' T(b)$.*

Definition 2.2. [7] *Let G, G' be groupoids then a map $T : G \to G'$ is called groupoid isomorphism if it is bijective and both T and T^{-1} are groupoid homomorphism.*

Proposition 2.3. *The composition of two groupoid homomorphisms is a groupoid homomorphism.*

Definition 2.4. [18] *A topological groupoid is a groupoid $(G, \star)$ together with a topology on G such that unary operation and its partial function are continuous functions.*

The Core fundamental groupoid of a topological space M is the disjoint union of the fundamental groups at points of M, and is denoted by $\bar{\pi}_1 M = \bigcup_{x \in M} \pi_1(M, x)$ and it is a topological groupoid under the quotient topology on it yielded by space M under standard projection p (i.e. the topology on $\bar{\pi}_1 M$ is the topology $\mathfrak{I}_{p_M} = \{p^{-1}(U) : \forall U \in \mathfrak{I}_M\}$, Moreover, the fibre of each element x under standard projection is $\pi_1(M, x)$ and it has indiscrete topology under subspace topology) [2], the same is used here. For each continuous map $f : M \to N$ the induced groupoid homomorphism is defined by $f_{\#} : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $f_{\#}([\gamma_x]) = [f \circ \gamma_x]$ for all equivalence classes containing loops in M, i.e. $[\gamma_x] \in \bar{\pi}_1 M$.

In general, a bundle is a triple (E, π, M), where π is a just surjection from total space E to base space M [1, 5, 16, 19]. One can consider topologies on respective space and projection as a continuous one. A fibre bundle is a quadruple (E, π, M, F) where E, M, F are topological spaces and for every element $x \in M$ there is an open set U containing x in M and a homeomorphism ϕ from $\pi^{-1}(U)$ to $U \times F$ such that $\pi_1 \circ \phi = \pi$ where π_1 is the first projection of $U \times F$. Similarly a principal G-bundle is a fibre bundle in which fibre space $F = G$ a topological group, and there is a continuous free right group action on total space, and the restriction of group action on each fibre is a regular group action. We are using fibre bundle maps, respective isomorphisms, and homotopy of bundle maps as in [16, 19]. A uniquely geodesic space is a metric space in which every pair of points has unique geodesic, infact they are topological spaces.

3. Core fundamental groupoid: a bundle

We have introduced the Core fundamental groupoid bundle and sections of it after being motivated by the standard notions of tangent bundle and vector fields of differential geometry, but intuitively, the Core fundamental groupoid bundle and its sections give more informative applications in the fields of topology and geometry, as we have seen using group actions. The Core fundamental groupoid contains the path homotopy classes of all loops based at each point of a topological space M, which is denoted by $\bar{\pi}_1 M$. A standard projection on the Core fundamental groupoid is defined by $p : \bar{\pi}_1 M \to M$, by $p([\gamma_x]) = x$ and it is a surjection. Thus, it is very clear that a triple $(\bar{\pi}_1 M, p, M)$ becomes a bundle, but not necessarily a fibre bundle. In some cases, one can see that $\bar{\pi}_1 M$ can be endowed with a fibre bundle structure, which we will see as a Proposition in this section. As we mentioned in the preliminary part, the standard projection p induces the quotient topology on the Core fundamental groupoid by base space, under which $\bar{\pi}_1 M$ becomes a topological groupoid

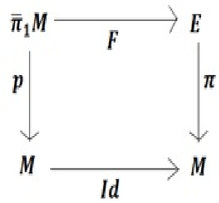
[2]. For each sub-groupoid of the Core fundamental groupoid of connected, locally path-connected and semi locally simply connected space M there is a covering space [3]. This is a way one can have a bundle (take such a covering map as a bundle) with the help of Core fundamental groupoid, but here we have shown the Core fundamental groupoid itself admits a bundle structure. Moreover, the topology on $\bar{\pi}_1 M$ as in [2] yields structures like fibre, principal G-fibre structures as follows.

Proposition 3.1. *Let M be a topological space such that each pair of fundamental groups of M be isomorphic, then $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a fibre bundle for some x_0 in M.*

Proof. Since each pair of fundamental groups of M is isomorphic, hence each fibre $p^{-1}(x_0)$ is isomorphic to all fibre at each point, so homeomorphic to indiscrete space $\pi_1(M, x_0)$. The standard projection $p : \bar{\pi}_1 M \to M$ from the fundamental groupoid to base space M is a surjection. From [2] it is a continuous map under the quotient topology. For each element $x \in M$, we can choose any open set U containing x, then define a map $\psi : p^{-1}(U) \to U \times \pi_1(M, x_0)$, by $\psi(\theta_x) = (x, T_{x,x_0}([\gamma_x]))$, where $\theta_x = [\gamma_x] \in p^{-1}(U)$ and T_{x,x_0} is one of the isomorphism from $\pi_1(M, x)$ to $\pi_1(M, x_0)$ (there are many isomorphisms between two such fundamental groups but for each choice, we will get a same expected behaviour of the function $\psi(\theta_x)$ even isomorphism is different), which is a well-defined homeomorphism. Because, bijection is due to bijection of T_{x,x_0}, and subspace topology on $p^{-1}(U)$ of $\bar{\pi}_1 M$ guarantees continuity of ψ and ψ^{-1}. It is true that, $p_1 \circ \psi = p$, where the projection $p_1 : U \times \pi_1(M, x_0) \to U$ is given by $p_1(x, [\gamma_x]) = x$. Thus the $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a fibre bundle. □

Proposition 3.2. *Let M be a topological space and each pair of fundamental groups of M be isomorphic, then the fibre bundle $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a trivial bundle.*

Proof. Define $(E, \pi, M, \pi_1(M, x_0))$ by $E = M \times \pi_1(M, x_0)$, and a projection $\pi : E \to M$, by $\pi(x, [\gamma_{x_0}]) = x$ then $(E, \pi, M, \pi_1(M, x_0))$ is a product bundle, in fact, it is bundle isomorphic to $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$, thus a trivial bundle. Because one can define a pair of maps $(F([\gamma_x]) = (x, T_{x,x_0}), Id_M)$ is an isomorphism of bundle map, where T_{x,x_0} is one of the isomorphism from $\pi_1(M, x)$ to $\pi_1(M, x_0)$ and following diagram commute. □



Proposition 3.3. *Let M be a connected topological manifold then $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a fibre bundle.*

Proof. Since fundamental groups between any two points of topological manifolds are isomorphic, therefore from Proposition 3.1 the result is true. □

Corollary 3.4. *If a topological space M has the trivial fundamental group at each point then $(\bar{\pi}_1 M, p, M, \{0\})$ is a fibre bundle.*

Proof. This is followed by Proposition 3.1. □

Corollary 3.5. *Let M be a simply connected space then $(\bar{\pi}_1 M, p, M, \{0\})$ is a fibre bundle.*

Proof. This is followed by Corollary 3.4. □

Proposition 3.6. *Let M be a uniquely geodesic space then bundle $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a principal $\pi_1(M, x_0)$-bundle for some x_0 in M.*

Proof. From Proposition 3.3, the $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a fibre bundle. Define a group action $\mu : \bar{\pi}_1 M \times \pi_1(M, x_0) \to \bar{\pi}_1 M$, by $\mu([\gamma_x], [\delta_{x_0}]) = [\gamma_x * \sigma * \delta_{x_0} * \overline{\sigma}]$ (or $[\sigma * \delta_{x_0} * \overline{\sigma} * \gamma_x]$) is a free right group action, where σ is the geodesic from x to x_0. Since, space M is uniquely geodesic space, therefore there exist unique geodesic between any two arbitrary elements of M, hence μ is well-defined. We can see $\mu([\gamma_x], [c_{x_0}]) = [\gamma_x * \sigma * c_{x_0} * \overline{\sigma}] = [\gamma_x]$ and $\mu(\mu([\gamma_x], [\beta_{x_0}]), [\delta_{x_0}]) = \mu([\gamma_x * \sigma * \beta_{x_0} * \overline{\sigma}], [\delta_{x_0}]) = [\gamma_x * \sigma * \beta_{x_0} * \overline{\sigma} * \sigma * \delta_{x_0} * \overline{\sigma}] = [\gamma_x * \sigma * \beta_{x_0} * \delta_{x_0} * \overline{\sigma}] = \mu([\gamma_x], [\beta_{x_0} * \delta_{x_0}])$. Whenever $\mu([\gamma_x], [\delta_{x_0}]) = [\gamma_x]$ this implies $[\gamma_x * \sigma * \delta_{x_0} * \overline{\sigma}] = [\gamma_x]$ or $\gamma_x * \sigma * \delta_{x_0} * \overline{\sigma} \simeq_p \gamma_x$ equivalently $\gamma_x * \sigma * \delta_{x_0} \simeq_p \gamma_x * \sigma$. This ensures that δ_{x_0} is contractible to x_0. Hence δ_{x_0} path homotopic to constant loop c_{x_0}, therefore $[\delta_{x_0}] = [c_{x_0}]$ identity. This implies group action is free.
Moreover, it is true that, $p \circ \mu = id' \circ p \times k$, where $p : \bar{\pi}_1 M \to M$ is the standard projection, $p \times k : \bar{\pi}_1 M \times \pi_1(M, x_0) \to M \times [c_{x_0}]$ defined by $p \times k([\gamma_x], [\delta_{x_0}]) = (p([\gamma_x]), [c_{x_0}])$ and $id' : M \times [c_{x_0}] \to M$ by $id'(x, [c_{x_0}]) = x$. Moreover, induced group action on each fibre i.e. $\mu : \pi_1(M, x) \times \pi_1(M, x_0) \to \pi_1(M, x)$ is obviously free and transitive. Let us have the transitivity of group action, since M is path-connected space hence all fundamental group are isomorphic to each other (In fact, one can see this $[\sigma * \delta_{x_0} * \overline{\sigma}]$ is a canonical isomorphism from $\pi_1(M, x_0)$ to $\pi_1(M, x)$). Now we compute,

$$
\begin{aligned}
orbit\ of\ ([\gamma_x]) &= \{\mu([\gamma_x], [\delta_{x_0}]) : [\delta_{x_0}] \in \pi_1(M, x_0)\} \\
&= \{[\gamma_x * \sigma * \delta_{x_0} * \overline{\sigma}] : [\delta_{x_0}] \in \pi_1(M, x_0)\}
\end{aligned}
$$

This $\sigma * \delta_{x_0} * \overline{\sigma}$ is bijective from $\pi_1(M, x_0)$ to $\pi_1(M, x)$, because it is a composition of translation by $[\gamma_x]$ with canonical isomorphism. Accordingly,

$$
orbit\ of\ ([\gamma_x]) = \pi_1(M, x)
$$

This concludes that the group action is transitive. Hence the result. □

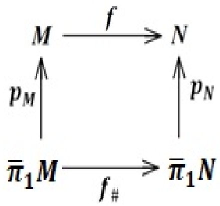
Corollary 3.7. *Let M be a simply connected space then $(\bar{\pi}_1 M, p, M, \{0\} = G)$ is a principal G-bundle or $(\bar{\pi}_1 M, p, M, \pi_1(M, x_0))$ is a principal $\pi_1(M, x_0)$-bundle for some x_0 in M.*

Proof. From Proposition 3.1, the $(\bar{\pi}_1 M, p, M, \{0\} = G)$ is a fibre bundle. Define a group action $\mu : \bar{\pi}_1 M \times \{0\} \to \bar{\pi}_1 M$, by $\mu([\gamma_x], 0) = [\gamma_x]$ is a trivial right group action, and this is free. This is easy to see that, $p \circ \mu = id' \circ p \times k$, where $p : \bar{\pi}_1 M \to M$ is the standard projection, $p \times k : \bar{\pi}_1 M \times \{0\} \to M \times \{0\}$ defined by $p \times k([\gamma_x], 0) = (p([\gamma_x]), 0)$ and $id' : M \times \{0\} \to M$ by $id'(x, 0) = x$. Moreover, the induced group action restricted to each fibre over $\bar{\pi}_1 M$ i.e. $\mu : \pi_1(M, x) \times \{0\} \to \pi_1(M, x)$ is obviously free and transitive. Hence the result.

Or, it is true that instead of G in the above group action, we can consider the fundamental group $\pi_1(M, x_0)$ for group action. For that define the $\mu : \bar{\pi}_1 M \times \pi_1(M, x_0) \to \bar{\pi}_1 M$, by $\mu([\gamma_x], [c_{x_0}]) = [\gamma_x]$, then all axioms regard principal $\pi_1(M, x_0)$ bundle will be satisfied. □

Proposition 3.8. *Let* $(\bar{\pi}_1 M, p_M, M, \pi_1(M, x_0)), (\bar{\pi}_1 N, p_N, N, \pi_1(N, y_0))$ *be fibre bundles, for some* x_0 *in* M *and* y_0 *in* N *and* $f : M \to N$ *be a continuous map then* $(f_{\#}, f)$ *is a fibre bundle map, where* $f_{\#} : (\bar{\pi}_1 M, \mathfrak{I}_{p_M}) \to (\bar{\pi}_1 N, \mathfrak{I}_{p_N})$ *is the induced groupoid homomorphism.*

Proof. From Proposition 3.36 in [2] induced groupoid homomorphism $f_{\#} : (\bar{\pi}_1 M, \mathfrak{I}_{p_M}) \to (\bar{\pi}_1 N, \mathfrak{I}_{p_N})$ is a topological groupoid homomorphism, for each continuous map $f : M \to N$. Also, from [2] the following diagram commute.



Thus $(f_{\#}, f)$ is a fibre bundle map. □

Corollary 3.9. *Let* $(\bar{\pi}_1 M, p_M, M, \pi_1(M, x_0)), (\bar{\pi}_1 N, p_N, N, \pi_1(N, y_0))$ *be fibre bundles, for some* x_0 *in* M *and* y_0 *in* N *and* $f : M \to N$ *be a homeomorphism then* $(f_{\#}, f)$ *is a fibre bundle isomorphism, where* $f_{\#} : (\bar{\pi}_1 M, \mathfrak{I}_{p_M}) \to (\bar{\pi}_1 N, \mathfrak{I}_{p_N})$ *is the induced groupoid homomorphism.*

Proposition 3.10. *Let* $(\bar{\pi}_1 M, p_M, M, \pi_1(M, x_0)), (\bar{\pi}_1 N, p_N, N, \pi_1(N, y_0))$ *be fibre bundles, for some* x_0 *in* M *and* y_0 *in* N *and* $f, g : M \to N$ *be homotopic, then bundle maps* $f_{\#}$ *and* $g_{\#}$ *are homotopic in the sense of bundle maps.*

Proof. It is followed by Propositions 4.30, 4.31 in [2]. □

Proposition 3.11. *Let* $(\bar{\pi}_1 M, p_M, M, \pi_1(M, x_0)), (\bar{\pi}_1 N, p_N, N, \pi_1(N, y_0))$ *be fibre bundles, for some* x_0 *in* M *and* y_0 *in* N *and* M *and* N *be same homotopic type then bundle* $(\bar{\pi}_1 M, p_M, M, \pi_1(M, x_0))$ *is same homotopic to bundle* $(\bar{\pi}_1 N, p_N, N, \pi_1(N, y_0))$

Proof. It is followed by Proposition 4.32 in [2]. □

4. SECTIONS OF CORE FUNDAMENTAL GROUPOID BUNDLE

A section is a more generalized notion of the vector field; a basic object in differential geometry [16, 19]. This is also a well-studied terminology in topology, generally a section of a bundle is a continuous map that gives an identity when it is composed with a standard projection from the bundle. Sections of the Core Fundamental groupoid bundle have important applications in group actions and classification of topological spaces, and also have rich consequences with pullback and pushforward. For instance, establishing this notion helps us to give a necessary condition for the existence of regular group action on any topological space, as well as smooth manifolds. Such an application will be discussed in a future research paper. Here, we define sections of the Core fundamental groupoid bundle for all kinds of bundles that we have introduced. In this context, a section assigns each point by a unique path homotopy class of a loop. See proposition 4.4 which shows continuity is obvious one. In our study, section means continuous section, therefore we include continuity in the definition of section and by keeping in mind a bundle $(\bar{\pi}_1 M, p, M)$ (by considering both $M, \bar{\pi}_1 M$ are topological spaces, but one can also define sections without continuity). Further, the definition is the same for both fibre bundle $(\bar{\pi}_1 M, p, M, F)$ and principal G-bundle $(\bar{\pi}_1 M, p, M, G)$ accordingly.

Definition 4.1. *A section of the Core fundamental groupoid bundle is a continuous map* $X : M \to \bar{\pi}_1 M$ *such that* $p \circ X = Id_M$.

Example 4.2. *In the Euclidean space* $\mathbb{R}^n$ *define* $X : \mathbb{R}^n \to \bar{\pi}_1 \mathbb{R}^n$*, by* $X(x) = [c_x]$*, where* c_x *is the constant loop based at* x*. Then it is easy to see that* X *is a section due to* $p \circ X = Id_{\mathbb{R}^n}$ *and continuity is from Proposition 4.4.*

Remark 4.3. *i) The zero/identity section of* $\bar{\pi}_1 M$ *is the continuous map* $X : M \to \bar{\pi}_1 M$ *defined by* $X(x) = [c_x]$*. Generally, it will be denoted by* E *or* O*.*
ii) For any simply connected space there is only one section, that is zero.

Proposition 4.4. *Let* M *be a topological space then every function* $X : M \to \bar{\pi}_1 M$ *satisfying* $p \circ X = Id_M$ *is a section of* $\bar{\pi}_1 M$*.*

Proof. It is enough to see only continuity of X. Take arbitrary open set D of $\bar{\pi}_1 M$, then $p^{-1}(V) = D$ for some open set is V in M. Consider $X^{-1}(D) = X^{-1}(p^{-1}(V)) = (p \circ X)^{-1}(V) = {Id_M}^{-1}(V) = V$, which is an open in M. Therefore X is continuous and a section of $\bar{\pi}_1 M$. □

Remark 4.5. *Different topologies on the Core fundamental groupoid other than this quotient topology need not give the result in Proposition 4.4.*

Definition 4.6. *Let* N *be a subspace of* M *then a section over* N *is a continuous map* $X : N \to \bar{\pi}_1 M$ *such that* $p_{|N} \circ X = Id_N$*, where* $p_{|N} : p^{-1}(N) \to N$ *is the restriction of* p*.*

Proposition 3.2 implies the existence of global sections, so we concentrate on them rather than sections over subspaces. Further, a local section X of $\bar{\pi}_1 M$ is a section defined on an open set U of M.

Here we concentrate on sections of the whole $\bar{\pi}_1 M$. For a given topological space M, the set of all sections of the Core fundamental groupoid of M is denoted by $\Gamma^0(\bar{\pi}_1 M)$. i.e., $\Gamma^0(\bar{\pi}_1 M) = \{X : M \to \bar{\pi}_1 M : X$ continuous and $p \circ X = Id_M\}$. The set $\Gamma^0(\bar{\pi}_1 M)$ has a nice algebraic structure coming from the fundamental groups.

Proposition 4.7. *For each* $[\gamma_{x_0}] \in \bar{\pi}_1 M$ *then there exists a section* X *of* $\bar{\pi}_1 M$ *such that* $X(x_0) = [\gamma_{x_0}]$.

Proof. Define $X : M \to \bar{\pi}_1 M$ by $X(x) = \begin{cases} [\gamma_{x_0}] & \text{for } x = x_0 \\ [c_x] & \text{otherwise} \end{cases}$, it is trivial to see $p \circ X = Id_M$. Hence by Proposition 4.4 it is a well-defined section and satisfies $X(x_0) = [\gamma_{x_0}]$. □

Proposition 4.8. *Let M be a topological space, and consider the following map $\circledast : \Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 M)$ by $(X, Y) \to X \circledast Y$, where $X \circledast Y : M \to \bar{\pi}_1 M$ defined by $(X \circledast Y)(x) = X(x) * Y(x)$ (it is the concatenation between the $X(x)$ and $Y(x)$) with this map the $\Gamma^0(\bar{\pi}_1 M)$ is a group.*

Proof. The operation $\circledast : \Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 M)$ by $(X, Y) \to X \circledast Y$, where $X \circledast Y : M \to \bar{\pi}_1 M$ defined by $(X \circledast Y)(x) = X(x) * Y(x)$ is clearly well-defined. Because $X(x) * Y(x) \in \pi_1(M, x) \subset \bar{\pi}_1 M$, well-defined function, also this is clear that $p \circ (X \circledast Y) = Id_M$ and by Proposition 4.4 $X \circledast Y$ is a section. Associativity $X \circledast (Y \circledast Z) = (X \circledast Y) \circledast Z$ will follow from the group structure of each fundamental group. The element zero section O is in $\Gamma^0(\bar{\pi}_1 M)$ and for every section X, we can see $O \circledast X = X \circledast O = X$, because, $O \circledast X(x) = O(x) * X(x) = X(x)$, for all x in M. Finally, given any section X there is a section $Y : M \to \bar{\pi}_1 M$ defined by $Y(x) = *^{-1}(X(x))$ (where $*^{-1}(X(x))$ is inverse of the element $X(x)$ in the fundamental group of M based at x), then it is obvious that $X \circledast Y = Y \circledast X = O$. This completes the proof. □

Remark 4.9. *i) If all fundamental groups over each point of the space are abelian then $\Gamma^0(\bar{\pi}_1 M)$ is an abelian group.*
ii) Since the fundamental group of each element of a topological group is an abelian (Eckamann-Hilton result). Therefore for any topological group G the $\Gamma^0(\bar{\pi}_1 G)$ becomes abelian.
iii) If M is simply connected then $\Gamma^0(\bar{\pi}_1 M)$ is an abelian group.
iv) For the Projective plane $\mathbb{R}P^2$ the $\Gamma^0(\mathbb{R}P^2)$ is an abelian group.

Proposition 4.10. *Let $f : M \to N$ be a continuous injection and X be a section of $\bar{\pi}_1 M$ then there exists a section Y in $\bar{\pi}_1 N$ such that $f_\# \circ X = Y \circ f$.*

Proof. Let $f : M \to N$ be a continuous injection, the continuity of f implies $f_\#$ is well-defined, and for the section $X : M \to \bar{\pi}_1 M$, we can define map say $Y : N \to \bar{\pi}_1 N$, by

$$Y(y) = \begin{cases} f_\# \left(X \left(f^{-1}(y) \right) \right) \text{ for } y \in f(M) \\ [c_y] \text{ for } y \in N \backslash f(M) \end{cases}$$

obviously, this becomes a section of $\bar{\pi}_1 N$ by Proposition 4.4. Moreover, this satisfy, $Y \circ f(x) = Y(f(x)) = f_\#(X(f^{-1}(f(x)))) = f_\#(X(x)) = f_\# \circ X(x)$, for $x \in M$. Hence the result. □

This result needs f to be injective. Suppose f is not injective such sections cannot be defined from codomain to its Core fundamental groupoid, because, in that case, induced groupoid homomorphism $f_\#$ assigns more than two elements to the elements of the same fundamental group of the same element of the codomain of f. For example, let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $f(z) = z^2$ which is not an injection but continuous. For $[\gamma_1 = e^{2\pi i t}]$ and $[\alpha_{-1} = -e^{4\pi i t}]$ element of $\bar{\pi}_1 \mathbb{S}^1$ and $f_\#([\gamma_1 = e^{2\pi i t}])$ and $f_\#([\alpha_{-1} = -e^{4\pi i t}])$ elements in $\pi_1(\mathbb{S}^1, 1) \subset \bar{\pi}_1 \mathbb{S}^1$ but they are not same elements due to $f \circ \gamma_1$ is not path homotopic to

$f \circ \alpha_{-1}$. Therefore for the uniqueness and non-triviality of such a section in Proposition 4.10 needs the map to be a homeomorphism.

Proposition 4.11. *Let $f : M \to N$ be a homeomorphism and X be a section of $\bar{\pi}_1 M$ then there exists a unique section Y in $\bar{\pi}_1 N$ such that $f_\# \circ X = Y \circ f$.*

Proof. Define a $Y : N \to \bar{\pi}_1 N$ by $Y(y) = f_\# \circ X \circ f^{-1}(y)$ which is a well-defined map and becomes a section obviously. Moreover, it satisfies $f_\# \circ X = Y \circ f$. For uniqueness, suppose there is another section Y' satisfying $Y' \circ f = f_\# \circ X$ implies $Y' = f_\# \circ X \circ f^{-1} = Y$. Hence it is unique. □

Proposition 4.12. *Let $f : M \to N$ be a homeomorphism and for any section Y of $\bar{\pi}_1 N$ then there exists a unique section X in $\bar{\pi}_1 M$ such that $f_\# \circ X = Y \circ f$.*

Proof. Define $X : M \to \bar{\pi}_1 M$ by $X(x) = f_\#^{-1} \circ Y \circ f(x)$ which is a section obviously. Uniqueness is due to the homeomorphism of f. □

Propositions 4.11, 4.12 help to define notions called f-relatedness, pushforward, pullback.

Definition 4.13. *Let $f : M \to N$ be a homeomorphism and X be a section of $\bar{\pi}_1 M$ then the section of $\bar{\pi}_1 N$ defined by $f^*(X) = f_\# \circ X \circ f^{-1}$ is called pushforward of section X under f. This gives a map $f^* : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 N)$, defined by $f^*(X) = f_\# \circ X \circ f^{-1}$ is called pushforward map.*

Definition 4.14. *Let $f : M \to N$ be a homeomorphism and Y be a section of $\bar{\pi}_1 N$ then the section of $\bar{\pi}_1 M$ defined by $f^\star(Y) = f_\#^{-1} \circ Y \circ f$ is called pullback of section Y under f. This gives a map $f^\star : \Gamma^0(\bar{\pi}_1 N) \to \Gamma^0(\bar{\pi}_1 M)$, defined by $f^\star(Y) = f_\#^{-1} \circ Y \circ f$ called pullback map.*

Proposition 4.15. *Let $f : M \to N$ be a homeomorphism then both $f^* : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 N)$ and $f^\star : \Gamma^0(\bar{\pi}_1 N) \to \Gamma^0(\bar{\pi}_1 M)$ are group homomorphisms.*

Proof. Let us see for, $f^* : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 N)$, choose arbitrary elements X_1, X_2 from $\Gamma^0(\bar{\pi}_1 M)$. We can have $f^*(X_1 \circledast X_2) = f_\# \circ (X_1 \circledast X_2) \circ f^{-1} = f_\# \circ X_1 \circ f^{-1} \circledast f_\# \circ X_2 \circ f^{-1}. = f^*(X_1) \circledast f^*(X_2)$, because for every $y \in N$ the

$$\begin{aligned} f_\# \circ (X_1 \circledast X_2) \circ f^{-1}(y) &= f_{\# f^{-1}(y)}(X_1(f^{-1}(y)) * X_2(f^{-1}(y))) \\ &= f_{\# f^{-1}(y)}(X_1(f^{-1}(y))) * f_{\# f^{-1}(y)}(X_2(f^{-1}(y))) \\ &= (f_\# \circ X_1 \circ f^{-1})(y) * (f_\# \circ X_2 \circ f^{-1})(y) \end{aligned}$$

Similarly, for, $f^\star : \Gamma^0(\bar{\pi}_1 N) \to \Gamma^0(\bar{\pi}_1 M)$, choose arbitrary elements Y_1, Y_2 from $\Gamma^0(\bar{\pi}_1 N)$. We can have $f^\star(Y_1 \circledast Y_2) = {f_\#}^{-1} \circ (Y_1 \circledast Y_2) \circ f = {f_\#}^{-1} \circ Y_1 \circ f \circledast {f_\#}^{-1} \circ Y_2 \circ f = f^\star(Y_1) \circledast f^\star(Y_2)$, because for every $x \in M$ the,

$$\begin{aligned} ({f_\#}^{-1} \circ (Y_1 \circledast Y_2) \circ f)(x) &= {f_{\# f(x)}}^{-1}(Y_1(f(x)) * Y_2(f(x))) \\ &= {f_{\# f(x)}}^{-1}(Y_1(f(x))) * {f_{\# f(x)}}^{-1}(Y_2(f(x))) \\ &= ({f_\#}^{-1} \circ Y_1 \circ f)(x) * ({f_\#}^{-1} \circ Y_2 \circ f)(x) \end{aligned}$$

□

Proposition 4.16. *Let $f : M \to N$ be a homeomorphism then,*
i) $f^{-1^*} = f^{*^{-1}} = f^\star$
ii) $f^{-1^\star} = f^{\star^{-1}} = f^*$

Proof. i) The homeomorphism $f : M \to N$ implies the inverse map $f^{-1} : N \to M$ is a homeomorphism. Therefore $f^{-1^*} : \Gamma^0(\bar{\pi}_1 N) \to \Gamma^0(\bar{\pi}_1 M)$ is well-defined and for any $Y \in \Gamma^0(\bar{\pi}_1 N)$ the $f^{-1^*}(Y) = {f^{-1}}_{\#} \circ Y \circ (f^{-1})^{-1} = {f_{\#}}^{-1} \circ Y \circ f = f^{*^{-1}}(Y) = f^\star(Y)$.
ii) This is similar to (i). □

Proposition 4.17. *Let $Id : M \to M$ be the identity map then*
i) $Id^* = Id_{\Gamma^0(\bar{\pi}_1 M)}$.
ii) $Id^\star = Id_{\Gamma^0(\bar{\pi}_1 M)}$.

Proof. i) Hypothesis $Id : M \to M$ is the identity map, so $Id^* : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 M)$ satisfy $Id^*(X) = Id_{\#} \circ X \circ Id^{-1} = X = Id_{\Gamma^0(\bar{\pi}_1 M)}(X)$, for every $X \in \Gamma^0(\bar{\pi}_1 M)$.
ii) Hypothesis $Id : M \to M$ is the identity map, so $Id^\star : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 M)$ satisfy $Id^\star(X) = {Id_{\#}}^{-1} \circ X \circ Id = X = Id_{\Gamma^0(\bar{\pi}_1 M)}(X)$, for every $X \in \Gamma^0(\bar{\pi}_1 M)$. □

Proposition 4.18. *Let $f : M \to N$ and $g : N \to R$ be homeomorphism then*
i) $(g \circ f)^* = g^* \circ f^*$.
ii) $(g \circ f)^\star = f^\star \circ g^\star$.

Proof. i) Both $f^* : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 N)$ and $g^* : \Gamma^0(\bar{\pi}_1 N) \to \Gamma^0(R)$ are composable and give $g^* \circ f^* : \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(R)$ as well-defined. Also, we have for every $X \in \Gamma^0(\bar{\pi}_1 M)$, the

$$\begin{aligned} (g \circ f)^*(X) &= (g \circ f)_{\#} \circ X \circ (g \circ f)^{-1} \\ &= g_{\#} \circ f_{\#} \circ X \circ f^{-1} \circ g^{-1} \\ &= g_{\#} \circ (f^*(X)) \circ g^{-1} = g^*(f^*(X)). \end{aligned}$$

Hence $(g \circ f)^* = g^* \circ f^*$.
ii) Both $f^\star : \Gamma^0(\bar{\pi}_1 N) \to \Gamma^0(\bar{\pi}_1 M)$ and $g^\star : \Gamma^0(R) \to \Gamma^0(\bar{\pi}_1 N)$ are composable and give $f^\star \circ g^\star : \Gamma^0(R) \to \Gamma^0(\bar{\pi}_1 M)$ as well-defined. Also, we have for every $Z \in \Gamma^0(R)$, the

$$\begin{aligned} (g \circ f)^\star(Z) &= {(g \circ f)_{\#}}^{-1} \circ Z \circ (g \circ f) \\ &= {f_{\#}}^{-1} \circ {g_{\#}}^{-1} \circ Z \circ g \circ f \\ &= {f_{\#}}^{-1} \circ (g^\star(Z)) \circ f \\ &= f^\star(g^\star(Z)). \end{aligned}$$

Hence $(g \circ f)^\star = f^\star \circ g^\star$. □

Proposition 4.19. *Let M be a topological space then $\omega : \Gamma^0(\bar{\pi}_1 M) \times \bar{\pi}_1 M \to \bar{\pi}_1 M$ defined by $\omega(X, [\gamma_x]) = [X(x) * \gamma_x]$ is a well-defined non-regular group action.*

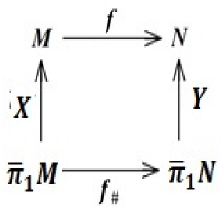
Proof. Since $X(x)$ and γ_x are elements of the same fundamental group $\pi_1(M, x)$, hence ω is well-defined. Let $[\gamma_x] \in \bar{\pi}_1 M$ and for zero section O, we have $\omega(O, [\gamma_x]) = [O(x) * \gamma_x] = [\gamma_x]$. And also for $[\gamma_x] \in \bar{\pi}_1 M$ and $X, Y \in \Gamma^0(\bar{\pi}_1 M)$ then $\omega(X, \omega(Y, [\gamma_x])) =$

$\omega(X, [Y(x) * \gamma_x]) = [X(x) * (Y(x) * \gamma_x)] = [(X(x) * Y(x)) * \gamma_x] = \omega(X \circledast Y, [\gamma_x])$. It is clear to see that ω is non-transitive and not free, so a non-regular group action. □

5. Section-related homeomorphisms, homeomorphism-related sections and their algebraic structures

Propositions 4.10, 4.11, and 4.12 are fundamental to define pullback and pushforward. In geometry, such vector field notions serve as motivations for the f-related vector field and the left-invariant vector field. Moreover, the same results motivate us to define homeomorphism related sections and section related to homeomorphisms on a given topological space. The rel sets acquire a group structure that helps us to have group actions in respective spaces. We are introducing such notions as given below.

Definition 5.1. [9, 10, 16] *Let $f : M \to N$ be a continuous map, a section $X \in \Gamma^0(\bar{\pi}_1 M)$ is said to be f-related to a section $Y \in \Gamma^0(\bar{\pi}_1 N)$, if $f_\# \circ X = Y \circ f$, or the following diagram commute,*



In this case, we call f is pair (X, Y)-related continuous map.
i) For a given $f \in C(M, N)$ then the set of all pairs of sections from $\Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 N)$ for which $f_\# \circ X = Y \circ f$, is called f-related pair of sections. We will denote it by $rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$.
ii) For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ and $Y \in \Gamma^0(\bar{\pi}_1 N)$ then set of all continuous maps f from M to N for which $f_\# \circ X = Y \circ f$, is called pair (X, Y)-related continuous maps, is denoted by $rel_{C(M,N)}(X, Y)$.

Remark 5.2. *With reference to Definition 5.1, we define a relation $R : C(M, N) \to \Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 N)$, given by $R(f) = (X, Y)$, if $f_\# \circ X = Y \circ f$. Further, we have the following results,*
We can see $R(\{f\}) = rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$ and $R^{-1}((X, Y)) = rel_{C(M,N)}(X, Y)$. For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ and $Y \in \Gamma^0(\bar{\pi}_1 N)$ the set $rel_{C(M,N)}(X, Y)$ may sometimes be empty, since the relation is not universal. However $rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$ is always non-empty due to $f_\# \circ O_M = O_N \circ f$. Interestingly for a given continuous map f, the $rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$ is a subgroup of $\Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 N)$, under coordinate-wise operation by $\circledast$.
The most useful notion of f-related is when f is a homeomorphism, which gives the idea of invariant sections. Therefore, we have highlighted the same in the below Remark.

Remark 5.3. *Suppose the function f in Definition 5.1 is a homeomorphism then we have the following concepts and results. We define a relation $Q : Homeo(M,N) \to \Gamma^0(\bar{\pi}_1 M)\times \Gamma^0(\bar{\pi}_1 N)$, by $Q(f) = (X,Y)$, if $f_{\#} \circ X = Y \circ f$.*
i) For a given $f \in Homeo(M,N)$ then set of all pairs of sections from $\Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 N)$ for which $f_{\#} \circ X = Y \circ f$, is called f-related pair of sections, is denoted by $rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$. Moreover, we can see $Q(\{f\}) = rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$.
ii) For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ and $Y \in \Gamma^0(\bar{\pi}_1 N)$ then set of all homeomorphisms f from M to N for which $f_{\#} \circ X = Y \circ f$, is called pair (X,Y)-related homeomorphisms, is denoted by $rel_{Homeo(M,N)}(X,Y)$. Moreover, we can see $Q^{-1}((X,Y)) = rel_{Homeo(M,N)}(X,Y)$.
For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ and $Y \in \Gamma^0(\bar{\pi}_1 N)$ the set $rel_{Homeo(M,N)}(X,Y)$ may sometimes be empty, since the relation is not universal. However $rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$ is always non-empty. Interestingly, for a given homeomorphism f, the $rel_{\Gamma^0(\bar{\pi}_1 M)\times\Gamma^0(\bar{\pi}_1 N)}(f)$ is a subgroup of $\Gamma^0(\bar{\pi}_1 M) \times \Gamma^0(\bar{\pi}_1 N)$, under coordinate-wise operation by $\circledast$.
Invariant of sections are only defined when f is homeomorphism on M, because both $f_{\#} \circ X$ and $X \circ f$ are sections of $\bar{\pi}_1 M$, hence we can compare them for the same behavior. Therefore, we have restricted it to homeomorphism for the notion of invariants of sections.

Remark 5.4. *Suppose the function f in Definition 5.1 is a homeomorphism between the same space and a section $X \in \Gamma^0(\bar{\pi}_1 M)$ satisfies the condition in Definition 5.1 then X is called f-related or f-invariant section. With reference to this definition, we have the following concepts and results.*
We define a relation $S : Homeo(M) \to \Gamma^0(\bar{\pi}_1 M)$, by $S(f) = X$, if $f_{\#} \circ X = X \circ f$.
i) For a given $f \in Homeo(M)$ then set of all sections of $\bar{\pi}_1 M$ for which $f_{\#} \circ X = X \circ f$, is called f-related sections or f-invariant sections, is denoted by $rel_{\Gamma^0(\bar{\pi}_1 M)}(f) = \{X \in \Gamma^0(\bar{\pi}_1 M) : f_{\#} \circ X = X \circ f\}$. Moreover, we can see $S(\{f\}) = rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$.
ii) For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ the set of all homeomorphisms f on M for which $f_{\#} \circ X = X \circ f$, is called X-related homeomorphisms, is denoted by $rel_{Homeo(M)}(X)$. Explicitly $rel_{Homeo(M)}(X) = \{f \in Homeo(M) : f_{\#} \circ X = X \circ f\}$, which is equal to $S^{-1}(\{X\})$.
For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ the set $rel_{Homeo(M)}(X)$ is always non-empty, due to $Id_{\#} \circ X = X \circ Id$ for any section, therefore, $Id \in rel_{Homeo(M)}(X)$. And also always set $rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$ is non-empty due to $f_{\#} \circ O = O \circ f$.

Proposition 5.5. *For a given $X \in \Gamma^0(\bar{\pi}_1 M)$ the set $rel_{Homeo(M)}(X)$ is a subgroup of $Homeo(M)$.*

Proof. Obviously $rel_{Homeo(M)}(X)$ is non-empty subset of all homeomorphisms on M. Take any two $f, g \in rel_{Homeo(M)}(X)$, consider $(f \circ g)_{\#} \circ X = f_{\#} \circ g_{\#} \circ X = f_{\#} \circ X \circ g = X \circ (f \circ g)$. Therefore $f \circ g \in rel_{Homeo(M)}(X)$. Associativity comes from the common property of the composition. Since $Id \in rel_{Homeo(M)}(X)$ and act as identity in $rel_{Homeo(M)}(X)$. And also for any $f \in rel_{Homeo(M)}(X)$ one can see $f^{-1} \in rel_{Homeo(M)}(X)$, because consider $(f^{-1})_{\#} \circ X = (f^{-1})_{\#} \circ X \circ f \circ f^{-1} = (f^{-1})_{\#} \circ f_{\#} \circ X \circ f^{-1} = X \circ f^{-1}$. Hence $rel_{Homeo(M)}(X)$ is a subgroup. □

Remark 5.6. *For $O \in \Gamma^0(\bar{\pi}_1 M)$ then $rel_{Homeo(M)}(O) = Homeo(M)$.*

Definition 5.7. *For a given $\mathcal{B} \subset \Gamma^0(\bar{\pi}_1 M)$ the set of all homeomorphisms f on M for which $f_\# \circ X = X \circ f$, for all $X \in \mathcal{B}$ is called $\mathcal{B}$-related homeomorphisms, denoted by $rel_{Homeo(M)}(\mathcal{B})$. Thus i.e. $rel_{Homeo(M)}(\mathcal{B}) = \{f \in Homeo(M) : f_\# \circ X = X \circ f$, for all $X \in \mathcal{B}\}$.*

Proposition 5.8. *For a given subset of sections $\mathcal{B} \subset \Gamma^0(\bar{\pi}_1 M)$ then set $rel_{Homeo(M)}(\mathcal{B})$ is a subgroup of $Homeo(M)$.*

Proof. Similar to Proposition 5.5. and indeed, it is an intersection of $rel_{Homeo(M)}(X)$ groups for all $X \in \mathcal{B}$. □

Remark 5.9. $rel_{Homeo(M)}(\Gamma^0(\bar{\pi}_1 M)) = \{Id\}$

The groups given by Propositions 5.5 and 5.8 can be seen as the isotropic subgroup of the point in $\Gamma^0(\bar{\pi}_1 M)$ or stabilizer of a section or intersection of the stabilizer of more points under a crucial group action on $\Gamma^0(\bar{\pi}_1 M)$ by $Homeo(M)$.

Proposition 5.10. *Let M be a topological space and let $\mu : Homeo(M) \times \Gamma^0(\bar{\pi}_1 M) \to \Gamma^0(\bar{\pi}_1 M)$ given by $\mu(f, X) = f_\# \circ X \circ f^{-1}$, then*
i) is non-transitive (If M is non-simply connected and for simply connected space it is transitive) group action (one can also define right group action).
ii) Isotropy$(X) = stab_{Homeo(M)}(X) = rel_{Homeo(M)}(X)$ for every $X \in \Gamma^0(\bar{\pi}_1 M)$.
iii)For a given subset of sections $\mathcal{B} \subset \Gamma^0(\bar{\pi}_1 M)$, then the $rel_{Homeo(M)}(\mathcal{B}) = \bigcap_{X \in \mathcal{B}} stab_{Homeo(M)}(X)$.
iv) Fixed point set $\Gamma^0(\bar{\pi}_1 M)^{Homeo(M)} = \{O\}$.

Proof. i) For a topological space M, the $Homeo(M)$ group acts by $\mu(f, X) = f_\# \circ X \circ f^{-1}$. It is clear that $Id_\# \circ X \circ Id^{-1} = X$ for all sections, and also that $\mu(f, \mu(g, X)) = \mu(f, g_\# \circ X \circ g^{-1}) = (f \circ g)_\# \circ X \circ (f \circ g)^{-1} = \mu(f \circ g, X)$.
The orbit of zero section under this group action then it is a singleton trivial section, that is $orb_\mu(O) = \{\mu(f, O)$: for all $f \in Homeo(M)\} = \{O\}$, hence the group action is non-transitive for non-simply connected space.
ii) For all $X \in \Gamma^0(\bar{\pi}_1 M)$ we compute $Isotropy(X) = Stab_{Homeo(M)}(X)$,

$$\begin{aligned} Stab_{Homeo(M)}(X) &= \{f \in Homeo(M) : \mu(f, X) = f_\# \circ X \circ f^{-1} = X\} \\ &= \{f \in Homeo(M) : f_\# \circ X = X \circ f\} \\ &= rel_{Homeo(M)}(X). \end{aligned}$$

It is well-known that isotropy(X) is a subgroup of $Homeo(M)$, so is $rel_{Homeo(M)}(X)$. Hence it is a subgroup.
iii) For a given set of sections $\mathcal{B} \subset \Gamma^0(\bar{\pi}_1 M)$, we compute,

$$\begin{aligned} rel_{Homeo(M)}(\mathcal{B}) &= \{f \in Homeo(M) : \mu(f, X) = f_\# \circ X \circ f^{-1} = X, \forall X \in \mathcal{B}\} \\ &= \{f \in Homeo(M) : f_\# \circ X = X \circ f, \forall X \in \mathcal{B}\} \\ &= \bigcap_{X \in \mathcal{B}} stab_{Homeo(M)}(X) \end{aligned}$$

iv) The set $\Gamma^0(\bar{\pi}_1 M)^{Homeo(M)} = \{X \in \Gamma^0(\bar{\pi}_1 M) : \mu(f, X) = f_\# \circ X \circ f^{-1} = X$, for every $f \in Homeo(M)\}$. Pushforward of the section under all homeomorphisms is only the zero section. Therefore the fixed point set is $\{O\}$. □

Proposition 5.11. *For a given $f \in Homeo(M)$ the set $rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$ is a subgroup of $\Gamma^0(\bar{\pi}_1 M)$.*

Proof. Obviously $rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$ is a non-empty subset of set of all sections of $\bar{\pi}_1 M$. Take any two $X, Y \in rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$, consider $f_\# \circ (X \circledast Y) = f_\# \circ X \circledast f_\# \circ Y = X \circ f \circledast Y \circ f = (X \circledast Y) \circ F$. Where it is clear that $f_\# \circ (X \circledast Y) = f_\# \circ X \circledast f_\# \circ Y$, because, for all $x \in M$ the

$$\begin{aligned} f_\# \circ (X \circledast Y)(x) &= f_{\# x}(X(x) * Y(x)) \\ &= f_{\# x}(X(x)) * f_{\# x}(Y(x)) \\ &= f_\# \circ X(x) * f_\# \circ Y(x) \\ &= (f_\# \circ X \circledast f_\# \circ Y)(x), \end{aligned}$$

therefore $X \circledast Y \in rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$. Also for every $X \in rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$ there is a $Y = X^{-1} \in rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$, defined by $Y(x) = *^{-1}(X(x))$ because for all $x \in M$ the

$$\begin{aligned} f_\# \circ X^{-1}(x) &= f_{\# x}(X^{-1}(x)) = (f_{\# x}(X(x)))^{-1} \\ &= (f_{\# x} \circ X(x))^{-1} = (X \circ f(x))^{-1} \\ &= (X(f(x)))^{-1} \\ &= X^{-1}(f(x)) \\ &= X^{-1} \circ f(x), \end{aligned}$$

this implies $f_\# \circ X^{-1} = X^{-1} \circ f$. Thus set $rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$ is a subgroup of $\Gamma^0(\bar{\pi}_1 M)$. □

Remark 5.12. *For the $Id \in Homeo(M)$ then $rel_{\Gamma^0(\bar{\pi}_1 M)}(Id) = \Gamma^0(\bar{\pi}_1 M)$.*

Definition 5.13. *Let $\mathcal{H} \subset Homeo(M)$ then the set of all sections $X \in \Gamma^0(\bar{\pi}_1 M)$ such that $f_\# \circ X = X \circ f$, for all $f \in \mathcal{H}$, is called $\mathcal{H}$-related sections or $\mathcal{H}$-invariant sections, denoted by $rel_{\Gamma^0(\bar{\pi}_1 M)}(\mathcal{H})$. Explicitly $rel_{\Gamma^0(\bar{\pi}_1 M)}(\mathcal{H}) = \{X \in \Gamma^0(\bar{\pi}_1 M) : f_\# \circ X = X \circ f$, for all $f \in \mathcal{H}\}$.*

For a given non-empty subset $\mathcal{H}$ of homeomorphism group on M the set $rel_{\Gamma^0(\bar{\pi}_1 M)}(\mathcal{H})$ is always non-empty, due to $f_\# \circ O = O \circ f$, for all $f \in \mathcal{H}$, therefore $O \in rel_{\Gamma^0(\bar{\pi}_1 M)}(\mathcal{H})$.

Proposition 5.14. *For a given $\mathcal{H} \subset Homeo(M)$ the set $rel_{\Gamma^0(\bar{\pi}_1 M)}(\mathcal{H})$ is a subgroup of $\Gamma^0(\bar{\pi}_1 M)$.*

Proof. This is similar to Proposition 5.11. (Indeed, it is an intersection of all subgroups $rel_{\Gamma^0(\bar{\pi}_1 M)}(f)$, where $f \in \mathcal{H}$). □

Remark 5.15. $rel_{\Gamma^0(\bar{\pi}_1 M)}(Homeo(M)) = \{O\}$.

Here there are some questions we will try to answer in future works,
i) For every non-trivial proper subgroup $\mathcal{B}$ of $\Gamma^0(\bar{\pi}_1 M)$ is $rel_{Homeo(M)}(\mathcal{B})$ always equal to a non-trivial subgroup of homeomorphisms that yield by a group action?

ii) For every non-trivial proper subgroup $\mathcal{H}$ of $Homeo(M)$ for which $rel_{\Gamma^0(\bar{\pi}_1 M)}(\mathcal{H})$ is it always a group which is isomorphic to a subgroup of the fundamental group of a point in M?
We are working on rel operations in both senses and formulating results by iteratively operating from rel. We will come up with results to the posed issues in a forthcoming research paper.

6. Conclusion

The Core fudamental groupoid is an algebraic structure on topological spaces and it is a sufficient topological invariant [2]. In general, the Core fundamental groupoid admits bundle structures but not fibre bundles. If the fundamental group of a topological space is isomorphic to any two points in the space, then its Core fundamental groupoid forms a fibre bundle. This is an important result that helps to conclude whether a space is a topological group or not. Further, the relatedness is a more general notion to the left-invariant vector fields in differential geometry. The relatedness in both senses is interrelated, and they absorb some special class (we will discuss them in a future paper) of continuous sections of the Core fundamental groupoid bundle when a topological space admits a group action. The Core fundamental groupoids of topological spaces are key to conclude the existence of regular group action and the existence of topological group structure on them.

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