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Similarity Measures based on the Novel Interval-valued Picture Hesitant Fuzzy Sets and their Applications in Pattern Recognition

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1. Introduction

In [62] defined the notion of fuzzy set (FS), which is based on a characteristic mapping having the grade membership for the element belonging to $[0, 1]$. In FS the grade of non-membership can also be calculated by the subtracting the membership grade from 1. To extend the notion of FS, Atanassov [2] gave the idea of the intuitionistic FS (IFS) by adding an additional grade of non-membership for an object with the condition that the sum of both grades should be from the interval $[0, 1]$. In the FS and IFS there are only single values for both grades and consequently the researchers extended the framework of the IFS to the interval-valued IFS (IvIFS). In mathematics the FS theory has great importance and attracted researchers to itself. Hence there are many developments of FS and IFS that have been introduced by them. Gorza l czany [11] introduced the idea of the IvFS and developed some operation for the numbers belonging to the IvFS. The element of IvFS has the characteristic mapping having grade in the form of interval from 0 to 1. In 1989, Atanassov and Gargov developed the concept of IvIFS [3] and also described some basic operations and their properties of IvIFS. Which has two functions, grade of membership and grade of non-membership whose values are intervals and it must be contained on the interval from 0 to 1. Further the sum of grade of membership and grade of non-membership must be contained on the interval from 0 to 1. Therefore, IvFS and IvIFS are the generalization of the notion of FS and IFS. Moreover, some novel interactive hybrid weighted was introduced by Li et al. [22] by discussing its application in the decision making. In [8] Garg introduced some aggregation operators by using trigonometric operation for q-rung orthopair fuzzy set. While an algorithm for selecting the anti virus mask for COVID-19 was developed by Yang et al. [60] by using the Spherical fuzzy set. Some interesting literature can be found in the [21, 9, 16, 10, 17, 59, 54, 14]. Because of some restrictions on the notion of FS and IFS, Cuong introduced the concept of PFS [5] such as in the human opinions where as an IFS can be only phenomena of yes or no types. But on the other hand PFS can describe the phenomena having four types, i.e., right, wrong, abstinence and refusal. There are three types of the mappings in PFS for an object to show its membership, abstinence and the non-membership to the set belonging from the interval from 0 to 1. In case of PFS the sum of grade of membership, grade of neutral membership and grade of non-membership must belong on the interval from 0 to 1. Thus PFS is the direct generalization in References [1,2]. Further we know that in case of PFS all functions are as single values. So due to these limitations Cuong defined the concept of IvPFS [6] and also introduced some basic operations of IvPFS. Similarly IvPFS has also three types functions, such as grade of membership, grade of neutral membership and grade of non-membership whose values are intervals and it must be contained on the interval from 0 to 1. Moreover, the sum of grade of membership, grade of neutral membership and grade of non-membership must be contained on the interval from 0 to 1. Therefore, IvPFS is the extension of the idea of PFS. The related literature can be found in [52] to [23]. The notion of the hesitant fuzzy set (HFS) was developed by the Torra and Narukawa [25] in 2009. In the HFS the membership mapping gives us a subset having elements from 0 to 1 when it is applied to the element of the universal set. Hence we may say that the HFS is the generalized form of the FS. Due to this characteristics the HFS has been widely used by researchers. For example in [26] Ullah et al. developed bipolar HFS and then Mahmood et al. [27] some aggregation operators by using cubic HFS with its application in decision making. Similarly, the HFS has been used by Alcantud et al. [1] and developed some theorems and extension peinciples. Further the interval-valued neutrosophic HF Einstein Choquet integral operator was developed by Kakati et al. [20]. Some interesting work on HFS can be found in the [28, 29, 30]. The idea of interval-valued Pythagorean HFS has developed an applied to the decision making by Zhang et al. in [63]. Some of the problems in the real world could not be specified by on the fuzzy framework. The intuitionistic HFS (IHFS) and its application in decision making have been discussed in [31]. Similarly, the idea of IvHFS has been proposed by the Farhadinia in [32]. Further the interval-valued IHFS has been developed by Zhang in [33] and then picture HFS has described and applied in the decision making by Wang and Li in [34]. Some remarkable literature can be found from[35] to [38]. The interesting and the important topic that describes the grades of thr similarity of the elements is SM. SM has vast application in real life problems like diagnosis problems, recognition of patterns, clustering etc. The concept of entropy and SM for the IvFS was developed by Zeng et al.

in [39]. In [40] applied the concept of IvHFS to the decision making by using the concept of the SM. The application of the SM in the pattern recognition for the framework of the IvIFS has been discussed in [41]. Similarly, some application of SM in the decision making problems for the framework of IvIFS by Liu et al. [42]. In addition, the SM for T-Spherical FS and its application is the pattern recognition have been discussed in [43] and Chen et al. [44] introduced distance and SM for IHFSs while Zhai et al. [45] proposed measures of probabilistic IvIHFSs and their applications in reducing excessive medical examinations. Recently, Ahmad et al. [46] developed the concept of SM for PHFSs and studied their applications in pattern recognition. Saikia et al. [51] investigated some intuitionistic hesitant fuzzy distance measure for MADM. Krishankumar et al. [47] studied MADM problem using double hierarchy in hesitant fuzzy linguistic environment while Garg and Arora [48] developed some McLaurin symmetric mean operators in using dual hesitant fuzzy soft sets. The interesting literature can be founf in [49, 50, 4, 56, 18, 61, 7, 53, 19, 15, 55]. In this article, we observed that there are some restrictions on the SM developed for PFSs in [57] and could not applicable to the information in the form of IvPHFSs. To deal with this limitations, we introduced the concept of IvPHFS and proposed some new SM such as cosine SM, grey SM and set-theoretic SM for IvPHFSs. Further some weighted SM are also introduced where weight of the attributes are considered. The SM in [57] become the special cases of the proposed SM. This manuscript has 8 different sections. In section first, we discussed the existing concepts in details. In section two, we studied some basic definitions of IvFS, IvIFS, IvPFS, IvHFS and IvIHFS. In section three, we proposed the concept of IvPHFS along with some basic operations and remarks. In section four, we introduced some new SM for IvPHFSs which are based on the existing SM of PFSs. Section five is based on the applications of SM of IvPHFSs in a building material recognition problem. In section six, the comparative study of the proposed work is established. In section seven, some advantages of the new work are discussed. Finally, the article ends with some future directions and conclusive remarks.

2. Preliminaries

In this section, we study some basic definitions and notions related to IvFS, IvIFS, IvPFS, IvHFS and IvIHFS. The means of X in this study is the universal set and of π, α and η denote the membership degree, neutral degree and non-membership degree for each element of the universal set X, whose values are intervals and it always contained on the interval from 0 to 1. Further we denote that $\pi^{-}, \alpha^{-}, \eta^{-}$ and $\pi^{+}, \alpha^{+}, \eta^{+}$ the lower and upper limit of π, α, η for each element of the universal set X respectively.

2.1. **Definition**[11]**.** An IvFS P on X is of the shape $P = \{\langle X, \pi_P(e)\rangle | e \in X\}$, where $\pi_P(e) = \left[\pi_P^{-}(e), \pi_P^{+}(e)\right] \subset [0,1]$ provided that $0 \leq \pi_P^{+}(e) \leq 1$ for all elements $e \in X$ and is called grade of membership of the element $s \in X$ to P. Moreover, (π) is said to be IvFN.

2.2. **Definition** [3]**.** An IvIFS P on X is of the form $P = \{\langle X, \pi_P(e), \eta_P(e)\rangle \mid sinX\}$, where $\pi_P(e) = \left[\pi_P^{-}(e), \pi_P^{+}(e)\right] \subset [0,1]$ and $\eta_P(e) = \left[\eta_A^{-}(e), \eta_P^{+}(e)\right] \subset [0,1]$ provided that $0 \leq \pi_P^{+}(e) + \eta_P^{+}(e) \leq 1$ for all $e \in X$ and are called grade of membership and grade of non-membership of $e \in X$ to P. Moreover, (π, η) is said to be IvIFN.

2.3. **Definition**[6]. An IvPFS P on X is of the form $P = \{< X, \pi_P(e), \alpha_P(e), \eta_P(e) > |e \in X\}$, where $\pi_P(s) = \left[\pi_P^-(e), \pi_P^+(e)\right] \subset [0,1], \alpha_P(e) = \left[\alpha_P^-(e), \alpha_P^+(e)\right] \subset [0,1]$ and $\eta_P(e) = \left[\eta_P^-(e), \eta_P^+(e)\right] \subset [0,1]$ given $0 \leq \pi_P^+(e) + \alpha_P^+(e) + \eta_P^+(e) \leq 1$ for all $e \in X$ and are called grade of membership, degree of neutral membership and grade of non-membership of the element $e \in X$ to P. Moreover, (π, α, η) is said to be IvPFN.

2.4. **Definition** [32]. An IvHFS P on X is of the form $P = \{\langle X, h_P(e)\rangle |e \in X\}$, where $h_P(e)$ is a set of some different interval values in $[0,1]$, denoting the grade of membership of the element $e \in X$ to P. Moreover, $h_P(e)$ is called interval-valued hesitant fuzzy number (IvHFN).

2.5. **Definition** [33]. An IvIHFS P on X is of the form $P = \{\langle X, h_P(e)\rangle |e \in X\}$, where $h_P(e)$ is a set of IvIFNs of some different interval values in $[0,1]$, denoting the grade of membership and grade of non-membership of the element $e \in X$ to P. Moreover, $h_P(e)$ is called interval-valued intuitionistic hesitant fuzzy number (IvIHFN).

2.6. **Definition** [16]. Let $[s_a, t_a]$, $[s_b, t_b] \in [0,1]$. Then we defined as:

a: $[s_a, t_a] \leq [s_b, t_b]$, iff $s_a \leq s_b, t_a \leq t_b$
b: $[s_a, t_a] \preceq [s_b, t_b]$, iff $s_a \preceq t_b, t_a \succeq t_b$
c: $[s_a, t_a] = [s_b, t_b]$, iff $s_a = s_b, t_a = t_b$

3. Interval-valued Picture Hesitant Fuzzy Sets

The aim of this section is to present the notion of IvPHFS as a generalization of IvIHFS. Some basic operations on IvPHFSs are also described and their results are studied. An IvPHFS has three types of functions, grade of membership denoted by π, grade of neutral membership denoted by α as well as grade of non-membership η for each element of the universal set X whose values are closed subintervals of [0, 1]. Further we establish that π^-, α^-, η^- denotes the lower limit of π, α, η and π^+, α^+, η^+ denotes the upper limit of π, α, η for each element of the universal set X respectievly. Moreover, IvPHFS(X) denote the set of all IvPHFSs of the universal set X respectively. The proposed concept of IvPHFS and their basic operations are demonstrated with the help of some examples and with the help of some remarks we prove that IvPHFS is the generalization of FS, IFS, PFS, IvHFS and IvIHFS.

3.1. **Definition.** An IvPHFS P on X is of the form $P = \{\langle X, h_P(e)\rangle |e \in X\}$, where $h_P(e)$ is a set of IvPFNs of some different interval values in $[0,1]$, denoting the grade of membership, grade of neutral and grade of non-membership of the element $e \in X$ to P. Moreover, $h_P(e)$ is called interval-valued picture hesitant fuzzy number (IvPHFN).

3.2. **Remark.** In definition 3.1, when

(1) $\alpha_P(e) = [0,0]$ for all $e \in X$. Then IvPHFS becomes to IvIHF
(2) $\alpha_P(e) = \eta_P(e) = [0,0]$ for all $e \in X$. Then IvPHFS becomes to IvHFS
(3) $\pi_P^-(e) = \pi_P^+(e), \alpha_P^-(e) = \alpha_P^+(e), \eta_P^-(e) = \eta_P^+(e)$ for all $s \in X$. Then IvPHFS becomes to PFS
(4) $\pi_P^-(e) = \pi_P^+(e)$ and $\eta_P^-(e) = \eta_P^+(e)$ and $\alpha_P(e) = [0,0]$ for all $e \in X$. Then IvPHFS becomes to IFS

(5) $\pi_P^-(e) = \pi_P^+(e)$ and $\alpha_P(e) = \eta_P(e) = [0,0]$ for all $e \in X$. Then IvPHFS becomes to FS

This remark clarifies that IvIHFS, IvHFS, PFS, IFS and FS are the special cases of IvPHFS. Another useful generalization in the literature is interval valued neutrosophic hesitant fuzzy set (IvNHFS) [20, 28] which also has grade of membership, grade of neutral membership and grade of non-membership. The main difference these two concepts is that in case of IvPHFS, we do have a refusal degree while in case of IvNHFS there is no concept of refusal degree. This addition of refusal degree drastically affects the results which explained in detail in the PhD thesis of Ullah [54].

3.3. **Example.** Let $X = \{e_1, e_2, e_3, e_4\}$. Then an IvPHFS P on X is defined as:

$$P = \left\{ \begin{array}{c} (e_1, [0.10, 0.15], [0.12, 0.20], [0.30, 0.50]), \\ (e_2, [0.11, 0.13], [0.30, 0.37], [0.20, 0.50]), \\ (e_3, [0.25, 0.30], [0.35, 0.50], [0.12, 0.20]), \\ (e_4, [0.25, 0.35], [0.30, 0.45], [0.12, 0.15]) \end{array} \right\}$$

3.4. **Definition.** For two IvPHFNs $P = (\pi_P, \alpha_P, \eta_P)$ and $Q = (\pi_Q, \alpha_Q, \eta_Q)$, we have
1. $P \subseteq Q$ iff $\pi_P^-(e) \leq \pi_Q^-(e), \pi_P^+(e) \leq \pi_Q^+(e), \alpha_P^-(e) \leq \alpha_Q^-(e), \alpha_P^+(e) \leq \alpha_P^+(e)$ and $\eta_P^-(e) \geq \eta_Q^-(e), \eta_P^+(e) \geq \eta_Q^+(e), \forall e \in X$,
2. $P = Q$ iff $P \subseteq Q$ and $Q \subseteq P$,
3. $$P \cup Q = \left\{ \left\langle \begin{array}{c} e, \left[\begin{array}{c} \max\left(\pi_P^-(e), \pi_Q^-(e)\right), \\ \max\left(\pi_P^+(e), \pi_Q^+(e)\right) \end{array} \right], \left[\begin{array}{c} \min\left(\alpha_P^-(e), \alpha_Q^-(e)\right), \\ \min\left(\alpha_P^+(e), \alpha_Q^+(e)\right) \end{array} \right] \\ , \left[\begin{array}{c} \min\left(\eta_P^-(e), \eta_Q^-(e)\right), \\ \min\left(\eta_P^+(e), \eta_Q^+(e)\right) \end{array} \right] \end{array} \right\rangle | s \in X \right\},$$
4. $$P \cap Q = \left\{ \left\langle \begin{array}{c} e, \left[\begin{array}{c} \min\left(\pi_P^-(e), \pi_Q^-(e)\right), \\ \min\left(\pi_P^+(e), \pi_Q^+(e)\right) \end{array} \right], \left[\begin{array}{c} \min\left(\alpha_P^-(e), \alpha_Q^-(e)\right), \\ \min\left(\alpha_P^+(e), \alpha_Q^+(e)\right) \end{array} \right] \\ , \left[\begin{array}{c} \max\left(\eta_P^-(e), \eta_Q^-(e)\right), \\ \max\left(\eta_P^+(e), \eta_Q^+(e)\right) \end{array} \right] \end{array} \right\rangle | e \in X \right\},$$
5. $P^c = \{\langle e, \eta_P(e), \alpha_P(e), \pi_P(e) \rangle | e \in X\}$

3.5. **Example.** Let $P = \{(e_1, [0.10, 0.30], [0.30, 0.40], [0.10, 0.20]), (e_2, [0.13, 0.17], [0.22, 0.27], [0.30, 0.40])\}$ and $Q = \{(e_1, [0.00, 0.20], [0.10, 0.20], [0.30, 0.60]), (e_2, [0.20, 0.25], [0.12, 0.30], [0.35, 0.45])\}$ be the two IvPHFNs. Then
1. $P^c = \{([0.10, 0.20], [0.30, 0.40], [0.10, 0.30]), ([0.30, 0.40], [0.22, 0.27], [0.13, 0.17])\}$
2. $P \cup B = \{([0.10, 0.30], [0.10, 0.20], [0.10, 0.20]), ([0.20, 0.25], [0.12, 0.27], [0.30, 0.40])\}$
3. $P \cap Q = \{([0.00, 0.20], [0.10, 0.20], [0.0.30, 0.60]), ([0.13, 0.17], [0.12, 0.27], [0.35, 0.45])\}$

4. Similarity Measures for IvPHFSs

The aim of this section is to develop some SM for IvPHFSs as generalization of SM of PFSs. In our study we denote the set of all IvPHFNs on the universal set X by IvPHFS(X).

Further with the help of some remarks we developed the SM for IvIHFSs. Moreover, we identify the concepts of SM for IvPHFS are demonstrated by the help of some examples.

4.1. **Cosine Similarity Measures.** In this portion, we shall propose some SM of IvPHFSs which are generalizations of corresponding work on PFS [57].

4.1.1. ***Definition.*** For $P, Q \in$IvPHFS(X), we define the cosine SM as:

$$C^1_{IvPHFS}(P,Q) = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{\begin{array}{c}\pi^-_P(e_i)\pi^-_Q(e_i) + \pi^+_P(e_i)\pi^+_Q(e_i) + \alpha^-_P(e_i)\alpha^-_Q(e_i) + \alpha^+_P(e_i)\alpha^+_Q(e_i)\\ +\eta^-_P(e_i)\eta^-_Q(e_i) + \eta^+_P(e_i)\eta^+_Q(e_i)\end{array}}{\sqrt{\begin{array}{c}(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2 +\\ (\alpha^-_P(e_i))^2 + (\alpha^+_P(e_i))^2 +\\ (\eta^-_P(e_i))^2 + (\eta^+_P(e_i))^2\end{array}} \cdot \sqrt{\begin{array}{c}\left(\pi^-_Q(e_i)\right)^2 + \left(\pi^+_Q(e_i)\right)^2 +\\ \left(\alpha^-_Q(e_i)\right)^2 + \left(\alpha^+_Q(e_i)\right)^2 +\\ \left(\eta^-_Q(e_i)\right)^2 + \left(\eta^+_Q(e_i)\right)^2\end{array}}} \right) \quad (1)$$

The cosine SM for IvPHFSs satisfies the following conditions of SM :

(1) $0 \leq C^1_{IvPHFS}(P,Q) \leq 1$
(2) $C^1_{IvPHFS}(P,Q) = C^1_{IvPHFS}(Q,P)$
(3) $C^1_{IvPHFS}(P,Q) = 1$ iff $P = Q, i = 1,2,3,...,n$
(4) $P \subseteq Q \subseteq C$, then $C^1_{IvPHFS}(P,C) \leq C^1_{IvPHFS}(P,Q)$ and $C^1_{IvPHFS}(P,C) \leq C^1_{IvPHFS}(Q,C)$

Proof. The proof of first two conditions is obvious. For condition no. (3), let A,B$\in$IvPHFS(X). When $P = Q$ that is $\pi_P(e_i) = \pi_Q(e_i), \alpha_P(e_i) = \alpha_Q(e_i), \eta_P(e_i) = \eta_Q(e_i)$ this implies that $\pi^-_P(e_i) = \pi^-_Q(e_i), \pi^+_P(e_i) = \pi^+_Q(e_i), \alpha^-_P(e_i) = \alpha^-_Q(e_i), \alpha^+_P(e_i) = \alpha^+_Q(e_i), \eta^-_P(e_i) = \eta^-_Q(e_i), \eta^+_P(e_i) = \eta^+_Q(e_i)$ for i=1,2,3 . . . ,n. Hence from equation (1), we know that

$$C^1_{IvPHFS}(P,Q) = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{\begin{array}{c}\pi^-_P(e_i)\pi^-_P(e_i) + \pi^+_P(e_i)\pi^+_P(e_i) + \alpha^-_P(e_i)\alpha^-_P(e_i) +\\ \alpha^+_P(e_i)\alpha^+_P(e_i) + \eta^-_P(e_i)\eta^-_P(e_i) + \eta^+_P(e_i)\eta^+_P(e_i)\end{array}}{\sqrt{\begin{array}{c}(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2\\ + (\alpha^-_P(e_i))^2 + (\alpha^+_P(e_i))^2\\ + (\eta^-_P(e_i))^2 + (\eta^+_P(e_i))^2\end{array}} \cdot \sqrt{\begin{array}{c}(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2\\ + (\alpha^-_P(e_i))^2 + (\alpha^+_P(e_i))^2\\ + (\eta^-_P(e_i))^2 + (\eta^+_P(e_i))^2\end{array}}} \right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\begin{array}{c}\left(\pi_P^-(E_i)\right)^2+\left(\pi_P^+(e_i)\right)^2+\left(\alpha_P^-(e_i)\right)^2+\\ \left(\alpha_P^+(e_i)\right)^2+\left(\eta_P^-(e_i)\right)^2+\left(\eta_P^+(e_i)\right)^2\end{array}}{\sqrt{\begin{array}{c}\left(\pi_P^-(e_i)\right)^2+\left(\pi_P^+(e_i)\right)^2\\ +\left(\alpha_P^-(e_i)\right)^2+\left(\alpha_P^+(e_i)\right)^2\\ +\left(\eta_P^-(e_i)\right)^2+\left(\eta_P^+(e_i)\right)^2\end{array}}\cdot\sqrt{\begin{array}{c}\left(\pi_P^-(e_i)\right)^2+\left(\pi_P^+(e_i)\right)^2\\ +\left(\alpha_P^-(e_i)\right)^2+\left(\alpha_P^+(e_i)\right)^2\\ +\left(\eta_P^-(e_i)\right)^2+\left(\eta_P^+(e_i)\right)^2\end{array}}}\right)$$

$$= 1$$

The fourth condition is obvious as geometrically, the angle of P, C is greater than that of P, Q and Q, C. Hence therefore, $C^1_{IvPHFS}(P, C) \leq C^1_{IvPHFS}(P, Q)$ and $C^1_{IvPHFS}(P, C) \leq C^1_{IvPHFS}(Q, C)$ □

4.1.2. ***Definition***. For $P, Q \in$IvPHFS(X), we define the weighted cosine SM as:

$$W^1_{IvPHFS}(P, Q)$$

$$= \sum_{i=1}^{n} w_i \left(\frac{\begin{array}{c}\pi_P^-(e_i)\,\pi_Q^-(e_i)+\pi_P^+(e_i)\,\pi_Q^+(e_i)+\alpha_P^-(e_i)\,\alpha_Q^-(e_i)+\\ \alpha_P^+(e_i)\,\alpha_Q^+(e_i)+\eta_P^-(e_i)\,\eta_Q^-(s_i)+\eta_P^+(s_i)\,\eta_B^+(e_i)\end{array}}{\sqrt{\begin{array}{c}\left(\pi_P^-(e_i)\right)^2+\left(\pi_P^+(e_i)\right)^2\\ +\left(\alpha_P^-(e_i)\right)^2+\left(\alpha_P^+(e_i)\right)^2\\ +\left(\eta_P^-(e_i)\right)^2+\left(\eta_P^+(e_i)\right)^2\end{array}}\cdot\sqrt{\begin{array}{c}\left(\pi_Q^-(e_i)\right)^2+\left(\pi_Q^+(e_i)\right)^2\\ +\left(\alpha_Q^-(e_i)\right)^2+\left(\alpha_Q^+(e_i)\right)^2\\ +\left(\eta_Q^-(e_i)\right)^2+\left(\eta_Q^+(e_i)\right)^2\end{array}}}\right) \quad (2)$$

By taking $w_i = \frac{1}{n}$ the equation (2) reduces to equation (1). The weighted cosine SM for IvPHFSs satisfies the properties of SM as follows:

(1) $0 \leq W^1_{IvPHFS}(P, Q) \leq 1$
(2) $W^1_{IvPHFS}(P, Q) = W^1_{IvPHFS}(Q, P)$
(3) $W^1_{IvPHFS}(P, Q) = 1$ iff $P = Q, i = 1, 2, 3..., n$

Proof. Proof is straightforward. □

4.1.3. ***Remark***. The definition 4.1.1 reduces to cosine SM of IvIHFS, if we assume that $\alpha_P = \alpha_Q = [0, 0]$ and we write it as:

$$C^{1}_{IvIHFS}\left(P,Q\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\begin{array}{c}\pi^{-}_{P}\left(e_i\right)\pi^{-}_{Q}\left(e_i\right)+\pi^{+}_{P}\left(e_i\right)\pi^{+}_{Q}\left(e_i\right)\\+\eta^{-}_{P}\left(e_i\right)\eta^{-}_{Q}\left(e_i\right)+\eta^{+}_{P}\left(e_i\right)\eta^{+}_{Q}\left(e_i\right)\end{array}}{\sqrt{\begin{array}{c}\left(\pi^{-}_{P}\left(e_i\right)\right)^2+\left(\pi^{+}_{P}\left(e_i\right)\right)^2\\+\left(\eta^{-}_{P}\left(e_i\right)\right)^2+\left(\eta^{+}_{P}\left(e_i\right)\right)^2\end{array}}\cdot\sqrt{\begin{array}{c}\left(\pi^{-}_{Q}\left(s_i\right)\right)^2+\left(\pi^{+}_{Q}\left(e_i\right)\right)^2\\+\left(\eta^{-}_{Q}\left(e_i\right)\right)^2+\left(\eta^{+}_{Q}\left(e_i\right)\right)^2\end{array}}}\right) \quad (3)$$

4.1.4. ***Example***. Let $P = \{(e_1, [0.00, 0.10], [0.10, 0.20], [0.20, 0.30]), (e_2, [0.10, 0.20], [0.20, 0.40], [0.30, 0.40]), (e_3, [0.20, 0.50], [0.00, 0.10], [0.20, 0.40])\}$ and $Q = \{(e_1, [0.20, 0.25], [0.25, 0.30], [0.35, 0.45]), (e_2, [0.10, 0.20], [0.40, 0.60], [0.00, 0.20]), (e_3, [0.12, 0.14], [0.30, 0.50], [0.20, 0.26])\}$ be the two IvPHFSs on the universal set $X = \{e_1, e_2, e_3, e_4\}$. Then by using Eq. (1), we get

$$C^{1}_{IvPHFS}\left(P,Q\right) = 0.7841$$

4.2. **Set – Theoretic SM..** In this section, we introduced another kind of SM and weighted SM. The work developed in this section is a generalization of a similar research on PFS proposed in [57].

4.2.1. ***Definition***.
For $P, Q \in$IvPHFS(X), we define set-theoretic SM as:

$$C^{2}_{IvPHFS}\left(A,B\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\begin{array}{c}\pi^{-}_{P}\left(e_i\right)\pi^{-}_{Q}\left(e_i\right)+\pi^{+}_{P}\left(e_i\right)\pi^{+}_{Q}\left(e_i\right)\\+\alpha^{-}_{P}\left(e_i\right)\alpha^{-}_{Q}\left(e_i\right)+\alpha^{+}_{P}\left(e_i\right)\alpha^{+}_{Q}\left(e_i\right)\\+\eta^{-}_{P}\left(e_i\right)\eta^{-}_{Q}\left(e_i\right)+\eta^{+}_{P}\left(e_i\right)\eta^{+}_{Q}\left(e_i\right)\end{array}}{\max\left\{\left(\begin{array}{c}\left(\pi^{-}_{P}\left(e_i\right)\right)^2+\left(\pi^{+}_{P}\left(e_i\right)\right)^2\\+\left(\alpha^{-}_{P}\left(e_i\right)\right)^2+\left(\alpha^{+}_{P}\left(e_i\right)\right)^2\\+\left(\eta^{-}_{P}\left(e_i\right)\right)^2+\left(\eta^{+}_{P}\left(e_i\right)\right)^2\end{array}\right),\left(\begin{array}{c}\left(\pi^{-}_{Q}\left(e_i\right)\right)^2+\left(\pi^{+}_{Q}\left(e_i\right)\right)^2\\+\left(\alpha^{-}_{Q}\left(e_i\right)\right)^2+\left(\alpha^{+}_{Q}\left(e_i\right)\right)^2\\+\left(\eta^{-}_{Q}\left(e_i\right)\right)^2+\left(\eta^{+}_{Q}\left(e_i\right)\right)^2\end{array}\right)\right\}}\right) \quad (4)$$

The set-theoretic SM for IvPHFSs satisfies the following properties of SM:

(1) $0 \leq C^{2}_{IvPHFS}\left(P,Q\right) \leq 1$
(2) $C^{2}_{IvPHFS}\left(P,Q\right) = C^{2}_{IvPHFS}\left(Q,P\right)$
(3) $C^{2}_{IvPHFS}\left(P,Q\right) = 1$ iff $P = Q, i = 1, 2, 3, ..., n$
(4) $P \subseteq Q \subseteq C$, then $C^{2}_{IvPHFS}\left(P,C\right) \leq C^{2}_{IvPHFS}\left(P,Q\right)$ and $C^{2}_{IvPHFS}\left(P,C\right) \leq C^{2}_{IvPHFS}\left(Q,C\right)$.

Proof. Similar □

4.2.2. ***Definition***.

For $P, Q \in$IvPHFS(X), we define the weighted set-theoretic similarity measure as:

$$W^2_{IvPHFS}(P,Q) = \sum_{i=1}^{n} W_i \left(\frac{\begin{array}{c}\pi^-_P(e_i)\pi^-_Q(e_i) + \pi^+_P(e_i)\pi^+_Q(e_i) + \alpha^-_P(e_i)\alpha^-_Q(e_i) + \\ \alpha^+_P(e_i)\alpha^+_Q(e_i) + \eta^-_P(e_i)\eta^-_Q(e_i) + \eta^+_P(e_i)\eta^+_Q(e_i)\end{array}}{\max\left\{\left(\begin{array}{c}(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2 \\ +(\alpha^-_P(e_i))^2 + (\alpha^+_P(e_i))^2 \\ +\left(\eta^-_Q(e_i)\right)^2 + (\eta^+_P(e_i))^2\end{array}\right), \left(\begin{array}{c}\left(\pi^-_Q(e_i)\right)^2 + \left(\pi^+_Q(e_i)\right)^2 \\ +\left(\alpha^-_Q(e_i)\right)^2 + \left(\alpha^+_Q(e_i)\right)^2 \\ +\left(\eta^-_Q(e_i)\right)^2 + \left(\eta^+_Q(e_i)\right)^2\end{array}\right)\right\}} \right) \quad (5)$$

By taking $w_i = \frac{1}{n}$ the equation (5) reduces to equation (4). The weighted set-theoretic SM for IvPHFSs satisfies the results of SM:

(1) $0 \leq W^2_{IvPHFS}(P,Q) \leq 1$
(2) $W^2_{IvPHFS}(P,Q) = W^2_{IvPHFS}(Q,P)$
(3) $W^2_{IvPHFS}(P,Q) = 1$ iff $P = Q, i = 1, 2, 3..., n$

Proof. Proof is straightforward □

4.2.3. ***Remark***.

The definition 4.2.1 reduces to cosine SM of IvIHFS, if we assume that $\alpha_P = \alpha_B = [0, 0]$ and we write it as:

$$C^2_{IvIHFS}(P,Q) = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{\begin{array}{c}\pi^-_P(e_i)\pi^-_Q(e_i) + \pi^+_P(e_i)\pi^+_Q(e_i) \\ +\eta^-_P(e_i)\eta^-_Q(e_i) + \eta^+_P(e_i)\eta^+_Q(e_i)\end{array}}{\max\left\{\left(\begin{array}{c}(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2 \\ +(\eta^-_P(e_i))^2 + (\eta^+_P(e_i))^2\end{array}\right), \left(\begin{array}{c}\left(\pi^-_Q(e_i)\right)^2 + \left(\pi^+_Q(e_i)\right)^2 \\ \left(\eta^-_Q(e_i)\right)^2 + \left(\eta^+_Q(e_i)\right)^2\end{array}\right)\right\}} \right) \quad (6)$$

4.2.4. ***Example***.

Let

$$A = \left\{ \begin{array}{c} \left(\begin{array}{c} e_1, [0.00, 0.10], [0.10, 0.20], \\ [0.20, 0.30] \end{array}\right), \left(\begin{array}{c} e_2, [0.10, 0.20], [0.20, 0.40], \\ [0.30, 0.40] \end{array}\right), \\ \left(\begin{array}{c} e_3, [0.20, 0.50], [0.00, 0.10], \\ [0.20, 0.40] \end{array}\right) \end{array} \right\}$$

and

$$Q = \left\{ \begin{array}{c} \left(\begin{array}{c} e_1, [0.20, 0.25], [0.25, 0.30], \\ [0.35, 0.45] \end{array}\right), \left(\begin{array}{c} e_2, [0.10, 0.20], [0.40, 0.60], \\ [0.00, 0.20] \end{array}\right), \\ \left(\begin{array}{c} e_3, [0.12, 0.14], [0.30, 0.50], \\ [0.20, 0.26] \end{array}\right) \end{array} \right\}$$

be the two IvPHFSs on the universal set $X = \{e_1, e_2, e_3, e_4\}$. Then by using Eq. (4), we get

$$C^2_{IvPHFS}(P, Q) = 0.6189$$

4.3. **Grey SM..**

The work develop in this section is a generalization of PFS which proposed in Reference [57].

4.3.1. ***Definition***.

For P,Q∈IvPHFS(X), we define the grey SM as:

$$C^3_{IvPHFS}(P, Q) = \frac{1}{3n} \sum_{i=1}^{n} \left(\frac{\Delta\pi_{\min} + \Delta\pi_{\max}}{\Delta\pi_i + \Delta\pi_{\max}} + \frac{\Delta\alpha_{\min} + \Delta\alpha_{\max}}{\Delta\alpha_i + \Delta\alpha_{\max}} + \frac{\Delta\eta_{\min} + \Delta\eta_{\max}}{\Delta\eta_i + \Delta\eta_{\max}} \right) \quad (7)$$

where

$$\begin{aligned}
\Delta\pi_i &= \left(\begin{array}{c} \left|\pi^-_P(e_i) - \pi^-_Q(e_i)\right| + \\ \left|\pi^+_P(e_i) - \pi^+_Q(e_i)\right| \end{array} \right), \Delta\pi_{\min} = \min \left\{ \begin{array}{c} \left|\pi^-_P(e_i) - \pi^-_Q(e_i)\right| + \\ \left|\pi^+_P(e_i) - \pi^+_Q(e_i)\right| \end{array} \right\}, \\
\Delta\pi_{\max} &= \max \left\{ \begin{array}{c} \left|\pi^-_P(e_i) - \pi^-_Q(e_i)\right| + \\ \left|\pi^+_P(e_i) - \pi^+_Q(e_i)\right| \end{array} \right\}, \\
\Delta\alpha_i &= \left(\begin{array}{c} \left|\alpha^-_P(e_i) - \alpha^-_Q(e_i)\right| + \\ \left|\alpha^+_P(e_i) - \alpha^+_Q(e_i)\right| \end{array} \right), \Delta\alpha_{\min} = \min \left\{ \begin{array}{c} \left|\alpha^-_P(e_i) - \alpha^-_Q(e_i)\right| + \\ \left|\alpha^+_P(e_i) - \alpha^+_Q(e_i)\right| \end{array} \right\}, \\
\Delta\alpha_{\max} &= \max \left\{ \begin{array}{c} \left|\alpha^-_P(e_i) - \alpha^-_Q(e_i)\right| + \\ \left|\alpha^+_P(e_i) - \alpha^+_Q(e_i)\right| \end{array} \right\}, \\
\Delta\eta_i &= \left(\begin{array}{c} \left|\eta^-_P(e_i) - \eta^-_Q(e_i)\right| + \\ \left|\eta^+_P(e_i) - \eta^+_Q(e_i)\right| \end{array} \right), \Delta\eta_{\min} = \min \left\{ \begin{array}{c} \left|\eta^-_P(e_i) - \eta^-_Q(e_i)\right| + \\ \left|\eta^+_P(e_i) - \eta^+_Q(e_i)\right| \end{array} \right\}, \\
\Delta\eta_{\max} &= \max \left\{ \begin{array}{c} \left|\eta^-_P(e_i) - \eta^-_Q(e_i)\right| + \\ \left|\eta^+_P(e_i) - \eta^+_Q(e_i)\right| \end{array} \right\}.
\end{aligned}$$

Obviously, the grey SM satisfy the following properties:

(1) $0 \leq C^3_{IvPHFS}(P, Q) \leq 1$
(2) $C^3_{IvPHFS}(P, Q) = C^3_{IvPHFS}(Q, P)$
(3) $C^3_{IvPHFS}(P, Q) = 1$ iff $P = Q, i = 1, 2, 3, ..., n$
(4) $P \subseteq Q \subseteq C$, then $C^3_{IvPHFS}(P, C) \leq C^3_{IvPHFS}(P, Q)$ and $C^3_{IvPHFS}(P, C) \leq C^3_{IvPHFS}(Q, C)$

Proof. Similar. □

4.3.2. ***Definition***.
For $P, Q \in$IvPHFS(X), we define the weighted grey SM as:

$$W^3_{IvPHFS}(P,Q) = \frac{1}{3}\sum_{i=1}^{n} w_i \left(\frac{\Delta\pi_{\min} + \Delta\pi_{\max}}{\Delta\pi_i + \Delta\pi_{\max}} + \frac{\Delta\alpha_{\min} + \Delta\alpha_{\max}}{\Delta\alpha_i + \Delta\alpha_{\max}} + \frac{\Delta\eta_{\min} + \Delta\eta_{\max}}{\Delta\eta_i + \Delta\eta_{\max}} \right) \quad (8)$$

where

$$\begin{aligned}
\Delta\pi_i &= \left(\begin{array}{c} \left|\pi_P^-(e_i) - \pi_Q^-(e_i)\right| + \\ \left|\pi_P^+(e_i) - \pi_Q^+(e_i)\right| \end{array} \right), \Delta\pi_{\min} = \min\left\{ \begin{array}{c} \left|\pi_P^-(e_i) - \pi_Q^-(e_i)\right| + \\ \left|\pi_P^+(e_i) - \pi_Q^+(e_i)\right| \end{array} \right\}, \\
\Delta\pi_{\max} &= \max\left\{ \begin{array}{c} \left|\pi_P^-(e_i) - \pi_Q^-(e_i)\right| + \\ \left|\pi_P^+(e_i) - \pi_Q^+(e_i)\right| \end{array} \right\}, \\
\Delta\alpha_i &= \left(\begin{array}{c} \left|\alpha_P^-(e_i) - \alpha_Q^-(e_i)\right| + \\ \left|\alpha_P^+(e_i) - \alpha_Q^+(e_i)\right| \end{array} \right), \Delta\alpha_{\min} = \min\left\{ \begin{array}{c} \left|\alpha_P^-(e_i) - \alpha_Q^-(e_i)\right| + \\ \left|\alpha_P^+(e_i) - \alpha_Q^+(e_i)\right| \end{array} \right\}, \\
\Delta\alpha_{\max} &= \max\left\{ \begin{array}{c} \left|\alpha_P^-(e_i) - \alpha_Q^-(e_i)\right| + \\ \left|\alpha_P^+(e_i) - \alpha_Q^+(e_i)\right| \end{array} \right\}, \\
\Delta\eta_i &= \left(\begin{array}{c} \left|\eta_P^-(e_i) - \eta_Q^-(e_i)\right| + \\ \left|\eta_P^+(e_i) - \eta_Q^+(e_i)\right| \end{array} \right), \Delta\eta_{\min} = \min\left\{ \begin{array}{c} \left|\eta_P^-(e_i) - \eta_Q^-(e_i)\right| + \\ \left|\eta_P^+(e_i) - \eta_Q^+(e_i)\right| \end{array} \right\}, \\
\Delta\eta_{\max} &= \max\left\{ \begin{array}{c} \left|\eta_P^-(e_i) - \eta_Q^-(e_i)\right| + \\ \left|\eta_P^+(e_i) - \eta_Q^+(e_i)\right| \end{array} \right\}.
\end{aligned}$$

where $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weighted vector of $e_i = (i = 1, 2, 3, ..., n)$, with $\sum_{i=1}^{n} w_i = 1$. In particular, if we take $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$. Then the weighted grey SM reduces to grey SM. The weighted grey SM for IvPHFSs satisfies the results of SM as follows:

(1) $0 \le W^3_{IvPHFS}(P,Q) \le 1$
(2) $W^3_{IvPHFS}(P,Q) = W^3_{IvPHFS}(Q,P)$
(3) $W^3_{IvPHFS}(P,Q) = 1$ iff $P = Q, i = 1, 2, 3..., n$

Proof. Proof is straightforward. □

4.3.3. ***Remark***.
The definition 4.3.1 reduces to grey SM of IvIHFS, if we assume that $\alpha_P = \alpha_Q = [0, 0]$ and we write it as:

$$C^3_{IvIHFS}(P,Q) = \frac{1}{3n}\sum_{i=1}^{n} \left(\frac{\Delta\pi_{\min} + \Delta\pi_{\max}}{\Delta\pi_i + \Delta\pi_{\max}} + \frac{\Delta\eta_{\min} + \Delta\eta_{\max}}{\Delta\eta_i + \Delta\eta_{\max}} \right) \quad (9)$$

where

$$\Delta\pi_i = \begin{pmatrix} \left|\pi_P^-(e_i) - \pi_Q^-(e_i)\right| + \\ \left|\pi_P^+(e_i) - \pi_Q^+(e_i)\right| \end{pmatrix}, \Delta\pi_{\min} = \min\left\{ \begin{array}{c} \left|\pi_P^-(e_i) - \pi_Q^-(e_i)\right| + \\ \left|\pi_P^+(e_i) - \pi_Q^+(e_i)\right| \end{array} \right\},$$

$$\Delta\pi_{\max} = \max\left\{ \begin{array}{c} \left|\pi_P^-(e_i) - \pi_Q^-(e_i)\right| + \\ \left|\pi_P^+(e_i) - \pi_Q^+(e_i)\right| \end{array} \right\},$$

$$\Delta\eta_i = \begin{pmatrix} \left|\eta_P^-(e_i) - \eta_Q^-(e_i)\right| + \\ \left|\eta_P^+(e_i) - \eta_Q^+(e_i)\right| \end{pmatrix}, \Delta\eta_{\min} = \min\left\{ \begin{array}{c} \left|\eta_P^-(e_i) - \eta_Q^-(e_i)\right| + \\ \left|\eta_Q^+(e_i) - \eta_Q^+(e_i)\right| \end{array} \right\},$$

$$\Delta\eta_{\max} = \max\left\{ \begin{array}{c} \left|\eta_P^-(e_i) - \eta_Q^-(e_i)\right| + \\ \left|\eta_P^+(e_i) - \eta_Q^+(e_i)\right| \end{array} \right\}.$$

4.3.4. ***Example.***
Let

$$A = \left\{ \begin{array}{c} \begin{pmatrix} e_1, [0.00, 0.10], [0.10, 0.20], \\ [0.20, 0.30] \end{pmatrix}, \begin{pmatrix} e_2, [0.10, 0.20], [0.20, 0.40], \\ [0.30, 0.40] \end{pmatrix}, \\ \begin{pmatrix} e_3, [0.20, 0.50], [0.00, 0.10], \\ [0.20, 0.40] \end{pmatrix} \end{array} \right\}$$

and

$$Q = \left\{ \begin{array}{c} \begin{pmatrix} e_1, [0.20, 0.25], [0.25, 0.30], \\ [0.35, 0.45] \end{pmatrix}, \begin{pmatrix} e_2, [0.10, 0.20], [0.40, 0.60], \\ [0.00, 0.20] \end{pmatrix}, \\ \begin{pmatrix} e_3, [0.12, 0.14], [0.30, 0.50], \\ [0.20, 0.26] \end{pmatrix} \end{array} \right\}$$

be the two IvPHFSs on the universal set $X = \{e_1, s_2, s_3, s_4\}$. Then by using Eq. (7), we get

$$C^3_{IvPHFS}(P, Q) = 0.7959$$

5. Applications

Following, the established SM defined in section 4 are applied to building material recognition is adopt in Reference [57].

5.1. **Building material recognition.**

We calculate the SM for unknown class of the building materials with the help of SM of IvPHFS. In such process we calculate the weighted SM of all known with that of unknown building material. After that we replace that building material with the building material having the greater SM as following:

Algorithm:

(1) We consider the known and unknown building material as a class in the form of IvPHFNs
(2) Calculate SM of each known and unknown material i.e., $P_i\ (1 \leq i \leq 4)$ and P
(3) Rank the SM of all known and unknown building material

(4) Identify the unknown material based on ranking

5.2. **Example.**

Consider four building material stone, steel, brick and muddy, which are represented by the IvPHFSs $P_i\,(1 \leq i \leq 4)$. Let $X = \{e_1, e_2, e_3, e_4, e_5\}$ be the space of attribute have weight $w = (0.17, 0.33, 0.12, 0.28, 0.10)^T$. Table 1 describes the class of unknown and known materials.

We identified that unknown material as follows.

Step 1: Class about unknown and known building material.

	P	P_1	P_2	P_3	P_4
e_1	$([.00, .10], [.10, .20], [.00, .30])$	$([.45, .47], [.15, .20], [.25, .33])$	$([.01, .07], [.10, .20], [.13, .15])$	$([.09, .12], [.00, .20], [.30, .60])$	$([.13, .16], [.20, .25], [.30, .45])$
e_2	$([.20, .30], [.20, .40], [.30, .40])$	$([.23, .28], [.31, .40], [.00, .20])$	$([.20, .25], [.33, .36], [.18, .22])$	$([.10, .50], [.00, .40], [.00, .10]))$	$([.22, .30], [.37, .40], [.20, .25])$
e_3	$([.10, .30], [.30, .50], [.10, .20])$	$([.00, .40], [.20, .40], [.00, .20])$	$([.13, .25], [.16, .20], [.25, .30])$	$([.22, .30], [.25, .40], [.00, .15])$	$([.33, .38], [.40, .42], [.00, .10])$
e_4	$([.00, .20], [.10, .40], [.20, .40])$	$([.51, .55], [.10, .20], [.20, .25])$	$([.15, .30], [.02, .07], [.04, .08])$	$([.13, .18], [.23, .30], [.00, .40])$	$([.16, .20], [.00, .50], [.08, .16])$
e_5	$([.30, .50], [.20, .30], [.00, .10]))$	$([.40, .60], [.00, .20], [.10, .20])$	$([.33, .37], [.48, .50], [.00, .10])$	$([.00, .35], .15, .20], [.20, .40])$	$([.43, .50], [.10, .15], [.20, .30])$

Table 1(SM of P_i with P)

Step 2: Comparison of SM.

SM	(P, P_1)	(P, P_2)	(P, P_3)	(P, P_4)
$W^1_{IvPHFS}(P, P_i)$	0.7773	0.8093	0.8232	0.8945
$W^2_{IvPHFS}(P, P_i)$	0.6157	0.6221	0.6846	0.7836
$W^3_{IvPHFS}(P, P_i)$	0.8539	0.8594	0.7998	0.8631

Table 2 (SM of P_i with P)

Step 3: Ranking of SM.

SM	Ranking of (P, P_i)
$W^1_{IvPHFS}(P, P_i)$	$(P, P_1) < (P, P_2) < (P, P_3) < (P, P_4)$
$W^2_{IvPHFS}(P, P_i)$	$(P, P_1) < (P, P_2) < (P, P_3) < (P, P_4)$
$W^3_{IvPHFS}(P, P_i)$	$(P, P_3) < (P, P_1) < (P, P_2) < (P, P_4)$

Table 3 (Ranking of SM of P_i with A)

Step 4: Upon ranking, it is noted that the SM of (P, P_1) is smaller than all other SM by using cosine SM and set-theoretic SM. However, if we apply grey SM, it seems that (P, P_3) has a smaller value among all other SM.

6. Comparative study

The proposed SM in this article is considered as generalization of SM for IvHFS, IvIHFS, PFSs and IFSs.

The following remarks show that the SM defined in Eq. (1) to Eq. (9) are generalization of SM for IvIHFSs, IvHFSs, PFSs, IFSs and FSs.

6.1. **Remark.**
In definition 4.1.2, when
1. $\alpha_P = \alpha_B = [0,0]$. Then Eq. (2) reduces to SM of IvIHFSs such as:

$$W^1_{IvIHFS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\begin{array}{c}\pi^-_P(e_i)\pi^-_Q(e_i) + \pi^+_P(e_i)\pi^+_Q(e_i) \\ +\eta^-_P(e_i)\eta^-_Q(e_i) + \eta^+_P(e_i)\eta^+_Q(e_i)\end{array}}{\sqrt{\begin{array}{c}(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2 \\ +(\eta^-_P(e_i))^2 + (\eta^+_P(e_i))^2\end{array}} . \sqrt{\begin{array}{c}(\pi^-_Q(e_i))^2 + (\pi^+_Q(e_i))^2 \\ +(\eta^-_Q(e_i))^2 + (\eta^+_Q(e_i))^2\end{array}}} \right) \tag{10}$$

2. If $\alpha_P = \alpha_Q = \eta_A = \eta_Q = [0,0]$. Then Eq. (2) reduces to SM of IvHFSs such as:

$$W^1_{IvHFS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\pi^-_P(e_i)\pi^-_Q(e_i) + \pi^+_P(e_i)\pi^+_Q(e_i)}{\sqrt{(\pi^-_P(e_i))^2 + (\pi^+_P(e_i))^2} . \sqrt{(\pi^-_Q(e_i))^2 + (\pi^+_Q(e_i))^2}} \right) \tag{11}$$

3. If $\pi^-_P = \pi^+_P, \alpha^-_P = \alpha^+_P, \eta^-_P = \eta^+_P$ and $\pi^-_Q = \pi^+_Q, \alpha^-_Q = \alpha^+_Q, \eta^-_Q = \eta^+_Q$. Then Eq. (2) reduces to SM of PFSs such as:

$$W^1_{PFS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\begin{array}{c}\pi_P(e_i)\pi_Q(e_i) + \alpha_P(e_i)\alpha_Q(e_i) \\ +\eta_P(e_i)\eta_Q(e_i)\end{array}}{\sqrt{\begin{array}{c}(\pi_P(e_i))^2 + (\alpha_P(e_i))^2 \\ +(\eta_P(e_i))^2\end{array}} . \sqrt{\begin{array}{c}(\pi_Q(e_i))^2 + (\alpha_Q(e_i))^2 \\ +(\eta_Q(e_i))^2\end{array}}} \right) \tag{12}$$

4. If $\pi^-_P = \pi^+_P, \eta^-_P = \eta^+_P$ and $\pi^-_Q = \pi^+_Q, \eta^-_Q = \eta^+_Q$ and $\alpha_P = \alpha_Q = [0,0]$. Then Eq. (2) reduces to SM of IFSs such as:

$$W^1_{IFS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\pi_P(e_i)\pi_Q(e_i) + \eta_P(e_i)\eta_Q(e_i)}{\sqrt{(\pi_P(e_i))^2 + (\eta_P(e_i))^2} . \sqrt{(\pi_Q(e_i))^2 + (\eta_Q(e_i))^2}} \right) \tag{13}$$

5. If $\pi^-_P = \pi^+_P$ and $\pi^-_Q = \pi^+_Q$ and $\alpha_P = \alpha_Q = \eta_P = \eta_Q = [0,0]$. Then Eq. (2) reduces to SM of FSs such as:

$$W^1_{FS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\pi_P(e_i)\pi_Q(e_i)}{\sqrt{(\pi_P(e_i))^2} . \sqrt{(\pi_Q(e_i))^2}} \right) \tag{14}$$

6.2. **Remark.**
In definition 4.2.2, when
1. $\alpha_P = \alpha_Q = [0, 0]$. Then Eq. (5) reduces to SM of IvIHFSs such as:

$$\begin{aligned} &W^2_{IvIHFS}(P,Q) \\ &= \sum_{i=1}^{n} w_i \left(\frac{\begin{array}{c}\pi_P^-(e_i)\,\pi_Q^-(e_i) + \pi_P^+(e_i)\,\pi_Q^+(e_i) \\ +\eta_P^-(e_i)\,\eta_Q^-(e_i) + \eta_P^+(e_i)\,\eta_Q^+(e_i)\end{array}}{\max\left\{ \left(\begin{array}{c} \left(\pi_P^-(e_i)\right)^2 + \left(\pi_P^+(e_i)\right)^2 \\ +\left(\eta_P^-(e_i)\right)^2 + \left(\eta_P^+(e_i)\right)^2 \end{array} \right), \left(\begin{array}{c} \left(\pi_Q^-(e_i)\right)^2 + \left(\pi_Q^+(e_i)\right)^2 \\ +\left(\eta_Q^-(e_i)\right)^2 + \left(\eta_Q^+(e_i)\right)^2 \end{array} \right) \right\}} \right) \end{aligned} \tag{15}$$

2. If $\alpha_P = \alpha_Q = \eta_P = \eta_Q = [0, 0]$. Then Eq. (5) reduces to SM of IvHFSs such as:

$$W^2_{IvHFS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\pi_P^-(e_i)\,\pi_Q^-(e_i) + \pi_P^+(e_i)\,\pi_Q^+(e_i)}{\max\left\{ \left(\left(\pi_P^-(e_i)\right)^2 + \left(\pi_P^+(e_i)\right)^2 \right), \left(\left(\pi_Q^-(e_i)\right)^2 + \left(\pi_Q^+(e_i)\right)^2 \right) \right\}} \right) \tag{16}$$

3. If $\pi_P^- = \pi_P^+, \alpha_P^- = \alpha_P^+, \eta_P^- = \eta_P^+$ and $\pi_Q^- = \pi_Q^+, \alpha_Q^- = \alpha_Q^+, \eta_Q^- = \eta_Q^+$. Then Eq. (5) reduces to SM of PFSs such as:

$$\begin{aligned} &W^2_{PFS}(P,Q) \\ &= \sum_{i=1}^{n} w_i \left(\frac{\begin{array}{c}\pi_P(e_i)\,\pi_Q(e_i) + \alpha_P(e_i)\,\alpha_Q(e_i) \\ +\eta_P(e_i)\,\eta_Q(e_i)\end{array}}{\max\left\{ \left(\begin{array}{c} (\pi_P(e_i))^2 + (\alpha_P(e_i))^2 \\ +(\eta_P(e_i))^2 \end{array} \right), \left(\begin{array}{c} (\pi_Q(e_i))^2 + (\alpha_Q(e_i))^2 \\ +(\eta_Q(e_i))^2 \end{array} \right) \right\}} \right) \end{aligned} \tag{17}$$

4. If $\pi_P^- = \pi_P^+, \eta_P^- = \eta_P^+$ and $\pi_Q^- = \pi_Q^+, \eta_Q^- = \eta_Q^+$ and $\alpha_P = \alpha_Q = [0, 0]$. Then Eq. (5) reduces to SM of IFSs such as:

$$W^2_{IFS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\pi_P(e_i)\,\pi_Q(e_i) + \eta_P(e_i)\,\eta_Q(e_i)}{\max\left\{ \left((\pi_P(e_i))^2 + (\eta_P(e_i))^2 \right), \left((\pi_Q(e_i))^2 + (\eta_Q(e_i))^2 \right) \right\}} \right) \tag{18}$$

5. If $\pi_P^- = \pi_P^+$ and $\pi_Q^- = \pi_Q^+$ and $\alpha_P = \alpha_B = \eta_P = \eta_Q = [0, 0]$. Then Eq. (5) reduces to SM of FSs such as:

(1) $$W^2_{FS}(P,Q) = \sum_{i=1}^{n} w_i \left(\frac{\pi_P(e_i)\pi_Q(e_i)}{\max\left\{ \left((\pi_P(e_i))^2 \right), \left((\pi_B(e_i))^2 \right) \right\}} \right) \tag{19}$$

6.3. **Remark.**
In definition 4.3.2, when
1. $\alpha_P = \alpha_Q = [0, 0]$. Then Eq. (8) reduces to SM of IvIHFSs such as:

$$W^3_{IvIHFS}(P,Q) = \frac{1}{3} \sum_{i=1}^{n} w_i \left(\frac{\Delta\pi_{\min} + \Delta\pi_{\max}}{\Delta\pi_i + \Delta\pi_{\max}} + \frac{\Delta\eta_{\min} + \Delta\eta_{\max}}{\Delta\eta_i + \Delta\eta_{\max}} \right) \tag{20}$$

where

$$\begin{aligned}
\Delta\pi_i &= \begin{pmatrix} \left|\pi_P^-(e_i)-\pi_Q^-(e_i)\right|+ \\ \left|\pi_P^+(e_i)-\pi_Q^+(e_i)\right| \end{pmatrix}, \Delta\pi_{\min} = \min\begin{Bmatrix} \left|\pi_P^-(e_i)-\pi_Q^-(e_i)\right|+ \\ \left|\pi_P^+(e_i)-\pi_Q^+(e_i)\right| \end{Bmatrix}, \\
\Delta\pi_{\max} &= \max\begin{Bmatrix} \left|\pi_P^-(e_i)-\pi_Q^-(e_i)\right|+ \\ \left|\pi_P^+(e_i)-\pi_Q^+(e_i)\right| \end{Bmatrix}, \Delta\eta_i = \begin{pmatrix} \left|\eta_P^-(e_i)-\eta_Q^-(e_i)\right|+ \\ \left|\eta_P^+(e_i)-\eta_Q^+(e_i)\right| \end{pmatrix}, \\
\Delta\eta_{\min} &= \min\begin{Bmatrix} \left|\eta_P^-(e_i)-\eta_Q^-(e_i)\right|+ \\ \left|\eta_P^+(e_i)-\eta_Q^+(e_i)\right| \end{Bmatrix}, \Delta\eta_{\max} = \max\begin{Bmatrix} \left|\eta_P^-(e_i)-\eta_Q^-(e_i)\right|+ \\ \left|\eta_A^+(e_i)-\eta_Q^+(e_i)\right| \end{Bmatrix}
\end{aligned}$$

2. If $\alpha_P = \alpha_Q = \eta_P = \eta_Q = [0,0]$. Then Eq. (8) reduces to SM of IvHFSs such as:

$$W_{IvHFS}^3(P,Q) = \frac{1}{3}\sum_{i=1}^{n} w_i\left(\frac{\Delta\pi_{\min}+\Delta\pi_{\max}}{\Delta\pi_i+\Delta\pi_{\max}}\right) \tag{21}$$

where

$$\begin{aligned}
\Delta\pi_i &= \begin{pmatrix} \left|\pi_P^-(e_i)-\pi_Q^-(e_i)\right|+ \\ \left|\pi_P^+(e_i)-\pi_Q^+(e_i)\right| \end{pmatrix}, \Delta\pi_{\min} = \min\begin{Bmatrix} \left|\pi_P^-(e_i)-\pi_Q^-(e_i)\right|+ \\ \left|\pi_P^+(e_i)-\pi_Q^+(e_i)\right| \end{Bmatrix}, \\
\Delta\pi_{\max} &= \max\begin{Bmatrix} \left|\pi_P^-(e_i)-\pi_P^-(e_i)\right|+ \\ \left|\pi_P^+(e_i)-\pi_Q^+(e_i)\right| \end{Bmatrix}
\end{aligned}$$

3. If $\pi_P^- = \pi_P^+, \alpha_P^- = \alpha_P^+, \eta_P^- = \eta_P^+$ and $\pi_Q^- = \pi_Q^+, \alpha_Q^- = \alpha_Q^+, \eta_Q^- = \eta_Q^+$. Then Eq. (8) reduces to SM of PFSs such as:

$$W_{PFS}^3(P,Q) = \frac{1}{3}\sum_{i=1}^{n} w_i\left(\frac{\Delta\pi_{\min}+\Delta\pi_{\max}}{\Delta\pi_i+\Delta\pi_{\max}} + \frac{\Delta\alpha_{\min}+\Delta\alpha_{\max}}{\Delta\alpha_i+\Delta\alpha_{\max}} + \frac{\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\eta_i+\Delta\eta_{\max}}\right) \tag{22}$$

where

$$\begin{aligned}
\Delta\pi_i &= (|\pi_P(s_i)-\pi_Q(s_i)|), \Delta\pi_{\min} = \min\{|\pi_P(s_i)-\pi_Q(e_i)|\}, \\
\Delta\pi_{\max} &= \max\{|\pi_P(e_i)-\pi_Q(e_i)|\}, \Delta\alpha_i = (|\alpha_P(e_i)-\alpha_Q(e_i)|), \\
\Delta\alpha_{\min} &= \min\{|\alpha_P(e_i)-\alpha_Q(e_i)|\}, \Delta\alpha_{\max} = \max\{|\alpha_P(e_i)-\alpha_Q(e_i)|\}, \\
\Delta\eta_i &= (|\eta_P(e_i)-\eta_Q(e_i)|), \Delta\eta_{\min} = \min\{|\eta_P(e_i)-\eta_Q(e_i)|\}, \\
\Delta\eta_{\max} &= \max\{|\eta_P(e_i)-\eta_Q(e_i)|\}
\end{aligned}$$

4. If $\pi_P^- = \pi_P^+, \eta_P^- = \eta_P^+$ and $\pi_Q^- = \pi_Q^+, \eta_Q^- = \eta_Q^+$ and $\alpha_P = \alpha_Q = [0,0]$. Then Eq. (8) reduces to SM of IFSs such as:

$$W_{IFS}^3(P,Q) = \frac{1}{3}\sum_{i=1}^{n} w_i\left(\frac{\Delta\pi_{\min}+\Delta\pi_{\max}}{\Delta\pi_i+\Delta\pi_{\max}} + \frac{\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\eta_i+\Delta\eta_{\max}}\right) \tag{23}$$

where

$$\begin{aligned}\Delta\pi_i &= \left(\left|\pi_P\left(e_i\right)-\pi_q\left(e_i\right)\right|\right), \Delta\pi_{\min}=\min\left\{\left|\pi_P\left(e_i\right)-\pi_Q\left(e_i\right)\right|\right\},\\ \Delta\pi_{\max} &= \max\left\{\left|\pi_P\left(e_i\right)-\pi_Q\left(e_i\right)\right|\right\}, \Delta\eta_i=\left(\left|\eta_P\left(e_i\right)-\eta_Q\left(e_i\right)\right|\right),\\ \Delta\eta_{\min} &= \min\left\{\left|\eta_P\left(e_i\right)-\eta_Q\left(e_i\right)\right|\right\}, \Delta\eta_{\max}=\max\left\{\left|\eta_P\left(e_i\right)-\eta_Q\left(e_i\right)\right|\right\}\end{aligned}$$

5. If $\pi_P^- = \pi_P^+$ and $\pi_Q^- = \pi_Q^+$ and $\alpha_P = \alpha_Q = \eta_P = \eta_B = [0, 0]$. Then Eq. (8) reduces to SM of FSs such as:

$$W_{IFS}^3(P,Q)=\frac{1}{3}\sum_{i=1}^{n}w_i\left(\frac{\Delta\pi_{\min}+\Delta\pi_{\max}}{\Delta\pi_i+\Delta\pi_{\max}}\right) \tag{24}$$

where

$$\begin{aligned}\Delta\pi_i &= \left(\left|\pi_A\left(e_i\right)-\pi_Q\left(e_i\right)\right|\right), \Delta\pi_{\min}=\min\left\{\left|\pi_P\left(e_i\right)-\pi_Q\left(e_i\right)\right|\right\},\\ \Delta\pi_{\max} &= \max\left\{\left|\pi_p\left(e_i\right)-\pi_Q\left(e_i\right)\right|\right\}\end{aligned}$$

7. Advantages of Proposed Study

Due some limitations of the existing SM proposed in Reference [57]. The main advantage of the proposed new similarity measures is that these SMs allow us to solve the problems that lie in the environment of IvPHFSs and PFS, HFS, IFS etc. On the other hand the existing SM which proposed in Reference [57] could not handle the problems that lie in the environment of IvPHFSs. If we take Example 5.2, the data is presented in the shape of IvPHFNs which cannot be processed by some existing SM.

Now, if we take the example from Reference [57]. It can be seen that proposed SM successfully solved this problem.

7.1. **Example.**

The data about unknown and known building material in table (4).

	P_1	P_2	P_3	P_4	P
e_1	$(.17, .53, .13)$	$(.51, .24, .21)$	$(.31, .39, .25)$	$(1, 0, 0)$	$(.91, .03, .05)$
e_2	$(.1, .81, .05)$	$(.62, .12, .07)$	$(.60, .26, .11)$	$(1, 0, 0)$	$(.78, .12, .07)$
e_3	$(.53, .33, .09)$	$(1, 0, 0)$	$(.91, .03, .02)$	$(.85, .09, .05)$	$(.90, .05, .02)$
e_4	$(.89, .08, .03)$	$(.13, .64, .21)$	$(.07, .09, .07)$	$(.74, .16, .10)$	$(.68, .08, .21)$
e_5	$(.42, .35, .18)$	$(.03, .82, .13)$	$(.04, .85, .10)$	$(.02, .89, .05)$	$(.05, .87, .06)$
e_6	$(.08, .89, .02)$	$(.73, .15, .08)$	$(.68, .26, .06)$	$(.08, .84, .06)$	$(.13, .75, .09)$
e_7	$(.33, .51, .12)$	$(.52, .31, .16)$	$(.15, .76, .07)$	$(.16, .71, .05)$	$(.15, .73, .08)$

Table 4 (Data of Patterns [61])

The above data in table (4) can be easily changed to the environment of IvPHFSs which is given in table (5).

	P_1	P_2	P_3	P_4	P
e_1	$([.17,.17], [.53,.53], [.13,.13])$	$([.51,.51], [.24,.24], [.21,.21])$	$([.31,.31], [.39,.39], [.25,.25])$	$([1,1], [0,0], [0,0])$	$([.91,.91], [.03,.03], [.05,.05])$
e_2	$([.10,.10], [.81,.81], [.05,.05])$	$([.62,.62], [.12,.12], [.07,.07])$	$([.60,.60], [.26,.26], [.11,.11])$	$([1,1], [0,0], [0,0])$	$([.78,.78], [.12,.12], [.07,.07])$
e_3	$([.53,.53]\ [.33,.33]\ [.09,.09])$	$([1,1], [0,0], [0,0])$	$([.91,.91], [.03,.03], [.02,.02])$	$([.85,.85], [.09,.09], [.05,.05])$	$([.90,.90], [.05,.05], [.02,.02])$
e_4	$([.89,.89], [.08,.08], [.03,.03])$	$([.13,.13], [.64,.64], [.21,.21])$	$([.07,.07], [.09,.09], [.07,.07])$	$([.74,.74], [.16,.16], [.10,.10])$	$([.68,.68], [.08,.08], [.21,.21])$
e_5	$([.42,.42], [.35,.35], [.18,.18])$	$([.03,.03], [.82,.82], [.13,.13])$	$([.04,.04], [.85,.85], [.10,.10])$	$([.02,.02], [.89,.89], [.05,.05])$	$([.05,.05], [.87,.87], [.06,.06])$
e_6	$([.08,.08], [.89,.89], [.02,.02])$	$([.73,.73], [.15,.15], [.08,.08])$	$([.68,.68], [.26,.26], [.06,.06])$	$([.08,.08], [.84,.84], [.06,.06])$	$([.13,.13], [.75,.75], [.09,.09])$
e_7	$([.33,.33], [.51,.51], [.12,.12])$	$([.52,.52], [.31,.31], [.16,.16])$	$([.15,.15], [.76,.76], [.07,.07])$	$([.16,.16], [.71,.71], [.05,.05])$	$([.15,.15], [.73,.73], [.08,.08])$

Table 5(Data of Patterns [61])

Then by using the proposed new SM and the results are showed in table (6).

SM	(P, P_1)	(P, P_2)	(P, P_3)	(P, P_4)
$W^1_{IvPHFS}(P, P_i)$	0.716	0.763	0.858	0.994
$W^2_{IvPHFS}(P, P_i)$	0.556	0.657	0.693	0.920
$W^3_{IvPHFS}(P, P_i)$	0.660	0.762	0.830	0.901

Table 6 (SM of P_i with P)

We can compute the SM of building materials which are presented in table 5, and these SM are same as [57]. In the same way we consider the information in the form of IvIHFS which can be converted in the form of IvPHFS. Then with the help of proposed SM such the information can be processed. Again, if the considered information is in the form of the IvHFS, then this form of information can also be processed easily by changing its form ton the PHFSs. Therefore our claim has proved.

8. Conclusion

In this manuscript, we defined the concept of IvPHFS demonstrated with the help of some remarks and examples. Noticing the shortcomings of previous similarities measures of PFSs, we generalized the concept of those SM to the environment of IvPHFSs. These SM consisting set-theoretic SM, grey SM and cosine SM. The proposed SM are demonstrated with the help of some examples. Moreover, some weighted SM are also described and applied to building material recognition problems and the results are discussed. The proposed new SM are compared with the existing SM and it is stated that previously defined SM become the special cases of proposed SM. The advantages of proposed work over the existing work have also been studied. In near future, the authors aim to develop some entropy measures, correlation coefficients, aggregation operators for the newly developed concept with some operations based on some t-norm and t-conorm in [12, 13].

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