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HZ-Coindex for the Sum-Graphs under Cartesian Product

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Abstract.:Let $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ be a simple connected graph with $V(\mathbb{G})$ and $E(\mathbb{G})$ as the vertex and edge sets respectively. The physical and structural properties of the graphs are discussed with different mathematical tools, One of them is topological index (TI) and it can be obtained directly from molecular structures and rapidly computed for large number of molecules. Graph operations play an important role in the development of various new classes of graphs. If G is a connected simple graph then we obtained new graphs by applying various operations on G such as S subdivided, R triangle parallel, Q line superposition and T total graphs operations on G. In this paper, we computed the hyper Zagreb coindex of derived graphs in the form of their factor graphs. At the end, we have analyzed our results by the numerical tables for the particular generalized D-sum graphs.

AMS (MOS) Subject Classification Codes: 05C09; 05C92
Key Words: Hyper Zagreb Coindex; Operations on graphs; D-sum graphs.

1. Introduction

A topological index (TI) is a function $F : \aleph \rightarrow \Re$, where $\Re$ is a set of real numbers and $\aleph$ is a class of molecular graphs. It assigns a numerical value to each graph unless the graphs are isomorphic. Chemical graph theory helps in predicting a large number of physico-chemical and structural properties of chemical compounds using these TIs such as volume, boiling point, melting point, freezing point, surface tension, density, solubility and heat of formation [29], [22].

TIs are categorized in three types such as degree, distance and polynomial based but mostly the degree based TIs are studied for details, see [14]. In (1947) Wiener used degree based TI to calculate boiling point of paraffin, see [32]. Gutman and Trinajstić (1972) introduced degree based Zagreb indices $M_1(G)$ and $M_2(G)$ to calculate total π-electron energy of hydrocarbons [15]. Klein *et al.* defined a molecular TI and investigate a close relation with the Wiener index [18]. Mendiratta *et al.* and Cornwell used Wiener's TI to study structural activity on antiviral 5-vinylpyrimidine nucleoside analogs, see [24]. Biye *et al.* wrote a novel on TI for QSPR/QSAR study of organic compounds, see [7].

In (2013) Shirdel *et al.* [30] defined a new TI known as the hyper-Zagreb index (HZI) and calculated the generalized results, Ghalavand *et al.* [28] determined extremal chemical trees with respect to HZI ,Menaka and Manikandan computed the (HZI) [23], Veylaki *et al.* [31] calculated HZ-coindex of some graph operations such as the composition, symmetric difference, disjunction, Cartesian product and tensor product. Natarajan *et al.* [25] computed the HZI of certain generalized thorn graphs. Gao *et al.* [13] calculated sharp bounds of the (HZI) for trees and cyclic graphs . Elumalai *et al.* [10] calculated bounds for simple connected graphs, Falahati-Nezhad and Azari computed upper and lower bounds in terms of different molecular descriptors for the (HZI) [11]. Liu and Tang calculated HZI of cacti with perfect matchings [20]. Alameri compared some TIS with the second HZI by using the strong correlation coefficient acquired from the chemical graphs of octane isomers [1]. Farahani computed HZI of circumcoronene series of benzenoid [12]. Rezaei *et al.* [27] calculated HZI for an infinite class of titania nanotubes. Luo computed HZI of the linear phenylene, nanotubes, hypercubes, zig-zag polyhex nanotube and dendrimer, see [21].

Graph operations play an important role in the development of various new classes of graphs, such as union, join, Cartesian product, corona product and derived. Basavanagoud and Patil computed HZ-coindex of some graph operations such as the union, Cartesian product and composition of graphs, see [5]. Alsharafi *et al.* calculated hyper-Zagreb coindices for different operations such as disjunction, symmetric difference, tensor product and strong product [3]. Basavanagoud and Desai computed forgotten and hyper-Zagreb TI of generalized transformation graphs [6].

Yan *et al.* [33] listed five graphs $D(G)$, where $D \in \{L, S, R, Q, T\}$ such as line graph $L(H)$, subdivided graph $S(H)$, line superposition graph $Q(H)$, triangle parallel $R(H)$ and total $T(H)$ by applying various operations on G and calculated the Wiener index of these graphs, see . Eliasi and Taeri [9] calculated the Wiener indices of subdivision related operations such as four sum-graphs $H_{1+D}H_2$, where $D \in \{S, R, Q, T\}$, Deng *et al.* [8] computed Zagreb indices, Javaid *et al.* [16] determine bounds for the second Zagreb coindex, Awais *et al.* [4] computed forgotten index, Li *et al.* [19] calculated bounds on general randic index, Javaid *et al.* [17] calculated the forgotten Index, Raza and Imran computed the reverse Zagreb indices of [26], Alanazi *et al.* [2] computing exact values for Gutman indices of these graphs.

In this paper, we compute HZ-coindex of $(A_{+D}B)$, where $D \in \{S, R, Q, T\}$ in the form of Zagreb indices and coindices, forgotten indices and coindices, hyper Zagreb indices and coindices of their basic graphs. At the end, the obtained results are also illustrated using examples for some particular D-sum graphs. In Section 2, we defined basic definitions and notions. In Section 3, the main results of work are explained and in Section 4, presents particular examples related to the main results.

2. Notations and Preliminaries

Let $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ be a simple connected graph, then $|V(\mathbb{G})| = n$ is called order of the graph and $|E(\mathbb{G})| = m$ is called size of the graph. The number of edges connecting to a vertex is called degree of that vertex. The complement of a graph G is denoted by $\bar{G}$ and defined as two vertices $uv \in E(\bar{G})$ iff $uv \notin E(G)$. In (1972) Gutman and Trinajstić [15] introduced the Zagreb indices denoted by $M_1(G)$ and $M_2(G)$ and defined as

$$M_1(G) = \sum_{x_1x_2 \in E(H)} [d_H(x_1) + d_H(x_2)], M_2(H) = \sum_{x_1x_2 \in E(H)} [d_H(x_1)d_H(x_2)]$$

The Zagreb coindices are denoted by $\bar{M}_1(G)$ and $\bar{M}_2(G)$ respectively see, [16]:

$$\bar{M}_1(G) = \sum_{x_1x_2 \notin E(G)} [d_G(x_1) + d_G(x_2)], \bar{M}_2(G) = \sum_{x_1x_2 \notin E(H)} [d_G(x_1)d_G(x_2)]$$

Shirdel *et al.* [30] introduced HZI that is defined as

$$HZ(G) = \sum_{r \in V(H)} [d_G(r)]^3 = \sum_{r_1r_2 \in E(G)} [d_G(r_1)^2 + d_G(r_2)^2]$$

Veylaki *et al.* [31] introduced HZ-coindex that is defined as

$$\bar{HZ} = \sum_{x_1x_2 \notin E(G)} [d_G(x_1) + d_G(x_2)]^2 \qquad (2.\ 1)$$

Let A be a simple connected graph, Then,

- $S(A)$ is a graph sketched by adding a new vertex in every edge of A,
- $R(A)$ is a graph sketched from $S(A)$ by using an edge between the adjacent vertices of G,
- $Q(A)$ is a graph formed from $S(A)$ by adding an edge between the pairs of new vertices which are on the adjacent edges of A,
- $T(A)$ is formed by performing both operations of $R(A)$ and $Q(A)$ on $S(A)$.

Suppose that A and B are two simple connected graphs, Then their D-sum graph is denoted by $A_{+D}B$ and defined with vertex set $|V(A_{+D}B| = V(A) \cup E(A) \times V(B)$ and edge set is defined as the vertices (x_1, x_2) and (y_1, y_2) of $A_{+D}B$ where $D \in \{S, R, Q, T\}$ are joined iff

- $x_1 = y_1 \in V(A)$ and $x_2 \backsim y_2 \in B$
- $x_2 = y_2 \in V(B)$ and $x_1 \backsim y_1 \in F(A)$

For details, see Figure 1 and Figure 2.

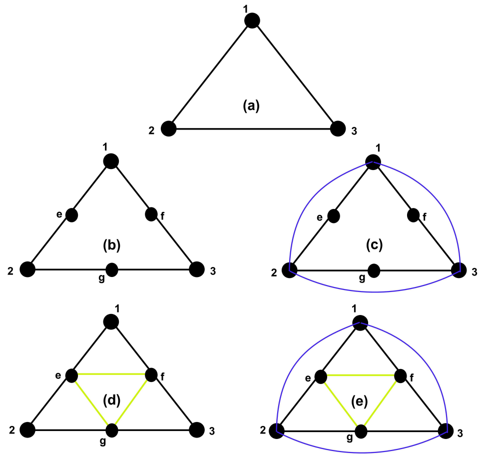

FIGURE 1. (a) $G \cong C_3$, (b) $S(G) \cong S(C_3)$, (c) $Q(G) \cong (C_3)$, (d) $R(G) \cong R(C_3)$ and (e) $T(G) \cong T(C_3)$

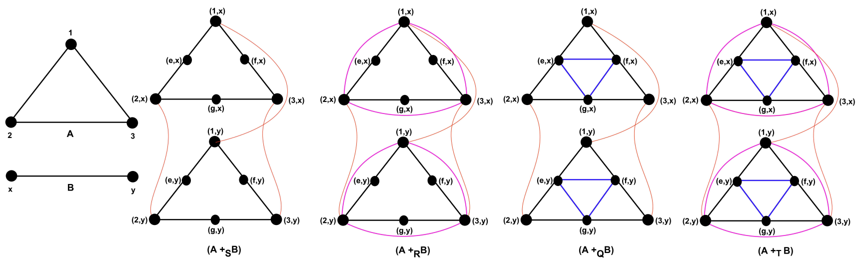

FIGURE 2. $A \cong C_3$, $B \cong P_2$ and $A_{+D}B \cong C_{3+D}P_2$, where $D \in \{S, R, Q, T\}$.

3. RESULTS

In this section, we discuss the main results of the hyper Zagreb coindex for D-sum graphs. Now, we assume some sums for $D \in \{S, R, Q, T\}$ that will be used our main results.

- $\alpha = 2(e_B + \bar{e}_B)$
- $\alpha_1 = \sum\limits_{\substack{r_1r_2 \notin E(D(A)) \\ r_1 \in V(A) \\ r_2 \in V(D(A)-V(A))}} [d_A(r_1) + d_{D(A)}(r_2)]^2$, $\alpha_2 = \sum\limits_{\substack{r_1r_2 \notin E(D(A)) \\ r_1 \in V(A) \\ r_2 \in V(D(A)-V(A))}} [d_A(r_1) + d_{D(A)}(r_2)]$
- $\alpha_3 = \sum\limits_{\substack{r_1r_2 \in E(D(A)) \\ r_1 \in V(A) \\ r_2 \in V(D(A)-V(A))}} [d_A(r_1) + d_{D(A)}(r_2)]^2$, $\alpha_4 = \sum\limits_{\substack{r_1r_2 \in E(D(A)) \\ r_1 \in V(A) \\ r_2 \in V(D(A)-V(A))}} [d_A(r_1) + d_{D(A)}(r_2)]$

$Graph$	$Order$	$Size$
A	n_a	e_a
B	n_b	e_b
$\bar{A}$	n_a	$\binom{n_a}{2} - e_a$
$\bar{B}$	n_b	$\binom{n_b}{2} - e_b$

TABLE 1. Useful notations that will use in Theorems.

Theorem 3.1. Let A and B be two simple connected graphs then HZ-coindex of $(A_{+S}B)$ is
$\bar{HZ}(A_{+S}B) = 8(n_2^2e_1^2 - n_2e_1) + 4\bar{e}_BM_1(A) + n_A\bar{F}(B) + 2n_B(M_2(A) + \bar{M}_2(A)) + 8e_A\bar{M}_1(B) + 2n_A\bar{M}_2(B) + n_B(F(A) + \bar{F}(A)) + 4M_1(B)(e_A + \bar{e}_A) + 8e_B(M_1(A) + \bar{M}_1(A) + [(e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 2((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B)) + M_1(A)(M_1(B) + \bar{M}_1(B)))] + [(e_B + \bar{e}_B)\bar{F}(A) + \bar{e}_A(F(B) + \bar{F}(B)) + 2((e_B + \bar{e}_B)\bar{M}_2(A) + \bar{e}_A(M_2(B) + \bar{M}_2(B)) + \bar{M}_1(A)(M_1(B) + \bar{M}_1(B)))] + n_B\alpha_1 + e_A(n_A - 2)M_1(B) + 4e_B\alpha_2 + n_B(n_B - 1)\alpha_1 + e_A(n_A - 2)M_1(B) + 4e_B(n_B - 1)\alpha_2 + 2\bar{e}_B\alpha_1 + e_A(n_A - 2)\bar{M}_1(B) + 8\bar{e}_B\alpha_2 + n_B(n_B - 1)HZ(S(A)) + 2e_AM_1(B) + 4e_B(n_B - 1)M_1(S(A)) + 2\bar{e}_BHZ(S(A)) + 2e_A\bar{M}_1(B) + 4\bar{e}_B(n_B - 1)M_1(S(A))$.

Proof. Using Equation (1), we have:

$$\bar{HZ}(H_{1+S}H_2) = \sum_{(x_1,x_2)(y_1,y_2)\notin E(A_SB}(d(x_1, y_1)+d(x_2, y_2))^2.$$

$$\bar{HZ}(H_{1+S}B) = \sum_{y_1,y_2\in V_B}\Big[\sum_{x_1,x_2\in(V(S(A)-V(A))} + \sum_{x_1,x_2\in V_A} + \sum_{\substack{x_1,x_2\in V(S(A))\\ x_1\in V(A)\\ x_2\in V(S(A)-V(A))}}\Big]$$

$$(d(x_1, y_1) + d(x_2, y_2))^2.$$

$$= \sum A + \sum B + \sum C. \tag{3. 2}$$

$$\sum A = \sum_{x_1,x_2\in V(S(A)-V(A))}\sum_{y_1,y_2\in V_B}(d(x_1, y_1) + d(x_2, y_2))^2$$

$$= \sum_{x_1,x_2\in V(S(A)-V(A))}\sum_{y_1,y_2\in V_B}(2+2)^2.$$

$$\sum A = 8(n_2^2e_1^2 - n_2e_1). \tag{3. 3}$$

$$\sum B = \sum B_1 + \sum B_2 + \sum B_3 + \sum B_4$$

$$\sum B_1 = \sum_{x\in V_A}\sum_{y_1y_2\notin E_B}(d(x, y_1)+d(x, y_2))^2 = \sum_{x\in V_A}\sum_{y_1y_2\notin E_B}(d(x_1, y_1)+d(x_2, y_2))^2.$$

$$= \sum_{x \in V_A} \sum_{y_1y_2 \notin E_B} [(2d(x))^2 + 2d(y_1)d(y_2) + 4d(x)(d(y_1) + d(y_2) + (d(y_1)^2 + d(y_2)^2))$$

$$= 4\bar{e}_B M_1(A) + n_A \bar{F}(B) + 8e_A \bar{M}_1(B) + 2n_A \bar{M}_2(B).$$

$$\sum B_2 = \sum_{y \in V_B} (\sum_{x_1x_2 \in E_A} + \sum_{x_1x_2 \notin E_A})[(d(x_1)^2 + d(x_2)^2) + (2d(y))^2 + 2d(x_1)d(x_2)$$
$$+4d(y)(d(x_1) + d(x_2))$$
$$= n_B(F(A)+\bar{F}(A))+4M_1(B)(e_A+\bar{e}_A)+2n_B(M_2(A)+\bar{M}_2(A))+8e_B(M_1(A)+\bar{M}_1(A).$$

$$\sum B_3 = [\sum_{x_1x_2 \in E_A} (\sum_{y_1y_2 \in E_B} + \sum_{y_1y_2 \notin E_B})](d(x_1, y_1) + d(x_2, y_2))^2)$$
$$= [\sum_{x_1x_2 \notin E_A} (\sum_{y_1y_2 \in E_B} + \sum_{y_1y_2 \notin E_B})]$$
$$+[(d(x_1)^2 + d(x_2)^2) + (d(y_1)^2$$
$$+d(y_2)^2) + 2(d(x_1)d(x_2) + d(y_1)d(y_2) + (d(x_1) + d(x_2))(d(y_1) + d(y_2))].$$

$= 2[M_1(A)(M_1(B) + \bar{M}_1(B))) + (e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 2((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B))]$

$$\sum B_4 = [\sum_{x_1x_2 \notin E_A} (\sum_{y_1y_2 \in E_B} + \sum_{y_1y_2 \notin E_B})]$$
$$+[(d(x_1)^2 + d(x_2)^2) + (d(y_1)^2 + d(y_2)^2) + 2(d(x_1)d(x_2) + d(y_1)d(y_2) + (d(x_1) + d(x_2))$$
$$(d(y_1) + d(y_2))].$$

$= [\bar{M}_1(A)(M_1(B) + \bar{M}_1(B))) + (e_B + \bar{e}_B)\bar{F}(A) + \bar{e}_A(F(B) + \bar{F}(B)) + 2((e_B + \bar{e}_B)\bar{M}_2(A) + \bar{e}_A(M_2(B) + \bar{M}_2(B))].$

$\sum B = 4\bar{e}_B M_1(A) + n_A\bar{F}(B) + 2n_A\bar{M}_2(B) + n_B(F(A) + \bar{F}(A)) + 4M_1(B)(e_A + \bar{e}_A) + 2n_B(M_2(A) + \bar{M}_2(A)) + 8e_B(M_1(A) + \bar{M}_1(A) + [M_1(A)(M_1(B) + \bar{M}_1(B))) + (e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 2((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B))] + [8e_A\bar{M}_1(B) + \bar{M}_1(A)(M_1(B) + \bar{M}_1(B)))$

$$+(e_B + \bar{e}_B)\bar{F}(A) + \bar{e}_A(F(B) + \bar{F}(B)) + 2((e_B + \bar{e}_B)\bar{M}_2(A) + \bar{e}_A(M_2(B) + \bar{M}_2(B))]. \tag{3. 4}$$

$$\sum C = \sum C_1 + \sum C_2 + \sum C_3.$$

$$\sum C_1 = \sum_{\substack{x_1x_2 \notin E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{s \in V_B} (d(x_1, s_1) + d(x_2, s_2))^2$$

$$= \sum_{\substack{x_1x_2 \notin E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y \in V_B} [(d_A(x_1)+d_{S(A)}(x_2))^2 + d_B(y)^2 + 2d(y)(d_A(x_1)+d_{S(A)}(x_2))]$$

$$= n_B\alpha_1 + e_A(n_A-2)M_1(B) + 4e_B\alpha_2.$$

$$\sum C_2 = \sum_{\substack{x_1x_2 \notin E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1,y_2 \in V_B} (d(x_1,y_1)+d(x_2,y_2))^2$$

$$= \sum_{\substack{x_1x_2 \notin E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1y_2 \in E_B} [(d_A(x_1)+d_{S(A)}(x_2))^2 + d_B(y)^2 + 2d(y)(d_A(x_1)$$
$$+d_{S(A)}(x_2))]$$

$$= n_B(n_B-1)\alpha_1 + e_A(n_A-2)M_1(B) + 4e_B(n_B-1)\alpha_2.$$

$$\sum C_3 = \sum_{\substack{x_1x_2 \notin E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1,y_2 \notin V_B} (d(x_1,y_1)+d(x_2,y_2))^2$$

$$= \sum_{\substack{x_1x_2 \notin E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1y_2 \notin E_B} [(d_A(x_1)+d_{S(A)}(x_2))^2 + d_B(y)^2 + 2d(y)(d_A(x_1)+d_{S(A)}(x_2))]$$

$$= 2\bar{e}_B\alpha_1 + e_A(n_A-2)\bar{M}_1(B) + 4\bar{e}_B(\bar{n}_B-1)\alpha_2.$$

$$\sum C_4 = \sum_{\substack{x_1x_2 \in E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1,y_2 \in V_B} (d(x_1,y_1)+d(x_2,y_2))^2$$

$$= \sum_{\substack{x_1x_2 \in E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1y_2 \in E_B} [(d_A(x_1)+d_{S(A)}(x_2))^2 + d_B(y)^2 + 2d(y)(d_A(x_1)+d_{S(A)}(x_2))]$$

$$= 2e_B HZ(S(A)) + 2e_A M_1(B) + 4e_B(n_B-1)M_1(S(A)).$$

$$\sum C_5 = \sum_{\substack{x_1x_2 \in E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1,y_2 \notin V_B} (d(x_1,y_1)+d(x_2,y_2))^2$$

$$= \sum_{\substack{x_1x_2 \in E(S(A)) \\ x_1 \in V(A) \\ x_2 \in V(S(A)-V(A))}} \sum_{y_1y_2 \notin E_B} [(d_A(x_1)+d_{S(A)}(x_2))^2 + d_B(y)^2 + 2d(y)(d_A(x_1)+d_{S(A)}(x_2))]$$

$$= 2\bar{e}_B HZ(S(A)) + 2e_A \bar{M}_1(B) + 4\bar{e}_B(\bar{n}_B-1)M_1(S(A)).$$

we obtained required result by substituting the values in Equation 2.

Theorem 3.2. Let A and B be two simple connected graphs then HZ-coindex of $(A_{+R}B)$ is: $\bar{HZ}(A_{+R}B) = 8(n_2^2e_1^2 - n_2e_1) + 8\bar{e}_BM_1(A) + n_A\bar{F}(B) + 16e_A\bar{M}_1(B) + 2n_A\bar{M}_2(B) + 4n_B(F(A) + \bar{F}(A)) + 4M_1(B)(e_A + \bar{e}_A) + 8n_B(M_2(A) + \bar{M}_2(A)) + 16e_B(M_1(A) + \bar{M}_1(A) + 2[2M_1(A)(M_1(B) + \bar{M}_1(B))) + 4(e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 8((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B))] + [2M_1(A)(M_1(B) + \bar{M}_1(B))) + 4(e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 8((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B))]n_B\alpha_1 + e_A(n_A - 2)M_1(B) + 4e_B\alpha_2 + \alpha\alpha_1 + e_A(n_A - 2)(M_1(B) + \bar{M}_1(B)) + 4(e_B(n_B - 1) + \bar{e}_B(\bar{n}_B - 1))\alpha_2 + \alpha\alpha_3 + 2e_A(M_1(B) + \bar{M}_1(B)) + 4(e_B + \bar{e}_B)(n_B - 1)\alpha_4$.

Proof. Using Equation (1), we have:

$$\bar{HZ}(A_{+R}B) = \sum_{(x_1,x_2)(y_1,y_2)\notin E(A_{+S}B)} (d(x_1,y_1) + d(x_2,y_2))^2.$$
$$= \sum A + \sum B + \sum C. \tag{3. 5}$$

Using Equation 3, $\sum A = 8(n_2^2e_1^2 - n_2e_1)$.

$$\sum B = \sum B_1 + \sum B_2 + \sum B_3 + \sum B_4$$
$$\sum B_1 = \sum_{x\in V_A}\sum_{y_1y_2\notin E_B} (d(x,y_1) + d(x,y_2))^2 = \sum_{x\in V_A}\sum_{y_1y_2\notin E_B} (d(x_1,y_1) + d(x_2,y_2))^2$$
$$= \sum_{x\in V_A}\sum_{y_1y_2\notin E_B} [(2d(x))^2 + (d(y_1)^2 + d(y_2)^2) + 2d(y_1)d(y_2) + 4d(x)(d(y_1) + d(y_2))$$
$$= 8\bar{e}_BM_1(A) + n_A\bar{F}(B) + 16e_A\bar{M}_1(B) + 2n_A\bar{M}_2(B).$$

$$\sum B_2 = \sum_{y\in V_B}(\sum_{x_1x_2\in E_A} + \sum_{x_1x_2\notin E_A})[(2d(y))^2 + 2d(x_1)d(x_2) + 4d(y)(d(x_1) + d(x_2)$$
$$+(d(x_1)^2 + d(x_2)^2))$$
$$= 4n_B(F(A)+\bar{F}(A))+4M_1(B)(e_A+\bar{e}_A)+8n_B(M_2(A)+\bar{M}_2(A))+16e_B(M_1(A)+\bar{M}_1(A).$$

$$\sum B_3 = [\sum_{x_1x_2\in E_A}(\sum_{y_1y_2\in E_B} + \sum_{y_1y_2\notin E_B})](d(x_1,y_1) + d(x_2,y_2))^2$$
$$= [\sum_{x_1x_2\notin E_A}(\sum_{y_1y_2\in E_B} + \sum_{y_1y_2\notin E_B})]$$
$$+[(d(x_1)^2 + d(x_2)^2) + (d(y_1)^2 + d(y_2)^2) + 2(d(x_1)d(x_2) + d(y_1)d(y_2) + (d(x_1) + d(x_2))$$
$$(d(y_1) + d(y_2))].$$

$= 2[2M_1(A)(M_1(B)+\bar{M}_1(B)))+4(e_B+\bar{e}_B)F(A)+e_A(F(B)+\bar{F}(B))+8((e_B+\bar{e}_B)M_2(A)+e_A(M_2(B)+\bar{M}_2(B))].$

$$\sum B_4 = [\sum_{x_1x_2\notin E_A}(\sum_{y_1y_2\in E_B}+\sum_{y_1y_2\notin E_B})]$$
$$(d(x_1,y_1)+d(x_2,y_2))^2$$
$$=[\sum_{x_1x_2\notin E_A}(\sum_{y_1y_2\in E_B}+\sum_{y_1y_2\notin E_B})]$$
$$+[(d(x_1)^2+d(x_2)^2)+(d(y_1)^2+d(y_2)^2)$$
$$+2(d(x_1)d(x_2)+d(y_1)d(y_2)+(d(x_1)+d(x_2))(d(y_1)+d(y_2))].$$

$= [2M_1(A)(M_1(B)+\bar{M}_1(B)))+4(e_B+\bar{e}_B)F(A)+e_A(F(B)+\bar{F}(B))+8((e_B+\bar{e}_B)M_2(A)+e_A(M_2(B)+\bar{M}_2(B))].$

$\sum B = 2n_A\bar{M}_2(B)+8\bar{e}_BM_1(A)+n_A\bar{F}(B)+16e_A\bar{M}_1(B)+4n_B(F(A)+\bar{F}(A))+4M_1(B)(e_A+\bar{e}_A)+16e_B(M_1(A)+\bar{M}_1(A)+8n_B(M_2(A)+\bar{M}_2(A))++[4(e_B+\bar{e}_B)F(A)+e_A(F(B)+\bar{F}(B))2[4(e_B+\bar{e}_B)F(A)+e_A(F(B)+\bar{F}(B))+8((e_B+\bar{e}_B)M_2(A)+e_A(M_2(B)+\bar{M}_2(B))+2M_1(A)(M_1(B)+\bar{M}_1(B)))]$

$$+8((e_B+\bar{e}_B)M_2(A)+e_A(M_2(B)+\bar{M}_2(B))+2M_1(A)(M_1(B)+\bar{M}_1(B)))]. \quad (3.\ 6)$$

$$\sum C = \sum C_1+\sum C_2+\sum C_3.$$
$$\sum C_1 = \sum_{\substack{x_1x_2\notin E(R(A))\\ x_1\in V(A)\\ x_2\in V(R(A)-V(A))}}\sum_{s\in V_B}(d(x_1,s_1)+d(x_2,s_2))^2$$

$$= \sum_{\substack{x_1x_2\notin E(R(A))\\ x_1\in V(A)\\ x_2\in V(R(A)-V(A))}}\sum_{y\in V_B}[(d_A(x_1)+d_{R(A)}(x_2))^2+d_B(y)^2+2d(y)(d_A(x_1)+d_{R(A)}(x_2))]$$
$$= n_B\alpha_1+e_A(n_A-2)M_1(B)+4e_B\alpha_2.$$

$$\sum C_2 = \sum_{\substack{x_1x_2\notin E(R(A))\\ x_1\in V(A)\\ x_2\in V(R(A)-V(A))}}\sum_{y_1,y_2\in V_B}(d(x_1,y_1)+d(x_2,y_2))^2$$
$$= \sum_{\substack{x_1x_2\notin E(R(A))\\ x_1\in V(A)\\ x_2\in V(R(A)-V(A))}}(\sum_{y_1y_2\in E_B}+\sum_{y_1,y_2\notin V_B})[(d_A(x_1)+d_{R(A)}(x_2))^2+d_B(y)^2$$
$$+2d(y)(d_A(x_1)+d_{R(A)}(x_2))]$$
$$= \alpha\alpha_1+e_A(n_A-2)(M_1(B)+\bar{M}_1(B))+4(e_B(n_B-1)+\bar{e}_B(\bar{n}_B-1))\alpha_2.$$

$$\sum C_3 = \sum_{\substack{x_1x_2\in E(R(A))\\ x_1\in V(A)\\ x_2\in V(R(A)-V(A))}}\sum_{y_1,y_2\in V_B}(d(x_1,y_1)+d(x_2,y_2))^2$$

$$= \sum_{\substack{x_1x_2\in E(R(A))\\ x_1\in V(A)\\ x_2\in V(R(A)-V(A))}} (\sum_{y_1y_2\in E_B} + \sum_{y_1,y_2\notin V_B})[(d_A(x_1)+d_{S(A)}(x_2))^2$$
$$+d_B(y)^2 + 2d(y)(d_A(x_1)+d_{R(A)}(x_2))]$$
$$= \alpha\alpha_3 + 2e_A(M_1(B)+\bar{M}_1(B)) + 4(e_B+\bar{e}_B)(n_B-1)\alpha_4.$$

we obtained required result by putting the values in Equation 5.

Theorem 3.3. Let A and B be two simple connected graphs then HZ-coindex of $(A_{+Q}B)$ is:
$\bar{HZ}(A_{+Q}B) = n_2\alpha_4 + 2(\bar{e}_2 + (n_2-1))(\alpha_7+\alpha_6+\alpha_5) + 8\bar{e}_BM_1(A) + n_A\bar{F}(B) + 16e_A\bar{M}_1(B) + 2n_A\bar{M}_2(B) + 16e_B(M_1(A)+\bar{M}_1(A)+4n_B(F(A)+\bar{F}(A))+4M_1(B)(e_A+\bar{e}_A) + 8n_B(M_2(A)+\bar{M}_2(A)) + 2[4(e_B+\bar{e}_B)F(A) + e_A(F(B)+\bar{F}(B)) + 8((e_B+\bar{e}_B)M_2(A)+e_A(M_2(B)+\bar{M}_2(B))+2M_1(A)(M_1(B)+\bar{M}_1(B)))] + [4(e_B+\bar{e}_B)F(A) + e_A(F(B)+\bar{F}(B)) + 8((e_B+\bar{e}_B)M_2(A)+e_A(M_2(B)+\bar{M}_2(B))+2M_1(A)(M_1(B)+\bar{M}_1(B)))] + n_B\alpha_1 + e_A(n_A-2)M_1(B) + 4e_B\alpha_2 + \alpha\alpha_1 + e_A(n_A-2)(M_1(B)+\bar{M}_1(B)) + 4(e_B(n_B-1)+\bar{e}_B(\bar{n}_B-1))\alpha_2 + \alpha\alpha_3 + 2e_A(M_1(B)+\bar{M}_1(B)) + 4(e_B(n_B-1)+\bar{e}_B(\bar{n}_B-1))\alpha_4$

Proof. Using Equation (1), we have:

$$\bar{HZ}(A_{+Q}B) = \sum_{(t_1,t_2)(y_1,y_2)\notin E(A_{+Q}B)} [d(t_1,y_1)+d(t_2,y_2)]^2.$$

$$\bar{HZ}(H_{1+Q}H_2) = [\sum_{x_1,x_2\in(V(Q(A)-V(A))}\sum_{y_1,y_2\in V_B} + \sum_{x_1,x_2\in V_A} + \sum_{\substack{x_1,x_2\in V(Q(A))\\ x_1\in V(A)\\ x_2\in V(Q(A)-V(A))}}]\sum_{y_1,y_2\in V_B}$$
$$[d(x_1,y_1)+d(x_2,y_2)]^2,$$
$$= \sum A + \sum B + \sum C. \quad (3.\ 7)$$

$$\sum A = \sum A_1 + \sum A_2 + \sum A_3 + \sum A_4.$$
$$\sum A_1 = \sum_{\substack{x_1x_2\notin E(Q(A))\\ x_1,x_2\in V(Q(A)-A)}} \sum_{y\in V_B} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2$$
$$= n_2 \sum_{\substack{x_1x_2\notin E(Q(A))\\ x_1,x_2\in V(Q(A)-A)}} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2 \ .$$

Take $\sum_{\substack{x_1x_2\notin E(Q(A))\\ x_1,x_2\in V(Q(A)-A)}} [d_{Q(A}(x_1)+d_{Q(A}(x_2)]^2 = \alpha_4$ so, $\sum A_1 = n_2\alpha_4$.

$$\sum A_2 = \sum_{x\in V(Q(A)-(A))}\sum_{y_1y_2\in E_B} [d_{Q(A}(x) + d_{Q(A}(x)]^2$$
$$= \sum_{x\in V(Q(A)-V(A))} (\sum_{y_1y_2\notin E_B} + \sum_{y_1y_2\in E_B})[d_{Q(A}(x)^2 + d_{Q(A}(x_2)^2]$$

$$= 2(\bar{e}_2 + (n_2 - 1)) \sum_{x \in V(Q(A)-(A))} [d_{Q(A}(x))]^2.$$

Take $\sum_{x \in V(Q(A)-(A))} [d_{Q(A}(x))]^2 = \alpha_5$, then $= 2(\bar{e}_2 + (n_2 - 1))\alpha_5$.

$$\sum A_3 = \sum_{\substack{x_1x_2 \in E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} \sum_{y_1y_2 \in E_B} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2$$

$$= \sum_{\substack{x_1x_2 \in E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} [\sum_{y_1y_2 \in E_B} (d_{Q(A}(x_1) + d_{Q(A}(x_2))^2 + \sum_{y_1y_2 \notin E_B} (d_{Q(A}(x_1) + d_{Q(A}(x_2))]^2$$

$$= 2(\bar{e}_2 + (n_2 - 1)) \sum_{\substack{x_1x_2 \in E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2$$

Take $\sum_{\substack{x_1x_2 \in E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2 = \alpha_6$, then $= 2(\bar{e}_2 + (n_2 - 1))\alpha_6$.

$$\sum A_4 = \sum_{\substack{x_1x_2 \notin E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} \sum_{y_1y_2 \in E_B} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2$$

$$= \sum_{\substack{x_1x_2 \notin E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} [\sum_{y_1y_2 \in E_B} (d_{Q(A}(x_1) + d_{Q(A}(x_2))^2 + \sum_{y_1y_2 \notin E_B} (d_{Q(A}(x_1) + d_{Q(A}(x_2))^2]$$

$$= 2(\bar{e}_2 + (n_2 - 1)) \sum_{\substack{x_1x_2 \notin E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2$$

Take $\sum_{\substack{x_1x_2 \notin E(Q(A)) \\ x_1,x_2 \in V(Q(A)-V(A))}} [d_{Q(A}(x_1) + d_{Q(A}(x_2)]^2 = \alpha_7$, then $= 2(\bar{e}_2 + (n_2 - 1))\alpha_7$.

Consequently,

$$\sum A = n_2\alpha_4 + 2(\bar{e}_2 + (n_2 - 1))(\alpha_7 + \alpha_6 + \alpha_5) \tag{3. 8}$$

Using Equation 6, for $\sum B$

$$\sum C = \sum C_1 + \sum C_2 + \sum C_3.$$

$$\sum C_1 = n_B\alpha_1 + e_A(n_A - 2)M_1(B) + 4e_B\alpha_2.$$

$$\sum C_2 = \alpha\alpha_1 + e_A(n_A - 2)(M_1(B) + \bar{M}_1(B)) + 4(e_B(n_B - 1) + \bar{e}_B(\bar{n}_B - 1))\alpha_2.$$

$$\sum C_3 = \alpha\alpha_3 + 2e_A(M_1(B) + \bar{M}_1(B)) + 4(e_B(n_B - 1) + \bar{e}_B(\bar{n}_B - 1))\alpha_4.$$

we obtained required result by putting the values in Equation 7.

Theorem 3.4. Let A and B be two simple connected graphs then HZ-coindex of $(A_{+T}B)$ is:
$\bar{HZ}(A_{+T}B) = n_2\alpha_4 + 2(\bar{e}_2 + (n_2 - 1))(\alpha_7 + \alpha_6 + \alpha_5) + 8\bar{e}_B M_1(A) + n_A \bar{F}(B) + 16e_A \bar{M}_1(B) + 2n_A \bar{M}_2(B) + 4n_B(F(A) + \bar{F}(A)) + 4M_1(B)(e_A + \bar{e}_A) + 8n_B(M_2(A) + \bar{M}_2(A)) + 16e_B(M_1(A) + \bar{M}_1(A) + 2[4(e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 8((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B)) + 2M_1(A)(M_1(B) + \bar{M}_1(B)))] + [2M_1(A)(M_1(B) + \bar{M}_1(B))) + 4(e_B + \bar{e}_B)F(A) + e_A(F(B) + \bar{F}(B)) + 8((e_B + \bar{e}_B)M_2(A) + e_A(M_2(B) + \bar{M}_2(B))] + n_B\alpha_1 + e_A(n_A - 2)M_1(B) + 4e_B\alpha_2 + \alpha\alpha_1 + e_A(n_A - 2)(M_1(B) + \bar{M}_1(B)) + 4(e_B(n_B - 1) + \bar{e}_B(\bar{n}_B - 1))\alpha_2 + \alpha\alpha_3 + 2e_A(M_1(B) + \bar{M}_1(B)) + 4(e_B + \bar{e}_B)(n_B - 1)\alpha_4.$

Proof. It follows from, Theorem 3.2 and Theorem 3.3.

4. APPLICATION EXAMPLE

We illustrate the results using particular D-sum graphs. Take two graphs $A \cong C_n$ and $B \cong P_2$ then values of hyper Zagreb coindex for their D-sum graph are given in Table 2. Table 2 and Figure 3, present that HZ-coindex of $C_{n+T}P_2$ is dominant than HZ-coindex of $C_{n+S}P_2$, $C_{n+R}P_2$, and $C_{n+Q}P_2$.

n, m	$\bar{HZ}(C_{n+S}P_m)$	$\bar{HZ}(C_{n+R}P_m)$	$\bar{HZ}(C_{n+Q}P_m)$	$\bar{HZ}(C_{n+T}P_m)$
3, 2	906	1656	1674	2496
4, 2	2512	3200	4496	5376
5, 2	4166	5520	7612	9280
6, 2	6216	8448	11496	14208
7, 2	8680	6584	17192	20160
8, 2	11552	16128	21664	27136
9, 2	14832	12880	27936	35136
10, 2	18520	23270	35000	44160
11, 2	22916	32208	42856	54208
12, 2	27120	38784	51504	65280
13, 2	32032	19202	60944	77376
14, 2	37384	53792	73920	90496
15, 2	45780	62160	84120	106560
16, 2	50216	74051	94212	124928
17, 2	55760	83088	106624	136000
18, 2	64512	91008	120024	153216
19, 2	70072	101840	134216	171456
20, 2	77840	114048	149968	192256

TABLE 2. For two graphs $A \cong C_n$ and $B \cong P_2$ values of hyper Zagreb coindex for their D-sum graphs.

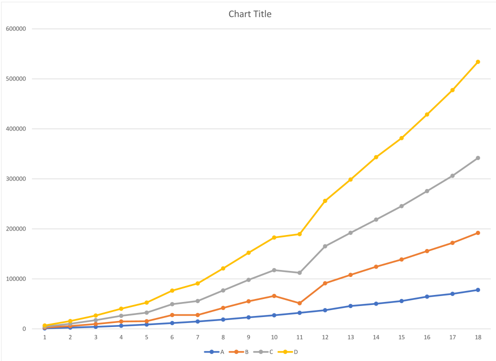

FIGURE 3. Graphical presentations of hyper Zagreb coindex $C_{n+D}P_2$ graphs such as $A_{+S}B$, $A_{+R}B$, $A_{+Q}B$ and $A_{+T}B$ are represented by blue, orange, gray, and yellow respectively.

5. CONCLUSION

In this paper, we studied D-sum graphs such as $A_{+D}B$, where $D \in \{S, R, Q, T\}$ using sub-division related operations and calculated HZ-coindex in the form of Zagreb indices and coincides, forgotten indices and coindices, HZ indices and coindices of their basic graphs A and B.

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