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Binomial Triple Sums with the Lucas Numbers

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Abstract.: In this paper, we consider binomial triple sums families whose coefficients are chosen as the Lucas numbers with indices in linear combination of the summation indices. These sums are expressed via certain linear combinations of terms of the Fibonacci and Lucas sequences $\{F_n, L_n\}$. Furthermore, we compute some kinds of alternating analogues of them whose powers are depend on the index or indices.

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1. Introduction

For $n > 1$, define the Fibonacci and Lucas sequences $\{F_n, L_n\}$ by

$$F_n = F_{n-1} + F_{n-2} \text{ and } L_n = L_{n-1} + L_{n-2}$$

with $F_0 = 0,\ F_1 = 1$ and $L_0 = 2$ and $L_1 = 1$, respectively.

The Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where α and β are $\left(1 \pm \sqrt{5}\right)/2$.

There are many types of sums formulas including binomial or Fibonomial coefficients, Fibonacci, Lucas, Pell and Pell-Lucas numbers (see [1, 8–15]).

Carlitz [1] derives various binomial sums including one binomial coefficient and the Fibonacci numbers.

Kılıç et. al. [5] compute the sums including one binomial coefficient and products of the Fibonacci or Lucas numbers as well as their alternating analogues of the forms

$$\sum_{i=0}^{n} \binom{n}{i} T_{k(a+bi)} T_{k(c+di)} \text{ and } \sum_{i=0}^{n} \binom{n}{i} (-1)^i T_{k(a+bi)} T_{k(c+di)},$$

where T_n is either a generalized Fibonacci or Lucas sequence.

Kılıç et. al. [2] consider some binomial sums including the powers of Fibonacci or Lucas numbers.

Kılıç and Belbachir [4] derive various double binomial sums. For example, they show that

$$\sum_{i,j} \binom{n-i}{j}\binom{n-j}{i} = F_{2n+2}.$$

Kılıç and Arıkan [3] compute many binomial sums including double sums and one binomial coefficient.

Kılıç and Taşdemir [6] consider and compute various families of binomial sums namely *binomial-double-sums* including one binomial coefficient and generalized Fibonacci or Lucas sequences.

Kılıç and Taşdemir [7] also consider some special families of binomial double sums including one binomial coefficient and the Fibonacci numbers of the form

$$\sum_{0\le i,j\le n} \binom{i}{j} F_{ri+tj}$$

for some integers r and t.

Taşdemir and Toska [14] compute the binomial double sums including the Lucas numbers as well as their alternating analogues. For example, they show that

$$\sum_{0\le i,j\le n} \binom{i}{j} L_{4ti+j} = \frac{1}{L_{2t+1}} \begin{cases} L_{(2t+1)n}L_{(2t+1)(n+1)} & \text{if } n \text{ is even,} \\ 5F_{(2t+1)n}F_{(2t+1)(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

Recently, Ömür and Duran [10] consider some special families of binomial triple sums including the Fibonacci numbers with indices in linear combination of the summation indices of the forms

$$\sum_{0\le i,j,k\le n} \binom{i}{j}\binom{j}{k} F_{ri+j+k.}$$

In this study, inspired from [10], we shall consider some binomial triple sums families whose coefficients will be chosen as the Lucas numbers. These sums will be expressed via certain linear combinations of terms of the Fibonacci and Lucas sequences $\{F_n, L_n\}$.

2. Binomial Triple Sums with the Lucas Numbers

We recall two auxiliary lemmas from [10, 15] before giving our results.

Lemma 2.1. *For any real numbers $x,\ y$ and z such that $x(1+y+yz)\neq 1$, we have*

$$\sum_{0\le i,j,k\le n} \binom{i}{j}\binom{j}{k} x^i y^j z^k = \frac{(x+xy+xyz)^{n+1}-1}{x+xy+xyz-1}.$$

Also, we give the following result which could be easily derived from the Binet formulas.

Lemma 2.2. *For integers m and n, we have*

$$L_{m+n} + (-1)^m L_{n-m} = L_m L_n \ \text{ and } \ L_{m+n} - (-1)^m L_{n-m} = 5F_m F_n.$$

Now we are ready to give our main results.

Theorem 2.3. *For a nonnegative integer n and an integer t, we have*

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}L_{(4t+1)i+j+k}=\frac{1}{3+2L_{4t+3}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}+5F_{\frac{(4t+3)(n+1)-3}{2}}F_{\frac{(4t+3)(n+1)+3}{2}}\right)+2L_{2t+1}L_{2t+2} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(10F_{\frac{(4t+3)n+1}{2}}F_{\frac{(4t+3)n-1}{2}}+L_{\frac{(4t+3)(n+1)}{2}}L_{\frac{(4t+3)(n+1)}{2}}\right)+2L_{2t+1}L_{2t+2} & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}L_{(4t+3)i+j+k}=\frac{1}{3+2L_{4t+5}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+5)n}{2}}L_{\frac{(4t+5)n}{2}}+L_{\frac{(4t+5)(n+1)-3}{2}}L_{\frac{(4t+5)(n+1)+3}{2}}\right)+10F_{2t+2}F_{2t+3} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(2L_{\frac{(4t+5)n-1}{2}}L_{\frac{(4t+5)n+1}{2}}+L_{\frac{(4t+5)(n+1)}{2}}L_{\frac{(4t+5)(n+1)}{2}}\right)+10F_{2t+2}F_{2t+3} & \text{if } n \text{ is odd.}\end{cases}$$

Proof. As a showcase, we only prove the first claim. The other claim can be proven similarly. By the Binet formula, we write

$$\begin{aligned}&\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}L_{(4t+1)i+j+k}\\ &=\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}\left(\alpha^{(4t+1)i+j+k}+\beta^{(4t+1)i+j+k}\right),\end{aligned}$$

which, by Lemma 2.1, equals

$$\frac{\left(\alpha^{4t+1}+\alpha^{4t+2}+\alpha^{4t+3}\right)^{n+1}-1}{\alpha^{4t+1}+\alpha^{4t+2}+\alpha^{4t+3}-1}+\frac{\left(\beta^{4t+1}+\beta^{4t+2}+\beta^{4t+3}\right)^{n+1}-1}{\beta^{4t+1}+\beta^{4t+2}+\beta^{4t+3}-1},$$

which, since $\alpha^2=\alpha+1$ and $\beta^2=\beta+1$, equals

$$\begin{aligned}&\frac{2^{n+1}\alpha^{(4t+3)(n+1)}-1}{2\alpha^{4t+3}-1}+\frac{2^{n+1}\beta^{(4t+3)(n+1)}-1}{2\beta^{4t+3}-1}\\ &=\frac{1}{-3-2L_{4t+3}}\left[-2^{n+2}L_{(4t+3)n}-2^{n+1}L_{(4t+3)(n+1)}-2L_{4t+3}+2\right]\\ &=\frac{1}{-3-2L_{4t+3}}\left[2^{n+1}(-2L_{(4t+3)n}-L_{(4t+3)(n+1)})-2L_{4t+3}+2\right].\end{aligned}$$

We consider two cases of n. First if $n=4r$, the previous expression becomes

$$\begin{aligned}&-\frac{2^{4r+1}(-2L_{16tr+12r}-L_{16tr+12r+4t+3})-2L_{4t+3}+2}{3+2L_{4t+3}}\\ &=\frac{2^{4r+1}\left[2(L_{16tr+12r}+L_0)+L_{16tr+12r+4t+3}-L_3\right]+2(L_{4t+3}-1)}{3+2L_{4t+3}}\end{aligned}$$

which, by Lemma 2.2, equals

$$\frac{2^{4r+1}\left(2L_{8tr+6r}L_{8tr+6r}+5F_{8tr+6r+2t}F_{8tr+6r+2t+3}\right)+2L_{2t+1}L_{2t+2}}{3+2L_{4t+3}}$$
$$=\frac{2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}+5F_{\frac{(4t+3)(n+1)-3}{2}}F_{\frac{(4t+3)(n+1)+3}{2}}\right)+2L_{2t+1}L_{2t+2}}{3+2L_{4t+3}}$$

Similarly, for $n=4r+2$, we obtain the same result. Thus, if n is even,

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}L_{(4t+1)i+j+k}=\frac{1}{3+2L_{4t+3}}$$
$$\times\left[2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}+5F_{\frac{(4t+3)(n+1)-3}{2}}F_{\frac{(4t+3)(n+1)+3}{2}}\right)+2L_{2t+1}L_{2t+2}\right],$$

as claimed. □

3. Alternative Binomial Triple Sums with the Lucas Numbers

Now we will give alternating analogues of the results given in the previous section.

Theorem 3.1. *For a nonnegative integer n and an integer t, we have*

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}(-1)^iL_{(4t+1)i+j+k}=\frac{1}{3-2L_{4t+3}}$$
$$\times\begin{cases}2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}-L_{\frac{(4t+3)(n+1)-3}{2}}L_{\frac{(4t+3)(n+1)+3}{2}}\right)-10F_{2t+1}F_{2t+2} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(5F_{\frac{(4t+3)(n+1)}{2}}F_{\frac{(4t+3)(n+1)}{2}}-10F_{\frac{(4t+3)n-1}{2}}F_{\frac{(4t+3)n+1}{2}}\right)-10F_{2t+1}F_{2t+2} & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}(-1)^iL_{(4t+3)i+j+k}=\frac{1}{3-2L_{4t+5}}$$
$$\times\begin{cases}2^{n+1}\left(2L_{\frac{(4t+5)n}{2}}L_{\frac{(4t+5)n}{2}}-5F_{\frac{(4t+5)(n+1)-3}{2}}F_{\frac{(4t+5)(n+1)+3}{2}}\right)-2L_{2t+2}L_{2t+3} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(5F_{\frac{(4t+5)(n+1)}{2}}F_{\frac{(4t+5)(n+1)}{2}}-2L_{\frac{(4t+5)n-1}{2}}L_{\frac{(4t+5)n+1}{2}}\right)-2L_{2t+2}L_{2t+3} & \text{if } n \text{ is odd.}\end{cases}$$

Proof. We only prove the first claim. The other claim can be proven similarly.

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}(-1)^iL_{(4t+1)i+j+k}$$
$$=\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}(-1)^i\left(\alpha^{(4t+1)i+j+k}+\beta^{(4t+1)i+j+k}\right),$$

which, by Lemma 2.1 and since $\alpha^2=\alpha+1$ and $\beta^2=\beta+1$, equals

$$\frac{(-1)^n\left(\alpha^{4t+1}+\alpha^{4t+2}+\alpha^{4t+3}\right)^{n+1}+1}{\alpha^{4t+1}+\alpha^{4t+2}+\alpha^{4t+3}+1}+\frac{(-1)^n\left(\beta^{4t+1}+\beta^{4t+2}+\beta^{4t+3}\right)^{n+1}+1}{\beta^{4t+1}+\beta^{4t+2}+\beta^{4t+3}+1}$$
$$=\frac{(-1)^n2^{n+1}\alpha^{(4t+3)(n+1)}+1}{2\alpha^{4t+3}+1}+\frac{(-1)^n2^{n+1}\beta^{(4t+3)(n+1)}+1}{2\beta^{4t+3}+1}$$
$$=\frac{(-1)^n2^{n+1}(-2L_{(4t+3)n}+L_{(4t+3)(n+1)})+2L_{4t+3}+2}{2L_{4t+3}-3}$$

which, for $n=4r$, equals

$$\frac{2^{4r+1}\left[-2(L_{16tr+12r}+L_0)+L_{16tr+12r+4t+3}+L_3\right]+2L_{4t+3}+2}{2L_{4t+3}-3}$$

which, by Lemma 2.2, gives us

$$\frac{2^{4r+1}\left(-2L_{8tr+6r}L_{8tr+6r}+L_{8tr+6r+2t}L_{8tr+6r+2t+3}\right)+2L_{4t+3}+2}{2L_{4t+3}-3}$$
$$=\frac{2^{4r+1}\left(2L_{8tr+6r}L_{8tr+6r}-L_{8tr+6r+2t}L_{8tr+6r+2t+3}\right)-10F_{2t+1}F_{2t+2}}{3-2L_{4t+3}}.$$

So for $n=4r$, it equals

$$\frac{2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}-L_{\frac{(4t+3)(n+1)-3}{2}}L_{\frac{(4t+3)(n+1)+3}{2}}\right)-10F_{2t+1}F_{2t+2}}{3-2L_{4t+3}}.$$

Similarly, for $n=4r+2$ we obtain the same result. Thus, the claim is true for n is even. $\square$

Now, we give the other results without proof. They could be proven by using again Lemma 2.1.

Theorem 3.2. *For a nonnegative integer n and an integer t, we have*

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}(-1)^jL_{4ti+j+k}=\frac{1}{3-2L_{4t+1}}$$
$$\times\begin{cases}2^{n+1}\left(2L_{\frac{(4t+1)n}{2}}L_{\frac{(4t+1)n}{2}}-5F_{\frac{(4t+1)(n+1)-3}{2}}F_{\frac{(4t+1)(n+1)+3}{2}}\right)-2L_{2t}L_{2t+1} & \textit{if } n \textit{ is even,}\\ 2^{n+1}\left(5F_{\frac{(4t+1)(n+1)}{2}}F_{\frac{(4t+1)(n+1)}{2}}-2L_{\frac{(4t+1)n-1}{2}}L_{\frac{(4t+1)n+1}{2}}\right)-2L_{2t}L_{2t+1} & \textit{if } n \textit{ is odd,}\end{cases}$$

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}(-1)^jL_{(4t+2)i+j+k}=\frac{1}{3-2L_{4t+3}}$$
$$\times\begin{cases}2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}-L_{\frac{(4t+3)(n+1)-3}{2}}L_{\frac{(4t+3)(n+1)+3}{2}}\right)-10F_{2t+1}F_{2t+2} & \textit{if } n \textit{ is even,}\\ 2^{n+1}\left(5F_{\frac{(4t+3)(n+1)}{2}}F_{\frac{(4t+3)(n+1)}{2}}-10F_{\frac{(4t+3)n-1}{2}}F_{\frac{(4t+3)n+1}{2}}\right)-10F_{2t+1}F_{2t+2} & \textit{if } n \textit{ is odd.}\end{cases}$$

Theorem 3.3. *For a nonnegative integer n and an integer t, we have*

$$\sum_{0\leq i,j,k\leq n}\binom{i}{j}\binom{j}{k}(-1)^{i+j}L_{4ti+j+k}=\frac{1}{3+2L_{4t+1}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+1)n}{2}}L_{\frac{(4t+1)n}{2}}+L_{\frac{(4t+1)(n+1)-3}{2}}L_{\frac{(4t+1)(n+1)+3}{2}}\right)+10F_{2t}F_{2t+1} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(2L_{\frac{(4t+1)n-1}{2}}L_{\frac{(4t+1)n+1}{2}}+L_{\frac{(4t+1)(n+1)}{2}}L_{\frac{(4t+1)(n+1)}{2}}\right)+10F_{2t}F_{2t+1} & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{0\leq i,j,k\leq n}\binom{i}{j}\binom{j}{k}(-1)^{i+j}L_{(4t+2)i+j+k}=\frac{1}{3+2L_{4t+3}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}+5F_{\frac{(4t+3)(n+1)-3}{2}}F_{\frac{(4t+3)(n+1)+3}{2}}\right)+2L_{2t+1}L_{2t+2} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(10F_{\frac{(4t+3)n-1}{2}}F_{\frac{(4t+3)n+1}{2}}+L_{\frac{(4t+3)(n+1)}{2}}L_{\frac{(4t+3)(n+1)}{2}}\right)+2L_{2t+1}L_{2t+2} & \text{if } n \text{ is odd.}\end{cases}$$

Theorem 3.4. *For a nonnegative integer n and an integer t, we have*

$$\sum_{0\leq i,j,k\leq n}\binom{i}{j}\binom{j}{k}(-1)^{j+k}L_{(4t+1)i+j+k}=\frac{1}{3+2L_{4t+1}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+1)n}{2}}L_{\frac{(4t+1)n}{2}}+L_{\frac{(4t+1)(n+1)-3}{2}}L_{\frac{(4t+1)(n+1)+3}{2}}\right)+10F_{2t}F_{2t+1} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(2L_{\frac{(4t+1)n-1}{2}}L_{\frac{(4t+1)n+1}{2}}+L_{\frac{(4t+1)(n+1)}{2}}L_{\frac{(4t+1)(n+1)}{2}}\right)+10F_{2t}F_{2t+1} & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{0\leq i,j,k\leq n}\binom{i}{j}\binom{j}{k}(-1)^{j+k}L_{(4t+3)i+j+k}=\frac{1}{3+2L_{4t+3}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}+5F_{\frac{(4t+3)(n+1)-3}{2}}F_{\frac{(4t+3)(n+1)+3}{2}}\right)+2L_{2t+1}L_{2t+2} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(10F_{\frac{(4t+3)n-1}{2}}F_{\frac{(4t+3)n+1}{2}}+L_{\frac{(4t+3)(n+1)}{2}}L_{\frac{(4t+3)(n+1)}{2}}\right)+2L_{2t+1}L_{2t+2} & \text{if } n \text{ is odd.}\end{cases}$$

Theorem 3.5. *For a nonnegative integer n and an integer t, we have*

$$\sum_{0\leq i,j,k\leq n}\binom{i}{j}\binom{j}{k}(-1)^{i+j+k}L_{(4t+1)i+j+k}=\frac{1}{3-2L_{4t+1}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+1)n}{2}}L_{\frac{(4t+1)n}{2}}-5F_{\frac{(4t+1)(n+1)-3}{2}}F_{\frac{(4t+1)(n+1)+3}{2}}\right)-2L_{2t}L_{2t+1} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(5F_{\frac{(4t+1)(n+1)}{2}}F_{\frac{(4t+1)(n+1)}{2}}-2L_{\frac{(4t+1)n-1}{2}}L_{\frac{(4t+1)n+1}{2}}\right)-2L_{2t}L_{2t+1} & \text{if } n \text{ is odd,}\end{cases}$$

$$\sum_{0\leq i,j,k\leq n}\binom{i}{j}\binom{j}{k}(-1)^{i+j+k}L_{(4t+3)i+j+k}=\frac{1}{3-2L_{4t+3}}$$
$$\times\begin{cases} 2^{n+1}\left(2L_{\frac{(4t+3)n}{2}}L_{\frac{(4t+3)n}{2}}-L_{\frac{(4t+3)(n+1)-3}{2}}L_{\frac{(4t+3)(n+1)+3}{2}}\right)-10F_{2t+1}F_{2t+2} & \text{if } n \text{ is even,}\\ 2^{n+1}\left(5F_{\frac{(4t+3)(n+1)}{2}}F_{\frac{(4t+3)(n+1)}{2}}-10F_{\frac{(4t+3)n-1}{2}}F_{\frac{(4t+3)n+1}{2}}\right)-10F_{2t+1}F_{2t+2} & \text{if } n \text{ is odd.}\end{cases}$$

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