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EXTENDED MITTAG-LEFFLER FUNCTIONS ASSOCIATED WITH WEYL FRACTIONAL CALCULUS OPERATORS

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Abstract.: This article deals with the family of extended Mittag-Leffler function in short ML-function defined in terms of extended Beta function, which depends upon the bounded sequence $\{\kappa_n\}$. The focus of the article is to define integral and differential operators of Weyl-type fractional operators associated with the proposed function. A new fractional calculus integral operator involving extended ML-function is also defined and its composition with the fractional calculus operators and some basic properties studied as well.

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1. INTRODUCTION

First time Swedish Mathematician, Mittag-Leffler [11] defined a function known as Mittag-Leffler by his own name and defined as

$$E_{\xi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\xi k + 1)} \ , \Re(\xi) > 0 \tag{1. 1}$$

This is the generalized form of the exponential function.

Due to vital role of this function in applied sciences, physics, mathematics and engineering, many researchers showed interest in its extension by introducing different parameters for getting more generalized form of the function. See, for example, each of the research monographs [6, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 36] and the references included in these sequels.

Srivastava et al. [35] introduced the function

$$\Theta(\{\kappa_n\}_{n\in N_0};z)=\left\{\begin{array}{ll}\sum_{k=0}^{\infty}\kappa_n\frac{z^n}{n!} & (|z|<\Re;0<\Re<\infty;\kappa_0:=1\\ m_0 z^{\varpi}\exp(z)[1+O(\frac{1}{z})] & (\Re(z)\to\infty;m_0>0;\varpi\in C\end{array}\right\}$$

where m_0 is a constant and the value of ϖ associated with the bounded sequence $\{\kappa_n\}_{n\in N_0}$ of real or complex number and $N_0 = N\bigcup\{0\}$. See, for details, Srivastava et al. [35] and Parmar [19] . Corresponding to the function $\Theta(\{\kappa_n\}_{n\in N_0};z)$ extended form of Gamma function, Beta function and Gauss hypergeometric function were defined. See, for details, Srivastava et al. [35]

The function $\Theta(\{\kappa_n\}_{n\in N_0};z)$ introduced by Srivasatava et al. [35] provides a path to researchers for the extension of number of special functions. Parmar [19] defined extension of generalized ML-function through extended Beta function corresponding to the function $\Theta(\{\kappa_n\}_{n\in N_0};z)$. Thus, the extension of ML-fucntion is defined as

$$E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(z;p)=\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{z^k}{\Gamma(\xi k+\zeta)} \tag{1. 2}$$

$$(\xi,\zeta,\gamma\in C;\Re(\xi)>0,\Re(\zeta)>0,\Re(\gamma)>0;p\geq 0)$$

where $B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}$ is known as extended Beta function and is defined as

$$B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\alpha,\beta;p):=\int_o^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{\kappa_n\}_{n\in N_0};-\frac{p}{t(1-t)}\right)dt$$

$$(\min\{\Re(\alpha),\Re(\beta)\}>0;\Re(p)\geq 0)$$

The above mentioned function $\Theta(\{\kappa_n\}_{n\in N_0};z)$ was systematically investigated by Srivastava et al. [34, 35] and used for the generalization of Mittag-Leffler function. Several further extensions of the function behave like the special cases of the form as in the present sequel.

Remark 1.1. *Different extension of Mittag-Leffler functions depends upon the choice of the sequence* $(\{\kappa_n\}_{n\in N_0})$

(i) In (1. 2), if we select

$$\{\kappa_n\}=1,\quad (n\in N)$$

Then extended function reduces to the form defined by Özarslan and Yilmaz [18] *(with* $c=1$*).*

$$E_{\xi,\zeta}^{\gamma}(z;p)=\sum_{k=0}^{\infty}\frac{B_p(\gamma+k,1-\gamma)}{B(\gamma,1-\gamma)}\frac{z^k}{\Gamma(\xi k+\zeta)} \tag{1. 3}$$

$$(\xi,\zeta,\gamma\in C;\Re(\xi)>0,\Re(\zeta)>0,\Re(\gamma)>1;p\geq 0)$$

(ii) If we select the sequence

$$\{\kappa_n\} = 0, \quad (n \in N)$$

This immediately reduces to Prabhakar's function [21]. *For any $z \in C$ this function is defined as*

$$E^{\gamma}_{\xi,\zeta}(z) = \frac{1}{\Gamma(z)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k) z^k}{k!\Gamma(\xi k+\zeta)}, \quad \xi, \zeta, \gamma \in C \quad \Re(\xi) > 0$$

(iii) For the selection of sequence

$$\{\kappa_n\} = \frac{(\rho)_n}{(\sigma)_n}, \quad (n \in N_0)$$

yields another form of the extended generalized Mittag-Leffler function:

$$E^{(\rho,\sigma);\gamma}_{\xi,\zeta}(z;p) = \sum_{k=0}^{\infty} \frac{B^{(\rho,\sigma)}_p(\gamma+k, 1-\gamma)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k+\zeta)} \tag{1. 4}$$

$$(\xi, \zeta, \gamma \in C; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma) > 1; p \geq 0)$$

By the selection of parameters $\xi = \zeta = 1$ in the equations (1. 2), (1. 3)and (1. 4) this immediately reduces to the extended confluent hypergeometric function [4].

In this article, we use the extension of ML-function defined by Parmar [19] as in (1. 2) in terms of extended Beta function . We investigate the composition of extended ML-function with Weyl fractional operators of integration and differentiation. In the second section, we introduce new integral operator containing extension of ML-function as a kernel and some of its basic properties will be discussed.

2. **Weyl Fractional calculus operators of Integration and Differntiation**

The theory of operators of fractional calculus has been extensively used due to its ability to deal with derivatives and integrals of arbitrary orders. Fractional integral operators containing various special functions, are extensively used in many fields of applied mathematics, physics and engineering like quantum mechanics, plasma, fluid dynamics, astrophysics, control theory and in statistical distribution theory etc. Due to its importance and popularity in recent years, many studies related to the fractional calculus found in papers of Choi and Agarwal [5], Geholt [7], Gupta and Parihar [8], Nadir and Khan [14, 15], Raina [25], Saigo [26], Saigo and Maeda [27], Samko et al. [28], Shishkina and Sitnik [29], Singh [30], Srivastava and Agarwal [33] and Srivastava et al. [37].

The objective of this article is to find the behavior of extended ML-function with Weyl fractional operators of integration and differentiation. Here, in this section, we are defining the class of fractional calculus of Weyl operators, (see, Samko et al. [28]). When the integral is left-sided $\int_a^x (.)$ then the corresponding fractional operators are called left-sided or first kind. When the integral is right-sided $\int_x^b (.)$ then the corresponding fractional integral and fractional derivative are called right-sided or second kind. Here a can be $-\infty$ and b

can be ∞. Thus, I_{0-}^{λ} and D_{0-}^{λ} are called left-sided Weyl integral operator and left-sided Weyl differential operator respectively and are as follows [9, 28].

$$(I_{0-}^{\lambda}\phi)(x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1}\phi(t)dt \qquad (2.\ 5)$$

and

$$(D_{0-}^{\lambda}\phi)(x) = (-1)^m \left(\frac{d}{dx}\right)^m \frac{1}{\Gamma(m-\lambda)} \int_x^{\infty} (t-x)^{m-\lambda-1}\phi(t)dt \qquad (2.\ 6)$$

where $\phi \in L(t,\infty), \lambda \in C$ and $\Re(\lambda) > 0$.

Weyl differential operator can be written in terms of integral operator

$$(D_{0-}^{\lambda}\phi)(x) = (-1)^m \left(\frac{d}{dx}\right)^m \left(I_{0-}^{m-\lambda}\phi\right)(x) \qquad (2.\ 7)$$

Now, we define an integral containing extended class of ML-function in its kernel. This new integral operator depends upon Weyl fractional integral operator.

$$\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\phi\right)(x) = \int_x^{\infty} (t-x)^{\zeta-1} E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\zeta};p]\phi(t)dt \qquad (2.\ 8)$$

where $\xi,\zeta,\gamma,\omega \in C; (\Re(\xi), \Re(\zeta) > 0), \phi(t)$ is such that the integral on right-side, exists. It is clear that when $\omega = 0$ then the proposed integral operator (2. 8) becomes the well-known classical Weyl operator. Bounded conditions of the new integral operator are defined in the space $L(t,\infty)$ of Lebesgue measurable functions on (t,∞).

$$L(t,\infty) = \left(h(x) := \|h\|_1 = \int_t^{\infty} |h(x)|dx < \infty\right) \qquad (2.\ 9)$$

Composition of Weyl fractional calculus with proposed integral operator, defined in (2. 8) is also establised.

3. First kind of Weyl Fractional calculus associated with Extended Mittag-Leffler Function

In this section, we consider differentiation of the family of the function $E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}$. Thus by using (1. 2)and differentiating term by term, we can easily derive

$$\left(\frac{d}{dx}\right)^m x^{\zeta-1} E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega x^{\xi};p) = x^{\xi-m-1} E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega x^{\xi};p) \qquad (3.\ 10)$$

where $\xi,\zeta,\gamma,\omega \in C; \min\{\Re(\xi), \Re(\zeta), \Re(\gamma)\} > 0$

Next, in the following, we determine the composition of a function $E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}$ with fractional integral operator I_{0-}^{λ} and fractional differential Weyl operator D_{0-}^{λ} of arbitrary order defined in (2. 5) and (2. 6). Thus, implementing term-by-term right side and considering the uniform convergence of the series.

Theorem 3.1. *Let $\xi, \zeta, \gamma, \omega, \lambda \in C; \min\{\Re(\xi), \Re(\zeta), \Re(\gamma), \Re(\lambda)\} > 0$, then the following relationship of Weyl fractional calculus formula of integration holds true.*

$$I_{0-}^{\lambda}\left[t^{-\lambda-\zeta}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega t^{-\xi};p)\right](x) = x^{-\xi}E_{\xi,\zeta+\lambda}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega x^{-\xi};p) \tag{3. 11}$$

and under the same conditions, the following relation for Weyl fractional derivative exit.

$$D_{0-}^{\lambda}\left[t^{\lambda-\zeta}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega t^{-\xi};p)\right](x) = x^{-\xi}E_{\xi,\zeta-\lambda}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega x^{-\xi};p) \tag{3. 12}$$

Proof. Using (1. 2),(2. 5) and due to uniform convergence of the series, we have $I_{0-}^{\lambda}\left[t^{-\lambda-\zeta}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega t^{-\xi};p)\right](x)$

$$= \frac{1}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}t^{-\lambda-\xi}\left(\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{\omega^k t^{-\xi k}}{\Gamma(\xi k+\zeta)}\right)dt$$

$$= \sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{\omega^k}{\Gamma(\xi k+\zeta)}\frac{1}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}t^{-\xi k-\lambda-\zeta}dt$$

where order of integration and summation are justified under the stated conditions i.e absolute convergence of the integral involved in any compact set C and the uniform convergence of the series involved.
Put

$$u = \frac{(t-x)}{t}$$

then the intergal becomes the type-2 beta integral and after some simplification, we get the desired result.
Now, we investigate the Weyl fractional derivative of extended Mittag-Leffler function. In an analogous manner, using (1. 2) and (2. 6) and uniform convergence of the series, we have

$$D_{0-}^{\lambda}\left[t^{\lambda-\zeta}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\omega t^{-\xi};p)\right](x) = (-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\lambda)}$$

$$\times\int_x^{\infty}(t-x)^{m-\lambda-1}t^{\lambda-\zeta}\left(\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{\omega^k t^{-\xi k}}{\Gamma(\xi k+\zeta)}\right)dt$$

$$= (-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\lambda)}\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{\omega^k}{\Gamma(\xi k+\zeta)}$$

$$\times\int_x^{\infty}(t-x)^{m-\lambda-1}t^{\lambda-\zeta-\xi k}dt$$

Changing the variable $u = \frac{(t-x)}{t}$, then we have the integral in the form of type-2 beta integral. After some simplification, we get the required result. □

4. Weyl fractional operator of integration associated with extended ML-function

Consider the new Weyl fractional calculus operator of integration containg extended ML-function in the kernel. First of all, we check the bounded condition and some basic properties of teh operator defined in (2. 8) in $L(t,\infty)$.

Theorem 4.1. *Let* $\xi,\zeta,\gamma,\omega\in C;\min\{\Re(\xi),\Re(\zeta),\Re(\gamma)\}>0$,*then for* $\sigma=t+a$, *we have* $\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\sigma^{\mu-1}\right)(x)$

$$=\frac{(x+a)^{\zeta-\mu}}{\Gamma(\mu)}\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}[\omega(x+a)^{\xi}]^k\Gamma(\mu-\xi k-\zeta)$$

Proof. From equation (2. 8) of our proposed integral

$$\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\sigma^{\mu-1}\right)(x)=\int_x^{\infty}(t-x)^{\zeta-1}\sigma^{\mu-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\zeta};p]dt$$

$$=\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{\omega^k}{\Gamma(\xi k+\zeta)}\int_x^{\infty}(t-x)^{\zeta-1}\sigma^{\mu-1}dt$$

By the selection of $u=\frac{t-x}{t+a}$, we approach the relevant result.

□

Theorem 4.2. *Let* $\xi,\zeta,\gamma,\omega\in C;\min\{\Re(\xi),\Re(\zeta),\Re(\gamma)\}>0$, *then* $\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\right)$ *is bounded on the space* $L(t,\infty)$ *and*

$$\|(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi)\|_1\le\wp\|\chi\|_1$$

where the constant $\wp(0<\wp<\infty)$ *is given by*

$$\wp=(b-a)^{\Re(\xi)}\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{|\omega(b-a)^{\xi}|^k}{|\Gamma(\xi k+\zeta)||\Re(\xi)k+\Re(\zeta)|k!} \tag{4. 13}$$

Proof. Let C_k denote the k^{th} term of (4. 13) , then by ratio test [1]

$$\left|\frac{C_{k+1}}{C_k}\right|=\left|\frac{\Gamma(\xi k+\zeta)}{\Gamma(\xi k+\xi+\zeta)}\right|\left|\frac{\Re(\xi)k+\Re(\zeta)}{\Re(\xi)k+\Re(\xi)+\Re(\zeta)}\right||\omega(b-a)^{\Re(\xi)}|$$

$$\approx\frac{|\omega(b-a)^{\Re(\xi)}|}{(|\xi|k)^{\Re(\xi)}}$$

as $k\to\infty$. Hence,$\left|\frac{C_{k+1}}{C_k}\right|\to 0$ as $k\to\infty$ the right side of the expression is convergent and finite under the given conditions.
Now, according to (2. 8), (2. 9) and by using Dirichlet formula, we get

$$\left\|\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)\right\|_1=\int_a^{\infty}\left|\int_x^{\infty}(t-x)^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\xi};p]\chi(t)dt\right|dx$$

$$\le\int_a^{\infty}\left[\int_a^t(t-x)^{\xi-1}\left|E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\xi};p]\right|dx\right]|\chi(t)|\,dt$$

by taking $u = t - x$

$$\left\|\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)\right\|_1 \leq \int_a^\infty \left[\int_x^{t-a} u^{\Re(\zeta)-1}\left|E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega u^\xi;p]\right|du\right]|\chi(t)|\,dt$$

$$\leq \int_a^\infty \left[\int_a^{b-a} u^{\Re(\zeta)-1}\left|E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega u^\xi;p]\right|du\right]|\chi(t)|\,dt$$

Let

$$\int_a^{b-a} u^{\Re(\zeta)-1}\left|E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega u^\xi;p]\right|du = \wp$$

Then

$$\wp = \sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)\left|\omega^k\right|}{B(\gamma,1-\gamma)\left|\Gamma(\xi k+\zeta)\right|}\int_0^{b-a} u^{\Re(\xi)k+\Re(\zeta)-1}du$$

$$= (b-a)^{\Re(\xi)}\sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)\left|\omega(b-a)^{\Re(\xi)}\right|^k}{B(\gamma,1-\gamma)\left|\Gamma(\xi k+\zeta)\right|\left|\Re(\xi)k+\Re(\zeta)\right|}$$

Thus, we have

$$\left\|\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)\right\|_1 \leq \int_a^\infty \wp\,|\chi(t)|\,dt \leq \wp\,\|\chi_1\|$$

□

Now, we determine composition of the integral operator $\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}$ defined in (2. 8) via Weyl fractional operator of integral I_{0-}^λ and the Weyl fractional operator of differentiation D_{0-}^λ.

Theorem 4.3. *Let* $\xi,\zeta,\gamma,\lambda,\omega \in C; \min\{\Re(\xi),\Re(\zeta),\Re(\gamma)\} > 0$, *then*

$$\left(I_{0-}^\lambda \epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x) = \left(\epsilon_{\xi,\zeta+\lambda,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x) = \left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}I_{0-}^\lambda\chi\right)(x)$$

For any arbitrary function $\chi \in L(t,\infty)$. *Under the same condition, following relations holds for the derivative of the integral operator.*

$$\left(D_{0-}^\lambda \epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x) = \left(\epsilon_{\xi,\zeta-\lambda,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x)$$

Proof. By (2. 5) and (2. 8) and using Dirichlet formula (see, for details, Samko et al. [28], we get

$$\left(I_{0-}^\lambda \epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x)$$

$$= \frac{1}{\Gamma(\lambda)}\left(\int_x^\infty (u-x)^{\lambda-1}\left[\int_u^\infty (t-u)^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^\xi]\chi(t)dt\right]\right)du$$

$$= \left(\int_x^\infty \frac{1}{\Gamma(\lambda)}\left[\int_x^t (u-x)^{\lambda-1}(t-u)^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^\xi]du\right]\right)\chi(t)dt$$

changing the variable $\tau = t - u$ then, we have

$$\left(I_{0-}^{\lambda}\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x)$$
$$=\left(\int_x^{\infty}\frac{1}{\Gamma(\lambda)}\left[\int_0^{t-x}(t-x-\tau)^{\lambda-1}\tau^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega\tau^{\xi};p]d\tau\right]\right)\chi(t)dt$$

Now, using the result of Parmar [19] pp-1078 of extended ML-function with the right-sided Riemann-Liouville fractional operator of integration I_{0-}^{λ}, we get

$$\left(I_{0-}^{\lambda}\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x)$$
$$=\int_x^{\infty}I_{0+}^{\lambda}\left[\tau^{\beta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega\tau^{\xi};p]\right](t-x)\chi(t)dt$$
$$=\int_x^{\infty}\left[(t-x)^{\zeta+\lambda-1}E_{\xi,\zeta+\lambda}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\xi};p]\right]\chi(t)dt$$

Thus, we reached the required assertion. To prove the second expression of the right side, we have

$$\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}I_{0-}^{\lambda}\chi\right)(x)$$
$$=\int_x^{\infty}\left[(t-x)^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\xi};p]\right]\frac{1}{\Gamma(\lambda)}\int_t^{\infty}[(u-t)^{\lambda-1}\chi(u)du]dt$$
$$=\int_x^{\infty}\frac{1}{\Gamma(\lambda)}\left[\int_x^{u}(t-x)^{\zeta-1}(u-t)^{\lambda-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(t-x)^{\xi};p]dt\right]\chi(u)du$$

Now, changing the variable $\tau=t-x$, we get

$$\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}I_{0-}^{\lambda}\chi\right)(x)$$
$$=\int_x^{\infty}\frac{1}{\Gamma(\lambda)}\left[\int_0^{u-x}(u-x-\tau)^{\lambda-1}\tau^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega t^{\xi};p]dt\right]\chi(u)du$$

Again by Parmar [19] pp-1078, we have

$$\left(\epsilon_{\xi,\zeta,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}I_{0-}^{\lambda}\chi\right)(x)$$
$$=\int_x^{\infty}I_{a+}^{\lambda}\left[\tau^{\zeta-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega t^{\xi};p]\right](u-x)\chi(u)du$$
$$=\int_x^{\infty}(u-x)^{\zeta+\lambda-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}[\omega(u-x)^{\xi};p]\chi(u)du$$
$$=\left(\epsilon_{\xi,\zeta+\lambda,\omega,\infty}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\chi\right)(x)$$

which completes the proof.

Now, we study the composition of Weyl differential operator with the proposed integral. Using (2. 6) and (2. 8), we have

$$\left(D^{\lambda}_{0-}\epsilon^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta,\omega,\infty}\chi\right)(x)$$
$$=(-1)^m\left(\frac{d}{dx}\right)^m\left(I^{m-\lambda}_{0-}\epsilon^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta,\omega,\infty}\chi\right)(x)$$

Using the result of the above theorem, we get

$$\left(D^{\lambda}_{0-}\epsilon^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta,\omega,\infty}\chi\right)(x)$$
$$=(-1)^m\left(\frac{d}{dx}\right)^m\int_x^{\infty}(t-x)^{\zeta+m-\lambda-1}E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta-k-\lambda}[\omega(t-x)^{\xi};p]\chi(t)dt$$

By using Dirichlet formula, we get

$$=(-1)^m\left(\frac{d}{dx}\right)^{m-1}\int_x^{\infty}\frac{\partial}{\partial x}(t-x)^{\zeta+m-\lambda-1}E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta+m-\lambda}[\omega(t-x)^{\xi};p]\chi(t)dt$$
$$+\lim(t-x)^{\zeta+m-\lambda-1}E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta+m-\lambda}[\omega(t-x)^{\xi};p]\chi(t)$$
$$=(-1)^m\left(\frac{d}{dx}\right)^{m-1}\int_x^{\infty}\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)\omega^k(\xi k+\zeta+m-\lambda-1)}{B(\gamma,1-\gamma)\Gamma(\xi k+\zeta-m-\lambda)}$$
$$\times(t-x)^{\xi k+\zeta+m-\lambda-2}\chi(t)dt$$
$$=(-1)^m\left(\frac{d}{dx}\right)^{m-1}\int_x^{\infty}(-1)(t-x)^{\zeta+k-\lambda-2}E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta+k-\lambda-1}[\omega(t-x)^{\xi};p]\chi(t)dt$$

Integrating $m-1$ times, we get

$$\left(D^{\lambda}_{0-}\epsilon^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta,\omega,\infty}\chi\right)(x)$$
$$=(-1)^m(-1)^m\int_x^{\infty}(t-x)^{\zeta-\lambda-1}E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta-\lambda}[\omega(t-x)^{\xi};p]\chi(t)dt$$
$$=\left(\epsilon^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta-\lambda,\omega,\infty}\chi\right)(x)$$

□

Now, numerous types of Mittag-Leffler functions can be deduced from the appropriate selection of arbitrary sequence κ_n. Here, we discuss only two examples by selecting appropriate values of parameters and bounded sequence κ_n.

In particular, $\kappa_n=1$ proposed function (1. 2) takes the form defined by $\ddot{O}$zarslan and Yilmaz [18]

$$E^{\gamma}_{\xi,\zeta}(z;p)=\sum_{k=0}^{\infty}\frac{B_p(\gamma+k,1-\gamma)}{B(\gamma,1-\gamma)}\frac{z^k}{\Gamma(\xi k+\zeta)}$$

and relation (3. 10), (3. 11)and (3. 12) yields the results for Mittag-Leffler function by $\ddot{O}$zarslan and Yilmaz [18]

Corollary 4.4. *Let $\xi, \zeta, \gamma, \omega \in C; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(p) > 0,$, then the relation for the usual differentiation holds*

$$\left(\frac{d}{dx}\right)^m x^{\zeta-1} E^{\gamma}_{\xi,\zeta}(\omega x^{\xi}; p) = x^{\zeta-m-1} E^{\gamma}_{\xi,\zeta-m}(\omega x^{\xi}; p) \tag{4. 14}$$

While composite relation of Weyl fractional integral and diffferential operators hold true.

$$I^{\lambda}_{0-}\left[t^{-\lambda-\zeta} E^{\gamma}_{\xi,\zeta}(\omega t^{-\xi}; p)\right](x) = x^{-\zeta} E^{\gamma}_{\xi,\zeta+\lambda}(\omega x^{-\xi}; p) \tag{4. 15}$$

$$D^{\lambda}_{0-}\left[t^{\lambda-\zeta} E^{\gamma}_{\xi,\zeta}(\omega t^{-\xi}; p)\right](x) = x^{-\zeta} E^{\gamma}_{\xi,\zeta-\lambda}(\omega x^{-\xi}; p) \tag{4. 16}$$

setting $\kappa_n = \frac{(\delta)_n}{(\mu)_n}, (n \in N_0)$ in (1. 2), then (3. 10), (3. 11)and (3. 12) yields the following results.

Corollary 4.5. *Let $\xi, \zeta, \gamma, \omega \in C; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma), \Re(p) > 0$, then the relation for the usual differentiation holds*

$$\left(\frac{d}{dx}\right)^m x^{\zeta-1} E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta}(\omega x^{\xi}; p) = x^{\zeta-m-1} E^{(\{\kappa_n\}_{n\in N_0};\gamma)}_{\xi,\zeta-m}(\omega x^{\xi}; p) \tag{4. 17}$$

While composite relation of Weyl fractional integral and diffferential operators hold true.

$$I^{\lambda}_{0-}\left[t^{-\lambda-\zeta} E^{(\delta,\mu);\gamma}_{\xi,\zeta}(\omega t^{-\xi}; p)\right](x) = x^{-\zeta} E^{(\delta,\mu);\gamma}_{\xi,\zeta+\lambda}(\omega x^{-\xi}; p) \tag{4. 18}$$

$$D^{\lambda}_{0-}\left[t^{\lambda-\zeta} E^{(\delta,\mu);\gamma}_{\xi,\zeta}(\omega t^{-\xi}; p)\right](x) = x^{-\zeta} E^{(\delta,\mu);\gamma}_{\xi,\zeta-\lambda}(\omega x^{-\xi}; p) \tag{4. 19}$$

Note: If we put $p = 0$, then the result of main theorem reduces for Prabhakar-type Mittag-Leffler function and on setting $\xi = \zeta = 1$ function reduces in term of extended confluent hypergeometric function. We have skipped the details.

Conclusion. In this paper, we have developed the images of Weyl fractional integral and differential operators involving extended form of ML-function which depends upon the bounded sequence. On suitable selections of bounded sequences and parameters involved, yields known and some new results. Further, a new integral operator of Weyl fractional is also defined while proposed extended function is used in its kernel.

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