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Generalized Fuzzy Filters in Quantales and Their Approximations

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Abstract.: The present paper represents the behaviour of fuzzy filters and (α,β)-fuzzy filters in Quantale. The detailed study of relationship among crisp filter, fuzzy filters and (α,β)-fuzzy filters in quantale are discussed. An important part is played by quantale homomorphism which shows inverse image of $(\in,\in\vee q)$-fuzzy filter is again $(\in,\in\vee q)$-fuzzy filter. Under (α,β)-fuzzy map, it is seen that inverse image of (α,β)-fuzzy filter is again a fuzzy filter under quantale homomorphism. The relationship between fuzzy filter and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy filters are also discussed. Further, generalized approximation of fuzzy filter, $(\in,\in\vee q)$-fuzzy filter and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy filter are discussed.

AMS (MOS) Subject Classification Codes: 08-XX; 08Axx; 08A99
Key Words: Filter; Fuzzy Filter; $(\in,\in\vee q)$-$f$uzzy $f$ilter; $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy filter.

1. INTRODUCTION

Quantale theory was proposed by Mulvey [16]. It was based on defining an algebraic structure on complete lattice. Since quantale was defined on complete lattice so there must be a correlation between linear logic and quantale theory which was studied by Yetter, in his study. He presented a new classes of models for linear intuitionistic logic [36]. In recent years quantale is applied in vast research areas, like algebraic theory [12], rough set theory [13, 20, 33, 35], topological theory [8], theoretical computer science [25] and linear logic [7].

Fuzzy set theory, at first proposed by Zadeh [37], had given an important scientific and mathematical tool to the description of those frameworks which are perplexing or uncertain. The importance of combination of fuzzy sets and algebraic structures in terms of belongingness and quasi-coincidence (presented by Ming and Ming [18]) had been observed by relating different fuzzy algebraic structures to the techniques of belongingness and quasi-coincidence. For illustration, the idea of (α, β)- fuzzy ideals of hemirings was proposed by Dudek $et\ al.$, [5]. In terms of $(\in, \in \vee q)$-fuzzy interior ideals, ordered semigroups was characterized by Khan $et\ al.$, [10]. Ma $et\ al.$ studied $(\in, \in \vee q)$-fuzzy filters of RO-algebras [14]. An $(\in_\gamma, \in_\gamma \vee q_\delta)$- fuzzy interior ideals in ordered semigroups was proposed by Khan $et\ al.$, [11]. The significance of these new types of notion is increased further by the work of Ma $et\ al.$ They presented the idea of $(\in_\gamma, \in_\gamma \vee q_\delta)$-types fuzzy ideals of BCI-algebras [15]. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in semigroups were investigated by Shabir and Ali jointly [27]. Further, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy substructures in quantales were discussed by Qurashi and Shabir [23].

Moreover, fuzzy substructures in quantales were first investigated in [13]. They defined rough fuzzy substructures of quantale. Fuzzy filters and their characteristics were presented by Wang and Liang [30]. Definition of L-fuzzy filters of quantales and their related properties were expressed by Shan and Liu [29]. Generalized rough fuzzy substructures in quantales were introduced by Qurashi and Shabir [19]. Some results related to fuzzy hyperideals of hyperquantales [6], were introduced by Farooq $et\ al.$ Several authors related fuzzy set theory to different algebraic structures like groups, rings, modules, semirings, semigroups and ordered semigroups, etc. Some studies about regular and intra-regular semirings in terms of bipolar fuzzy ideals, was investigated by Shabir $et\ al.$ [28].

Rough set theory, introduced in 1982 by Pawlak [17], has a mathematical approach to imperfect knowledge. Many authors applied the concept of rough set theory to algebraic and fuzzy algebraic structures. Roughness in crisp substructures like Quantale, Quantale module and Rings, were introduced by Yang and Xu [35], Qurashi and Shabir [20] and Davvaz [3], respectively. Roughness in Hemirings was introduced by Ali $et\ al.$[1]. Rough Pythagorean fuzzy ideals in semigroups, were discussed by Hussain $et\ al.$ [9]. Generalization of approximation of fuzzy substructures in quantales in the form of $(\in, \in \vee q)$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ were studied by Qurashi and Shabir [21, 22]. Substructures of Γ-semihypergroups [34], in terms of rough prime bi-Γ-hyperideals were studied by Yaqoob $et\ al.$ Generalized roughness in $(\in, \in \vee q)$-fuzzy types in quantale and hemirings were introduced by Qurashi and Shabir [22] and Rameez $et\ al.$ [24], respectively. Concluding the above discussion, it is the first attempt to investigate generalized fuzzy filters and their approximations in quantale. In this study, it is important to observe that how complete congruence plays an important role while studying approximation of $(\in, \in \vee q)$-fuzzy and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy types substructures.

The whole paper is organized as follows. After introduction, some related definitions and results are presented in section 2 while section 3 presents fuzzy filters and (α, β)-fuzzy filters in quantales. The detailed study of relationship among crisp filter, fuzzy filters and (α, β)-fuzzy filters in quantale are discussed. Moreover, under (α, β)-fuzzy map, it is seen that inverse image of (α, β)-fuzzy filter is again a fuzzy filter under quantale homomorphism. Further, these concepts are applied to $(\in, \in \vee q)$-fuzzy filters of quantales in section 4 and important part is played by quantale homomorphism which shows inverse of $(\in, \in \vee q)$-fuzzy filter is again $(\in, \in \vee q)$-fuzzy filter. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filters and its relationship with fuzzy filters are stated in section 5. Moreover, lower approximations and upper approximations of fuzzy filters, $(\in, \in \vee q)$-fuzzy filter and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filters are studied in section 6, under complete congruence and congruence relations.

2. PRELIMINARIES

In the preliminaries, some definitions are introduced. These are necessary for further discussion.

Definition 2.1. [26] *A quantale $K = (K, \otimes)$ is a complete lattice K having an associative binary operation "$\otimes$" satisfying*

$$y \otimes (\vee_{i\in I} k_i) = \vee_{i\in I} (y \otimes k_i) \text{ and } (\vee_{i\in I} y_i) \otimes k = \vee_{i\in I} (y_i \otimes k)$$

for all $y, k \in K$ and $\{y_i\}, \{k_i\} \subseteq K$ $(i \in I)$ where I is an indexing set. Let $A_1, A_2 \subseteq K$. Then the following are defined as;

$$\begin{aligned} A_1 \otimes A_2 &= \{a_1 \otimes a_2 \mid a_1 \in A_1,\ a_2 \in A_2\}; \\ A_1 \vee A_2 &= \{a_1 \vee a_2 \mid a_1 \in A_1,\ a_2 \in A_2\}; \\ \vee_{i\in I} A_i &= \{\vee_{i\in I} a_i \mid a_i \in A_i\}. \end{aligned}$$

Let $\emptyset \neq Q_1 \subseteq K$. Then Q_1 is known as a subquantale of K if it is closed under $\otimes$ and arbitrary sup.

Throughout the paper, the symbol K will represent for quantale. The symbol $\perp$ and $\top$ will show the bottom and top element of quantale, unless stated otherwise.

Definition 2.2. [31] *Let K be a quantale. A non-empty subset F_r of K is said to be a filter of K if F_r is an upper set and closed under $\otimes$. That is*
(1) *For all $w \in K$ and for all $k \in F_r$, $k \leq w$ implies $w \in F_r$;*
(2) *$k, w \in F_r$ implies $k \otimes w \in F_r$ for all $k, w \in K$.*

Definition 2.3. [33, 35] *Let K be a quantale. An equivalence relation Ω on K is called a congruence on K if for all $k, w, x, y, k_i, w_i \in K$ where $i \in I$, we have $k\Omega w, x\Omega y \Longrightarrow (k \otimes x)\Omega(w \otimes y)$ and $k_i \Omega w_i$ $(i \in I) \Longrightarrow (\vee_{i\in I} k_i)\Omega(\vee_{i\in I} w_i)$.*

Definition 2.4. [33, 35] *A congruence Ω on a quantale K is called $\vee$-complete if $[k\vee w]_\Omega = [k]_\Omega \vee [w]_\Omega$ for all $k, w \in K$ and is called $\otimes$-complete if it satisfies $[k\otimes w]_\Omega = [k]_\Omega \otimes [w]_\Omega$ for all $k, w \in K$. A congruence which is both $\vee$-complete and $\otimes$-complete is called a complete congruence.*

Example 2.5. *Let $(K, \otimes)$ be a quantale where K is complete lattice shown in Figure 1 and $\otimes$ on the quantale is the same as the meet operation in the lattice K as shown in Table 1.*

Table. 1

$\otimes$	$\perp$	e	f	k	h	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
e	$\perp$	e	$\perp$	e	$\perp$	e
f	$\perp$	$\perp$	f	$\perp$	f	f
k	$\perp$	e	$\perp$	k	$\perp$	k
h	$\perp$	$\perp$	f	$\perp$	h	h
$\top$	$\perp$	e	f	k	h	$\top$

The subsets $F_1 = \{e, k, \top\}$, $F_2 = \{f, h, \top\}$ and $F_3 = \{k, \top\}$ of quantale K are examples of filters of K.

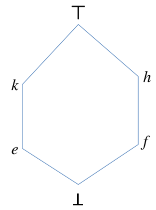

FIGURE 1

A fuzzy subset Γ of a quantale K is a function $\Gamma : K \longrightarrow [0,1]$. Throughout this paper, we shall employ Max for maximum and Min for minimum in $[0,1]$, unless stated otherwise. Moreover, the supremum and infimum for the elements of a quantale K will be represented by the symbols $\vee$ and $\wedge$, respectively.

Definition 2.6. [30] *Let Γ be a $fsst$ of a quantale K. Then Γ is called a fuzzy filter of K if*
(1) $\Gamma(k) \leq \Gamma(w)$ *if* $k \leq w$;
(2) $\Gamma(k \otimes w) \geq Min\{\Gamma(k), \Gamma(w)\}$ *for all* $k, w \in K$.

From here onward, we will write a fuzzy subset and fuzzy filter by $fsst$ and FFR. If Γ_1 and Γ_2 are $fsst$ of K. Then, $\Gamma_1 \subseteq \Gamma_2$ if and only if $\Gamma_1(k) \leq \Gamma_2(k)$ for all $k \in K$ and intersection of two $fsst$ are defined as $(\Gamma_1 \Cap \Gamma_2)(k) = Min\{\Gamma_1(k), \Gamma_2(k)\}$.

Definition 2.7. *Let K_{F_r} represent the characteristic function of a crisp subset F_r of a quantale K. Then $K_{F_r} : K \longrightarrow [0,1]$ is defined by*

$$K_{F_r}(k) = \begin{cases} 1, & if\ k \in F_r, \\ 0, & if\ k \notin F_r. \end{cases}$$

It is obvious that $\emptyset \neq F_r \subseteq K$ is a filter if and only if the characteristic function K_{F_r} of F_r is a FFR of K.

Definition 2.8. [26] *Let $(K_1, \otimes)$ and $(K_2, \otimes')$ be two quantales. Then $\xi : K_1 \longrightarrow K_2$ is called a quantale homomorphism (QHM) if,*
(1) $\xi(k \otimes b) = \xi(k) \otimes' \xi(b)$;
(2) $\xi(\vee_{i\in I} k_i) = \vee_{i \in I} \xi(k_i)$ *for all* $k, b \in K_1$ *and* $\{k_i\} \subseteq K_1$ $(i \in I)$.

A quantale homomorphism (QHM), $\xi : K_1 \longrightarrow K_2$ is called an epimorphism if ξ is onto K_2 and ξ is called a monomorphism if ξ is one-one. If ξ is bijective, then it is called an isomorphism. It is clear that if $k \leq b$, then $\xi(k) \leq \xi(b)$.

Definition 2.9. [32] *Let $\xi : K_1 \longrightarrow K_2$ be a mapping from a quantale K_1 to a quantale K_2, and let Γ and Γ' be fsst in K_1 and K_2, respectively. Then the image of Γ under ξ and the pre-image of Γ' under ξ are the fssts $\xi(\Gamma)$ and $\xi^{-1}(\Gamma')$, respectively, defined as follows:*

(1) $\xi(\Gamma)(k) = \begin{cases} \underset{x \in \xi^{-1}(k)}{Sup} \ \Gamma(x), & if\ \xi^{-1}(k) \neq \emptyset\ \forall\ k \in K_2; \\ 0, & otherwise \end{cases}$

(2) $\xi^{-1}(\Gamma')(k) = \Gamma'(\xi(k))$ *for all* $k \in K_1$.

If ξ *is a* QHM*, then* $\xi(\Gamma)$ *is called the homomorphic image of* Γ *under* ξ *and* $\xi^{-1}(\Gamma')$ *is called the homomorphic pre-image of* Γ'.

3. (α, β)-FUZZY FILTERS IN QUANTALES

Let α, β represent one of $\in, q, \in \vee q$ and $\in \wedge q$. Further, we will express FFR and (α, β)-FFR for fuzzy filter and (α, β)-fuzzy filter, respectively.

The following discussion is about the concept of belongingness and quasi-coincidence of a fuzzy point with a $fsst$.

A $fsst$ Γ of K is called a fuzzy point if

$$\Gamma(k) = \begin{cases} p, & \text{if } k = y \\ 0, & \text{otherwise} \end{cases}$$

$\forall\ k, y \in K$ and $p \in (0, 1]$ is its value where y is the support of Γ is represented by the symbol y_p is used to represent fuzzy point for a relation between fuzzy point k_p and a $fsst$ Γ in a set K, the meaning of the symbol $k_p \alpha g$ was explained by Pu and Liu, where;

(1) $k_p \in \Gamma$ means that k_p belongs to Γ if $\Gamma(k) \geq p$.

(2) $k_p q \Gamma$ means that k_p is quasi-coincident with Γ if $\Gamma(k) + p > 1$.

(3) $k_p(\in \vee q)\Gamma$ means k_p belongs to Γ or k_p is a quasi-coincident with Γ that is $\Gamma(k) \geq p$ or $\Gamma(k) + p > 1$. Also, $k_p(\in \wedge q)\Gamma$ denotes that $k_p \in \Gamma$ and $k_p q \Gamma$.

For a $fsst$ Γ of K such that $\Gamma(k) \leq 0.5$ for any $k \in K$ in the case $k_p(\in \wedge q)\Gamma$, we have $\Gamma(k) \geq p$ and $\Gamma(k) + p > 1$. Thus, $1 < \Gamma(k) + p \leq \Gamma(k) + \Gamma(k) = 2\Gamma(k)$. This shows that $\Gamma(k) \geq 0.5$. Hence, $\{k_p : k_p(\in \wedge q)\Gamma\} = \emptyset$. Thus, the case $\alpha = \in \wedge q$ is omitted.

If $k_p \in \Gamma$, $k_p q \Gamma$ or $k_p\ (\in \vee q)\Gamma$ does not hold, then we write as $k_p\ \overline{\in}\ \Gamma$, $k_p\ \overline{q}\ \Gamma$ or $k_p\ \overline{(\in \vee q)}\ \Gamma$, respectively. Thus, $k_p \overline{\alpha} \Gamma$ means that $k_p \alpha \Gamma$ does not hold. Each $fsst$ Γ defined on K can be characterized by level subsets. That is by the sets of the form $L(\Gamma; p) = \{k \in K : \Gamma(k) \geq p\}$ where $p \in [0, 1]$. An important part is played by the support of Γ, that is set $\Gamma_\circ = \{k \in K : \Gamma(k) > 0\}$.

Proposition 3.1. *Let* Γ_1 *and* Γ_2 *be* $FFRs$ *of a quantale* K. *Then* $(\Gamma_1 \Cap \Gamma_2)$ *is a* FFR *of* K.

Proof. Let $k_1, k_2 \in K$ with $k_1 \leq k_2$. As Γ_1 and Γ_2 are the FFR of K, so

$\Gamma_1(k_1) \leq \Gamma_1(k_2)$ and $\Gamma_2(k_1) \leq \Gamma_2(k_2)$

$\Longrightarrow Min(\Gamma_1(k_1), \Gamma_2(k_1)) \leq Min(\Gamma_1(k_2), \Gamma_2(k_2))$

$\Longrightarrow (\Gamma_1 \Cap \Gamma_2)(k_1) \leq (\Gamma_1 \Cap \Gamma_2)(k_2)$.

Next, as $\Gamma_1(k_1 \otimes k_2) \geq Min(\Gamma_1(k_1), \Gamma_1(k_2))$ and $\Gamma_2(k_1 \otimes k_2) \geq Min(\Gamma_2(k_1), \Gamma_2(k_2))$.

$\Longrightarrow Min(\Gamma_1(k_1 \otimes k_2), \Gamma_2(k_1 \otimes k_2)) \geq Min(Min(\Gamma_1(k_1), \Gamma_1(k_2)), Min(\Gamma_2(k_1), \Gamma_2(k_2))$

$\Longrightarrow Min(\Gamma_1(k_1 \otimes k_2), \Gamma_2(k_1 \otimes k_2)) \geq Min(Min\{\Gamma_1(k_1), \Gamma_2(k_1)\}, Min\{\Gamma_1(k_2), \Gamma_2(k_2)))$

$\Longrightarrow (\Gamma_1 \Cap \Gamma_2)(k_1 \otimes k_2) \geq Min((\Gamma_1 \Cap \Gamma_2)(k_1), (\Gamma_1 \Cap \Gamma_2)(k_2))$.

Therefore, $(\Gamma_1 \Cap \Gamma_2)$ is a FFR of K. □

The next Proposition has simple proof and so is omitted.

Proposition 3.2. *A $fsst$, Γ of K is a FFR of a quantale K if and only if $\emptyset \neq L(\Gamma; p)$ for all $p \in (0, 1]$ is a filter of K.*

Example 3.3. *Consider the quantale discussed in Example 2.5. Filters of K are K, $\{f, h, \top\}$ $\{\top\}$ and $\{h, \top\}$. Define a fsst, $\Gamma : K \longrightarrow (0, 1]$ by $\Gamma = \frac{0.5}{\bot} + \frac{0.5}{e} + \frac{0.6}{f} + \frac{0.5}{k} + \frac{0.7}{h} + \frac{0.9}{\top}$. Then*

$$L(\Gamma; p) = \begin{cases} K & if \;\; 0 < p \leq 0.5 \\ \{f, h, \top\} & if \;\; 0.5 < p \leq 0.6 \\ \{h, \top\} & if \;\; 0.6 < p \leq 0.7 \\ \{\top\} & if \;\; 0.7 < p \leq 0.9 \\ \emptyset & if \;\; 0.9 < p \leq 1 \end{cases}$$

Thus, by Proposition 3.2, Γ is a FFR of K.

Theorem 3.4. *Let Γ be a $fsst$ of K. Then $\emptyset \neq L(\Gamma; p)$ is a filter of K for all $p \in (0.5, 1]$ if and only if Γ satisfies the following:*
(1) $Max(\Gamma(y), 0.5) \geq \Gamma(k)$ *with* $k \leq y$;
(2) $Max(\Gamma(k \otimes y), 0.5) \geq Min(\Gamma(k), \Gamma(y))$ *for all* $k, y \in K$.

Proof. Let $L(\Gamma; p)$ be a filter of K for all $p \in (0.5, 1]$. Then if there exist $k, w \in K$ with $k \leq w$ such that the condition (1) is not valid, then $Max(\Gamma(w), 0.5) < \Gamma(k) = r$. Then $r \in (0.5, 1]$, $k \in U(\Gamma; r)$. But $r > \Gamma(w)$ implies that $w \notin L(\Gamma; r)$, we get a contradiction. Hence condition (1) is valid.

If there are $k, w \in K$ such that $Min(\Gamma(k), \Gamma(w)) = s > Max(\Gamma(k \otimes w), 0.5)$, then $k, w \in U(\Gamma; s)$ and $s \in (0.5, 1]$. But $\Gamma(k \otimes w) < s$. Thus $k \otimes w \notin L(\Gamma; s)$, a contradiction. Hence condition (2) is valid.

Conversely, let conditions (1) and (2) be satisfied. Let $w, k \in K$ with $w \leq k$ be such that $w \in L(\Gamma; p)$ for some $p \in (0.5, 1]$. Then $\Gamma(w) \geq p$. Since $w \leq k$ so it follows by condition (1)

$$Max(\Gamma(k), 0.5) \geq \Gamma(w) \geq p > 0.5$$

so that $\Gamma(k) \geq p$, i.e., $k \in L(\Gamma; p)$. Now, for $w, k \in U(\Gamma; p)$, we have,

$$Max(\Gamma(w \otimes k), 0.5) \geq Min(\Gamma(w), \Gamma(k)) \geq p > 0.5$$

and so $\Gamma(w \otimes k) \geq p$. It follows that $w \otimes k \in L(\Gamma; p)$. Thus $L(\Gamma; p)$ is a filter of K for all $p \in (0.5, 1]$. □

The next definition is about (α, β)-FFR.

Definition 3.5. *Let Γ be a $fsst$ of a quantale K. Then Γ is called an (α, β)-FFR of K if it satisfies:*
(1) $z_p \alpha \Gamma \rightarrow y_p \beta \Gamma$ *for* $z \leq y$;
(2) $z_p \alpha \Gamma$ *and* $y_v \alpha \Gamma \rightarrow (z \otimes y)_{Min(p,v)} \beta \Gamma$ *for all* $z, y \in K$ *and* $p, v \in (0, 1]$.

Proposition 3.6. *Let K be a quantale and Γ be a non-zero (α, β)-FFR of K. Then $\Gamma_\circ = \{k \in K \mid \Gamma(k) > 0\}$ is a filter of K.*

Proof. Let $k, u \in K$ with $k \leq u$ and $k \in \Gamma_\circ$. Then $0 < \Gamma(k)$. Suppose that $\Gamma(u) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $k_{\Gamma(k)} \alpha \Gamma$ but $u_{\Gamma(u)} \overline{\beta} \Gamma$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Moreover, $k_1 q \Gamma$, but $u_1 \overline{\beta} \Gamma$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence

$\Gamma(u) > 0$, that is $u \in \Gamma_\circ$. Now let $k, u \in \Gamma_\circ$. Then $\Gamma(k) > 0$ and $\Gamma(u) > 0$. Assume that $\Gamma(k \otimes u) = 0$ and let $\alpha \in \{\in, \in \vee q\}$, then $k_{\Gamma(k)}\alpha g$ and $u_{\Gamma(u)}\alpha g$ but $(k \otimes u)_{inf(\Gamma(k),\Gamma(u))}\overline{\beta}\Gamma$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Also $k_1 qg$ and $u_1 qg$ but $(k \otimes u)_1 \overline{\beta}\Gamma$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Thus, $\Gamma(k \otimes u) > 0$ and $k \otimes u \in \Gamma_\circ$. Therefore $\Gamma_\circ$ is a filter of K. □

Proposition 3.7. *Let K be a quantale and F_r be a filter of K. Then a $fsst$ Γ of K such that*

$$\Gamma(z) = \begin{cases} \geq 0.5 & if\ z \in F_r \\ 0 & if\ z \in K \setminus F_r. \end{cases}$$

is an $(\alpha, \in \vee q)$-FFR of K.

Proof. Suppose F_r is a filter of K.

(**i**) Let $k, y \in K$ with $k \leq y$ and $m \in (0, 1]$ be such that $k_m \in \Gamma$. Then $k \in F_r$ and we have $y \in F_r$. If $m \leq 0.5$ then $\Gamma(y) \geq 0.5 \geq m$ implies $\Gamma(y) \geq m$, and so $y_m \in \Gamma$. If $m > 0.5$ then $\Gamma(y) + m > 0.5 + 0.5 = 1$ and $y_m q\Gamma$. Hence $y_m(\in \vee q)\Gamma$. Let $m, r \in (0, 1]$ and $k, y \in K$ with $k_m \in \Gamma$ and $y_r \in \Gamma$. Thus $k, y \in F_r$ and we have $k \otimes y \in F_r$. If $Min(m, r) \leq 0.5$ then $\Gamma(k \otimes y) \geq 0.5 \geq Min(m, r)$ and so $\Gamma(k \otimes y) \geq Min(m, r)$ implies $(k \otimes y)_{Min(m,r)} \in \Gamma$. If $Min(m, r) > 0.5$ then $\Gamma(k \otimes y) + Min(m, r) > 0.5 + 0.5 = 1$ and so $(k \otimes y)_{Min(m,r)} q\Gamma$. Hence $(k \otimes y)_{Min(m,r)}(\in \vee q)\Gamma$.

(**ii**) Let $m \in (0, 1]$ and $k, y \in K$ with $k \leq y$ be such that $k_m q\Gamma$. Then $k \in F_r$ and $y \geq k \in F_r$ implies $y \in F_r$. If $m \leq 0.5$ then $\Gamma(y) \geq 0.5 \geq m$ implies $\Gamma(y) \geq m$ and so $y_m \in \Gamma$. If $m > 0.5$ then $\Gamma(y) + m > 0.5 + 0.5 = 1$ and $y_m q\Gamma$. Hence $y_m(\in \vee q)\Gamma$. Let $k, y \in K$ and $m, r \in (0, 1]$ be such that $k_m qg$ and $y_r qg$. Then $k, y \in F_r$ and so $k \otimes y \in F_r$. If $Min(m, r) \leq 0.5$ then $\Gamma(k \otimes y) \geq 0.5 \geq Min(m, r)$ and so $\Gamma(k \otimes y) \geq Min(m, r)$ implies $(k \otimes y)_{Min(m,r)} \in \Gamma$. If $Min(m, r) > 0.5$ then $\Gamma(k \otimes y) + Min(m, r) > 0.5 + 0.5 = 1$ and so $(k \otimes y)_{Min(m,r)} q\Gamma$. Hence $(k \otimes y)_{Min(m,r)}(\in \vee q)\Gamma$.

(**iii**) Let $m, v \in (0, 1]$ and $y, k \in K$ be such that $y_m \in \Gamma$ or $k_v q\Gamma$. Then $\Gamma(y) \geq m$ and $\Gamma(k) + v > 1$. Thus, $y, k \in F_r$ and so $y \otimes k \in F_r$, we have $\Gamma(y \otimes k) \geq 0.5$. Thus, $(y \otimes k)_{Min(m,v)} \in \Gamma$ for $Min(m, v) \leq 0.5$ and $(y \otimes k)_{Min(m,v)} q\Gamma$ for $Min(m, v) > 0.5$. Thus $(y \otimes k)_{Min(m,v)}(\in \vee q)\Gamma$. □

Definition 3.8. [2] *Let Γ_1 and Γ_2 be two fsst of K_1 and K_2 respectively and ξ be a mapping of K_1 into K_2. Then ξ is said to be an (α, β)-fuzzy map from Γ_1 to Γ_2 if for all $k \in K_1$ and $t \in (0, 1]$, $k_t \alpha \Gamma_1$ implies $(\xi(k))_t \beta \Gamma_2$.*

Proposition 3.9. *Let Γ be a fsst of a quantale K and $\xi : K_1 \longrightarrow K_2$ be a QHM. Then $(\xi(k))_p \alpha \Gamma$ if and only if $k_p \alpha \xi^{-1}(\Gamma)$ for all $k \in K$ and $p \in (0, 1]$.*

Proof. Let $\alpha = \in$. Then $(\xi(k))_p \in \Gamma \Longleftrightarrow \Gamma(\xi(k)) \geq p \Longleftrightarrow \xi^{-1}(\Gamma)(k) \geq p \Longleftrightarrow k_p \in \xi^{-1}(\Gamma)$. Let $\alpha = q$, then $(\xi(k))_p q\Gamma \Longleftrightarrow \Gamma(\xi(k)) + p > 1 \Longleftrightarrow \xi^{-1}(\Gamma)(k) + p > 1 \Longleftrightarrow k_p q \xi^{-1}(\Gamma)$. Similarly, the other cases can be obtained. □

Theorem 3.10. *Let $\xi : K_1 \longrightarrow K_2$ be a QHM. Let Γ_1 and Γ_2 be (α, β)-FFR of K_1 and K_2, respectively. If ξ is an (α, α)-fuzzy map from Γ_1 to Γ_2, then $\xi^{-1}(\Gamma_2)$ is an (α, β)-FFR of K_1.*

Proof. Let $z, w \in K_1$. As ξ is a QHM, so $\xi(z), \xi(w) \in K_2$. Since Γ_1 is an (α, β)-FFR of K_1, so

(a) $z_t \alpha \Gamma_1$ and $z \leq w \longrightarrow w_t \beta \Gamma_1$;

(b) $y_p \alpha \Gamma_1, w_v \alpha \Gamma_1 \longrightarrow (y \otimes w)_{Min(p,v)} \beta \Gamma_1$ for all $y, z, w \in K_1$ and $p, v, t \in (0, 1]$.

Also, since ξ is an (α, α)-fuzzy map from Γ_1 to Γ_2, we have,

(c) $(\xi(z))_t \alpha \Gamma_2$ and $\xi(z) \leq \xi(w)$ (order preserving);

(d) $(\xi(y))_p \alpha \Gamma_2$ and $(\xi(w))_v \alpha \Gamma_2$;

By Proposition 3.9, we obtain

(e) $z_t \alpha \xi^{-1}(\Gamma_2)$;

(f) $y_p \alpha \xi^{-1}(\Gamma_2)$ and $w_v \alpha \psi^{-1}(\Gamma_2)$;

As Γ_2 is an (α, β)-FFR of K_2, hence we have,

(g) $\xi(w)_t \beta \Gamma_2$;

(h) $(\xi(y) \otimes' \xi(w))_{Min(p,v)} \beta \Gamma_2$;

but ξ is a QHM as well, hence

(i) $(\xi(y) \otimes' \xi(w))_{Min(p,v)} \beta \Gamma_2 \longrightarrow (\xi(y \otimes w))_{Min(p,v)} \beta \Gamma_2$

By Proposition 3.9, we obtain $(y \otimes w)_{Min(p,v)} \beta \xi^{-1}(\Gamma_2)$ and $w_t \beta \xi^{-1}(\Gamma_2)$. Hence $\xi^{-1}(\Gamma_2)$ is a (α, β)-FFR of K_1. □

4. $(\in, \in \vee q)$-Fuzzy Filters of Quantale

$(\in, \in \vee q)$-fuzzy filter in quantale and characterization the filters of quantale in terms of $(\in, \in \vee q)$-fuzzy filter are introduced in this section. Next the shortened form $(\in, \in \vee q)$-FFR will be written for $(\in, \in \vee q)$-fuzzy filter.

Definition 4.1. *A fsst, Γ of a quantale K is called an $(\in, \in \vee q)$-FF of K if it satisfies:*

(1) $z \leq y, z_p \in \Gamma \longrightarrow y_p (\in \vee q) \Gamma$;

(2) $z_p \in \Gamma, y_v \in \Gamma \longrightarrow (z \otimes y)_{Min(p,v)} (\in \vee q) \Gamma$ *for all* $z, y \in K$ *and* $p, v \in (0, 1]$.

Example 4.2. *Consider the quantale in Example 2.5. Let* $\Gamma = \frac{0.5}{\bot} + \frac{0.6}{e} + \frac{0.65}{f} + \frac{0.6}{k} + \frac{0.7}{h} + \frac{0.9}{\top}$. *Then Γ is an $(\in, \in \vee q)$-FFR of K. But*

(1) Γ *is not an* $(\in, \in)$-FFR *of* K, *since* $e_{0.58} \in \Gamma$ *and* $f_{0.63} \in \Gamma$ *but* $(e \otimes f)_{Min(0.63,0.58)} = \bot_{0.58} \overline{\in} \Gamma$.

(2) Γ *is not an* $(q, \in)$-FFR *of* K, *since* $f_{0.52} \in \Gamma$ *and* $k_{0.51} \in \Gamma$ *but* $(f \otimes k)_{Min(0.52,0.51)} = \bot_{0.5} \overline{\in} \Gamma$.

(3) Γ *is not an* $(\in, q)$-FFR *of* K, *since* $k_{0.57} \in \Gamma$ *and* $h_{0.4} \in \Gamma$ *but* $(k \otimes h)_{Min(0.57,0.4)} \in \Gamma = \bot_{0.4} \overline{q} \Gamma$.

Lemma 4.3. *A $fsst$, Γ in a quantale K is a FFR of K if and only if it satisfies*;

(1) $w_v \in \Gamma$ *and* $w \leq k \longrightarrow k_v \in \Gamma$;

(2) $k_p, w_v \in \Gamma \longrightarrow (k \otimes w)_{Min(p,v)} \in \Gamma$ *for all* $k, w \in K$ *and* $p, v \in (0, 1]$.

Proof. Let Γ be a FF of K. Let $w_v \in \Gamma$ for some $v \in (0, 1]$. Then $\Gamma(w) \geq v$. Since Γ is a FF of K so, for $w \leq k$, we have $v \leq \Gamma(w) \leq \Gamma(k)$. This shows that $\Gamma(k) \geq v$. Hence $k_v \in \Gamma$. Consider $k, w \in K$, $p, v \in (0, 1]$ be such that $k_p \in \Gamma$ and $w_v \in \Gamma$. Then $\Gamma(k) \geq p$ and $\Gamma(w) \geq v$. But Γ is a FF of K so, we have $\Gamma(k \otimes w) \geq Min(\Gamma(k), \Gamma(w)) \geq Min(p, v)$. Thus $\Gamma(k \otimes w) \geq Min(p, v)$. This implies that $(k \otimes w)_{Min(p,v)} \in \Gamma$.

Conversely, suppose that Γ satisfies the above two conditions. To show that for all $k, w \in K$ and $k \leq w$ implies $\Gamma(k) \leq \Gamma(w)$. Let $\Gamma(k) > \Gamma(w)$ for some $k, w \in K$. Then there exists $v \in (0,1]$ such that $\Gamma(k) \geq v > \Gamma(w)$. Then $k_v \in \Gamma$ but $w_v \overline{\in} \Gamma$, a contradiction to the hypothesis (1). Now we show that $Min(\Gamma(k), \Gamma(w)) \leq \Gamma(k \otimes w)$ for all $w, k \in K$. On contrary suppose that $\Gamma(a \otimes c) < Min(\Gamma(a), \Gamma(c))$ for some $a, c \in K$. Let $p \in (0,1]$ be such that $\Gamma(a \otimes c) < p \leq Min(\Gamma(a), \Gamma(c))$. Then $\Gamma(a) > p$ and $\Gamma(c) > p$ but $(a \otimes c)_p$ $\overline{\in}$ Γ. This contradicts our hypothesis (2). Thus $\Gamma(k \otimes w) \geq Min(\Gamma(k), \Gamma(w))$ for all $k, w \in K$. Hence Γ is a FFR of a quantale K. □

Remark 4.4. *A $fsst, \Gamma$ of a quantale K is a FF of K if and only if Γ is an $(\in, \in)$-FF of K.*

Theorem 4.5. *A $fsst$ Γ of K is an $(\in, \in \vee q)$-FFR of K if and only if the following conditions are satisfied:*
(1) $\Gamma(y) \geq Min\{\Gamma(k), 0.5\}$ *for* $k \leq y$;
(2) $\Gamma(k \otimes y) \geq Min\{\Gamma(k), \Gamma(y), 0.5\}$ *for all* $k, y \in K$.

Proof. Let Γ be an $(\in, \in \vee q)$-FFR and $k, y \in K$ be such that $k \leq y$. If $\Gamma(k) = 0$, then $\Gamma(y) \geq Min(\Gamma(k), 0.5)$. Let $\Gamma(k) \neq 0$ and assume, on the contrary that $\Gamma(y) < Min(\Gamma(k), 0.5)$. Take $v \in (0,1]$ such that $\Gamma(y) < v \leq Min(\Gamma(k), 0.5)$. Case-1 If $\Gamma(k) < 0.5$, then $\Gamma(y) < v \leq \Gamma(k)$ and so $k_v \in \Gamma$ but $y_v \overline{\in} \Gamma$. Also $\Gamma(y) + v < 0.5 + 0.5 = 1$ so $y_v \overline{q} \Gamma$. Thus, $k_v \in \Gamma$ but $y_v (\overline{\in \vee q}) \Gamma$, a contradiction. Case-2 If $\Gamma(k) \geq 0.5$ then $\Gamma(y) < 0.5$ and so $k_{0.5} \in \Gamma$ but $y_{0.5} \overline{\in} \Gamma$ and $\Gamma(y) + 0.5 < 1$, i.e., $y_{0.5} \overline{q} \Gamma$, again a contradiction. Hence $\Gamma(y) \geq Min(\Gamma(k), 0.5)$ for all $k, y \in Q_v$ with $k \leq y$. Let $w, y \in K$ be such that $\Gamma(w \otimes y) < Min(\Gamma(w), \Gamma(y), 0.5)$. Take $p \in (0,1]$ such that $\Gamma(w \otimes y) < p \leq Min(\Gamma(w), \Gamma(y), 0.5)$. Case-1 If $Min(\Gamma(w), \Gamma(y)) < 0.5$ then $\Gamma(w \otimes y) < p \leq Min(\Gamma(w), \Gamma(y))$ and $w_p, y_p \in \Gamma$ but $(w \otimes y)_p \overline{\in} \Gamma$. Also we have, $\Gamma(w \otimes y) + p < 0.5 + 0.5 = 1$, so $(w \otimes y)_p \overline{q} \Gamma$, a contradiction. Let $0.5 \leq Min(\Gamma(w), \Gamma(y))$. Then $w_{0.5}, y_{0.5} \in \Gamma$ but $(w \otimes y)_{0.5} \overline{\in} \Gamma$ and $\Gamma(w \otimes y) + 0.5 < 1$, i.e., $(w \otimes y)_{0.5} \overline{q} \Gamma$, again a contradiction. Thus, $\Gamma(w \otimes y) \geq Min(\Gamma(w), \Gamma(y), 0.5)$ for all $w, y \in K$.

Conversely suppose that the conditions (1) and (2) are satisfied. Let $w, k \in K$ and $w_v \in \Gamma$ with $w \leq k$ for some $v \in (0,1]$. Then $\Gamma(w) \geq v$. By hypothesis, $\Gamma(k) \geq Min(\Gamma(w), 0.5) \geq Min(v, 0.5)$. Case-1. If $v \leq 0.5$, then $\Gamma(k) \geq v$ and $k_v \in \Gamma$. If $v > 0.5$ then $\Gamma(k) + v > 0.5 + 0.5 = 1$ and so $k_v q \Gamma$, i.e., k_v $(\in \vee q)\Gamma$. Let $v_1, v_2 \in (0,1]$ and $w, k \in K$ be such that w_{v_1}, $k_{v_2} \in \Gamma$. Then $\Gamma(w) \geq v_1$ and $\Gamma(k) \geq v_2$ and so by hypothesis we have, $Min(v_1, v_2, 0.5) \leq Min(\Gamma(w), \Gamma(k), 0.5) \leq \Gamma(w \otimes k)$. Case-1. If $Min(v_1, v_2) \leq 0.5$ then $\Gamma(w \otimes k) \geq Min(v_1, v_2)$ and $(w \otimes k)_{Min(v_1, v_2)} \in \Gamma$. Case-2. If $Min(v_1, v_2) > 0.5$ then $\Gamma(w \otimes k) + Min(v_1, v_2) > 0.5 + 0.5 = 1$ and so $(w \otimes k)_{Min(v_1, v_2)} qg$. Hence $(w \otimes k)_{Min(v_1, v_2)}$ $(\in \vee q)\Gamma$. Consequently, Γ is an $(\in, \in \vee q)$-FFR of K. □

Remark 4.6. *A $fsst, \Gamma$ of a quantale K is an $(\in, \in \vee q)$-FFR of K if and only if conditions (1) and (2) of Theorem 4.5 are satisfied.*

We have the following Corollary from the above Definition.

Corollary 4.7. *Every $(\in, \in)$-FF of K is an $(\in, \in \vee q)$-FF of K.*

Proof. The proof is simple. □

For $(\in, \in \vee q)$-FFR to be an $(\in, \in)$-FFR of K, some condition is imposed in the next Proposition.

Proposition 4.8. *Let Γ be an $(\in, \in \vee q)$-FFR of K such that $\Gamma(z) < 0.5$ for all $z \in K$. Then Γ is an $(\in, \in)$-FF of K.*

Proof. Let Γ be an $(\in, \in \vee q)$-FFR of K such that $\Gamma(z) < 0.5$ for all $z \in K$. Then by Theorem 4.5, if $z \leq y$ then $\Gamma(y) \geq Min(\Gamma(z), 0.5) = \Gamma(z)$. Now if $z, w \in K$ then $\Gamma(z \otimes y) \geq Min(\Gamma(z), \Gamma(y), 0.5) = Min(\Gamma(z), \Gamma(y)$. Hence Γ is an $(\in, \in)$-FFR of K by Lemma 4.3. □

Using Theorem 4.5, the following characterizations of FFR of quantale are suggested.

Lemma 4.9. *Let $(K, \otimes)$ be a quantale and $\emptyset \neq F_r \subseteq K$. Then the K_{F_r}(characteristic function) is an $(\in, \in)$-FFR of K if and only if F_r is a filter of K.*

Proof. Let $w, k \in K$ be such that $k \leq w$ and $k_p \in K_{F_r}$ where $p \in (0, 1]$. Then $K_{F_r}(k) \geq p > 0$, and so $K_{F_r}(k) = 1$, i.e., $k \in F_r$. Since F_r is a filter, we have $w \in F_r$ and so $K_{F_r}(w) = 1 \geq p$. Therefore $w_p \in K_{F_r}$. Suppose $p, v \in (0, 1]$ and $w, k \in K$ be such that $w_p \in K_{F_r}$ and $k_v \in K_{F_r}$. Then $K_{F_r}(w) \geq p > 0$ and $K_{F_r}(k) \geq v > 0$, which show that $K_{F_r}(w) = K_{F_r}(k) = 1$. Thus $w, k \in F_r$ and F_r is a filter so $w \otimes k \in F_r$. It shows that $K_{F_r}(w \otimes k) = 1 \geq Min(p, v)$ so that $(w \otimes k)_{Min(p,v)} \in K_{F_r}$ and consequently K_{F_r} is an $(\in, \in)$-FFR of K.

Conversely, let K_{F_r} be an $(\in, \in)$-FFR of K and $w, k \in F_r$. Then $w_1 \in K_{F_r}$ and $k_1 \in K_{F_r}$ which show that $(w \otimes k)_1 = (w \otimes k)_{Min(1,1)} \in K_{F_r}$. Hence $K_{F_r}(w \otimes k) = 1$, and so $w \otimes k \in F_r$. Let $w, k \in K$ and $w \leq k$ be such that $w \in F_r$. Then $K_{F_r}(w) = 1$, and thus $w_1 \in K_{F_r}$. Since K_{F_r} is an $(\in, \in)$-FFR, so we have $k_1 \in K_{F_r}$. Thus $K_{F_r}(k) = 1$ and $k \in F_r$. Hence F_r is a filter of K. □

Theorem 4.10. *The K_{F_r} is an $(\in, \in \vee q)$-FFR of K if and only if F_r is a filter of K for $\emptyset \neq F_r \subseteq K$.*

Proof. Suppose K_{F_r} be an $(\in, \in \vee q)$-FFR of K and $w, k \in F_r$. Then $w_1 \in K_{F_r}$ and $k_1 \in K_{F_r}$ which show that $(w \otimes k)_1 = (w \otimes k)_{Min(1,1)}$ $(\in \vee q) K_{F_r}$. Hence $K_{F_r}(w \otimes k) > 0$, and so $w \otimes k \in F_r$. Let $w, k \in K$ and $k \in F_r$ be such that $k \leq w$. Then $K_{F_r}(k) = 1$, and thus $k_1 \in K_{F_r}$. Since K_{F_r} is an $(\in, \in \vee q)$-FFR, so we have $w_1 \in K_{F_r}$. Thus $K_{F_r}(w) = 1$. Hence $w \in F_r$.

Conversely, if F_r is a filter of K, then K_{F_r} is an $(\in, \in)$-FF of K by lemma 4.9, and therefore K_{F_r} is an $(\in, \in \vee q)$-FFR of K by Corollary 4.7. □

Theorem 4.11. *A $fsst$ Γ of K is an $(\in, \in \vee q)$-FFR of K if and only if $L(\Gamma; p) = \{k \in K : \Gamma(k) \geq p\}$ is a filter of K for all $p \in (0, 0.5]$.*

Proof. Suppose Γ is an $(\in, \in \vee q)$-FF of K. Let $w, b \in K$ be such that $w \leq b$, and let $p \in (0, 0.5]$ be such that $w \in L(\Gamma; p)$. Then $\Gamma(w) \geq p$ and it is clear from Theorem $4.5(1)$ that

$$\Gamma(b) \geq Min(\Gamma(w), 0.5) \geq inf(p, 0.5) = p$$

and so $b \in L(\Gamma; p)$. Let $w, a \in L(\Gamma; p)$ for some $p \in (0, 0.5]$. Then from Theorem $4.5(2)$, we have $\Gamma(w \otimes a) \geq Min(\Gamma(w), \Gamma(a), 0.5) \geq Min(p, 0.5) = p$, and so $w \otimes a \in L(\Gamma; p)$.

Conversely, let $L(\Gamma; p)$ be a filter of K for all $p \in (0, 0.5]$. If there exist $a, y \in K$ with $a \leq y$ such that $\Gamma(y) < Min(\Gamma(a), 0.5)$, select $v \in (0, 0.5]$ such that $\Gamma(y) <$

$v \leq Min(\Gamma(a), 0.5)$, then $a \in L(\Gamma; v)$ but $y \notin L(\Gamma; v)$, a contradiction. Hence $\Gamma(y) \geq Min(\Gamma(a), 0.5)$ for all $a, y \in K$ with $a \leq y$. If there exist $k, y \in K$ such that $\Gamma(k \otimes y) < Min(\Gamma(k), \Gamma(y), 0.5)$. Choose $s \in (0, 0.5]$ such that $Min(\Gamma(k), \Gamma(y), 0.5) \geq s > \Gamma(k \otimes y)$. Then $k, y \in L(\Gamma; s)$ but $k \otimes y \notin L(\Gamma; s)$, a contradiction. Hence $Min(\Gamma(k), \Gamma(y), 0.5) \leq \Gamma(k \otimes y)$ for all $k, y \in K$. By Theorem 4.5, Γ is an $(\in, \in \vee q)$-FFR of K. □

Theorem 4.12. *Let K_1 and K_2 be two quantales and $\xi : K_1 \longrightarrow K_2$ be a QHM. Let Γ be $(\in, \in \vee q)$-FFR of K_2. Then $\xi^{-1}(\Gamma)$ is an $(\in, \in \vee q)$-FFR of K_1.*

Proof. Let $k, y \in K$ be such that $y \leq k$. Then $\xi(y) \leq \xi(k)$.

$$\begin{aligned} \xi^{-1}(\Gamma)(k) &= \Gamma(\xi(k)) \\ &\geq Min\{\Gamma(\xi(y)), 0.5\} \\ &= Min\{\xi^{-1}(\Gamma)(y), 0.5\}. \end{aligned}$$

Hence, $\xi^{-1}(\Gamma)(k) \geq inf\{\xi^{-1}(\Gamma)(y), 0.5\}$

Now,

$$\begin{aligned} \xi^{-1}(\Gamma)(k \otimes w) &= \Gamma(\xi(k \otimes w)) \\ &= \Gamma(\xi(k) \otimes' \sigma_t(w)),\ \xi \text{ is a } QHM \\ &\geq Min\{\Gamma(\xi(k)), \Gamma(\xi(w)), 0.5\} \\ &= Min\{\xi^{-1}(\Gamma)(k), \xi^{-1}(\Gamma)(w), 0.5\}. \end{aligned}$$

Thus, $\xi^{-1}(\Gamma)(k \otimes w) \geq Min\{\xi^{-1}(\Gamma)(k), \xi^{-1}(\Gamma)(w), 0.5\}$.

By Theorem 4.5, we have $\xi^{-1}(\Gamma)$ is an $(\in, \in \vee q)$-FFR of K_1. □

5. $(\in_\gamma, \in_\gamma \vee q_\delta)$- FUZZY FILTERS OF QUANTALE

In the present section, the more general forms of $(\in, \in \vee q)$-FFR are introduced.

Throughout the remaining paper $\gamma, \delta \in [0, 1]$, where $\gamma < \delta$ and $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$. Let k_p be a fuzzy point and Γ be a a $fsst$ of K. Then

(1) $k_p \in_\gamma \Gamma$ if $\Gamma(k) \geq p > \gamma$.
(2) $k_p q_\delta \Gamma$ if $\Gamma(k) + p > 2\delta$.
(3) $k_p(\in_\gamma \vee q_\delta)\Gamma$ if $k_p \in_\gamma \Gamma$ or $k_p q_\delta \Gamma$.
(4) $k_p(\in_\gamma \wedge q_\delta)\Gamma$ if $k_p \in_\gamma \Gamma$ and $k_p q_\delta \Gamma$.
(5) $k_p \overline{\alpha} \Gamma$ if $k_p \alpha g$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

Note that the case when $\alpha = \in_\gamma \wedge q_\delta$ is omitted. Suppose that Γ is a $fsst$ of a quantale K such that $\Gamma(k) \leq \delta$ for all $k \in K$. Suppose $k \in K$ and $p \in [0, 1]$ be such that $k_p(\in_\gamma \wedge q_\delta)\Gamma$. Then it follows that $\Gamma(k) \geq p > \gamma$ and $\Gamma(k) + p > 2\delta$. Hence, $2\delta < \Gamma(k) + p \leq \Gamma(k) + \Gamma(k) = 2\Gamma(k)$, that is $\Gamma(k) > \delta$. This means that $\{k_p : k_p(\in_\gamma \wedge q_\delta)\Gamma\} = \emptyset$. Therefore, we are not taking the case when $\alpha = \in_\gamma \wedge q_\delta$.

From here onward, we will write $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filter.

Definition 5.1. *Let Γ be a $fsst$ of a quantale K. Then Γ is said to be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K, if*

$(*)_1$ $w_v \in_\gamma \Gamma \longrightarrow z_v(\in_\gamma \vee q_\delta)\Gamma$ *with* $w \leq z$;

$(*)_2$ $z_p \in_\gamma \Gamma$, $w_v \in_\gamma \Gamma \longrightarrow (z \otimes w)_{inf(p,v)}(\in_\gamma \vee q_\delta)\Gamma$ *for all* $z, w \in K$ *and* $p, v \in (\gamma, 1]$.

Example 5.2. *Consider Example* 2.5. *Let* $\Gamma = \frac{0.5}{\perp} + \frac{0.6}{e} + \frac{0.65}{f} + \frac{0.6}{k} + \frac{0.72}{h} + \frac{0.91}{\top}$. *Then* Γ *is an* $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FFR$ *of* K.

Theorem 5.3. *Let Γ be a $fsst$, of a quantale K and Γ be an $(q_\delta, \in_\gamma \vee q_\delta)$-$FFR$ of K. Then the following conditions hold:*

(1) $Max(\Gamma(w), \gamma) \geq Min(\Gamma(k), \delta)$ *with* $k \leq w$;

(2) $Max(\Gamma(k \otimes w), \gamma) \geq Min(\Gamma(k), \Gamma(w), \delta)$ *for all* $k, y, w \in K$.

Proof. Let $k, w \in K$ be such that $Max(\Gamma(w), \gamma) < Min(\Gamma(k), \delta)$ with $k \leq w$. Then for all $\gamma < p \leq 1$ such that

$$2\delta - Max(\Gamma(w), \gamma) > p \geq 2\delta - Min(\Gamma(k), \delta)$$

we have,

$$2\delta - \Gamma(w) \geq 2\delta - Max(\Gamma(w), \gamma) > p \geq Max(2\delta - \Gamma(k), \delta)$$

That is, $2\delta - \Gamma(w) > p$, $2\delta - \Gamma(k) < p$
and so,

$$\Gamma(k) + p > 2\delta,\ \Gamma(w) + p < 2\delta$$

and $\Gamma(w) < \delta < p$. Hence $k_p q_\delta \Gamma$ but $w_p \overline{(\in_\gamma \vee q_\delta)} \Gamma$, a contradiction. Hence $Max(\Gamma(w), \gamma) \geq Min(\Gamma(k), \delta)$ with $k \leq w$.

If there exist $k, w \in K$ such that $Max(\Gamma(k \otimes w), \gamma) < Min\ (\Gamma(k), \Gamma(w), \delta)$. Then for all $\gamma < v \leq 1$ such that

$$2\delta - Max(\Gamma(k \otimes w), \gamma) > v \geq 2\delta - Min(\Gamma(k), \Gamma(w), \delta)$$

we have,

$$2\delta - \Gamma(k \otimes w) \geq 2\delta - Max(\Gamma(k \otimes w), \gamma) > v \geq Max(2\delta - \Gamma(k), 2\delta - \Gamma(w), \delta)$$

We have, $2\delta - \Gamma(k \otimes w) > v$, $2\delta - \Gamma(k) < v$, $2\delta - \Gamma(w) < v$
and so,

$$\Gamma(k) + v > 2\delta,\ \Gamma(w) + v > 2\delta,\ \Gamma(k \otimes w) + v < 2\delta$$

and $\Gamma(k \otimes w) < \delta < v$. Hence $w_v q_\delta \Gamma, k_v q_\delta \Gamma$ but $(k \otimes w)_v \overline{(\in_\gamma \vee q_\delta)} \Gamma$, a contradiction. Therefore $Max(\Gamma(k \otimes w), \gamma) \geq Min(\Gamma(k), \Gamma(w), \delta)$ for all $k, w \in K$. □

Theorem 5.4. *Let Γ be a fsst of a quantale K. Then Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K if and only if the conditions below hold:*

(1) $k \leq w$ *implies* $Max(\Gamma(w), \gamma) \geq Min(\Gamma(k), \delta)$;

(2) $Max(\Gamma(k \otimes w), \gamma) \geq Min(\Gamma(k), \Gamma(w), \delta)$ *for all* $k, w \in K$.

Proof. $(*)_1 \Longrightarrow (1)$. If there exist $k, w \in K$ with $k \leq w$ such that $Max(\Gamma(w), \gamma) < p \leq Min(\Gamma(k), \delta)$. Then, $\Gamma(k) \geq p > \gamma$, $\Gamma(w) < p$ and $\Gamma(w) + p < 2p \leq 2\delta$. This implies that $k_p \in_\gamma \Gamma$ but $w_p \overline{(\in_\gamma \vee q_\delta)} \Gamma$, a contradiction. Hence (1) is valid.

$(1) \Longrightarrow (*)_1$. Assume that there exist $k, w \in K$ with $k \leq w$ and $v \in (\gamma, \delta]$ such that $k_p \in_\gamma \Gamma$ but $w_p \overline{(\in_\gamma \vee q_\delta)} \Gamma$, then $\Gamma(k) \geq p > \gamma$ and $\Gamma(w) < p$ and $\Gamma(w) + p \leq 2\delta$. It follows that $\Gamma(w) < \delta$ and hence, $Max(\Gamma(w), \gamma) < Min(\Gamma(k), \delta)$, a contradiction.

$(*)_2 \Longrightarrow (2)$. If there exist $k, w \in K$ such that $Max(\Gamma(k \otimes w), \gamma) < v \leq Min(\Gamma(k), \Gamma(w), \delta)$. Then $\Gamma(k) \geq v > \gamma$, $\Gamma(w) \geq v > \gamma$, but $\Gamma(k \otimes w) < v$ and $\Gamma(k \otimes w) + v < 2v \leq 2\delta$, i.e., $k_v \in_\gamma \Gamma$, $w_v \in_\gamma \Gamma$ but $(k \otimes w)_v \overline{(\in_\gamma \vee q_\delta)} \Gamma$, a contradiction. Hence $Max(\Gamma(k \otimes w), \gamma) \geq Min(\Gamma(k), \Gamma(w), \delta)$ for all $k, w \in K$.

(2) $\Longrightarrow (*)_2$. Suppose there exist $k, w \in K$ and $u, v \in (\gamma, \delta]$ such that $k_u \in_\gamma \Gamma$ and $w_v \in_\gamma \Gamma$ but $(k \otimes w)_{Min(u,v)} \overline{(\in_\gamma \vee q_\delta)}\Gamma$, then $\Gamma(k) \geq u > \gamma$, $\Gamma(w) \geq v > \gamma$, $\Gamma(k \otimes w) < Min(u, v)$ and $\Gamma(k \otimes w) + Min(u, v) \leq 2\delta$. It concludes that $\Gamma(k \otimes w) < \delta$ and so $Max(\Gamma(k \otimes w), \gamma) < Min(\Gamma(k), \Gamma(w), \delta)$, a contradiction. Hence $(*)_2$ is valid. □

Corollary 5.5. *Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta, \gamma' < \delta'$, $\gamma' < \gamma$ and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-$FFR$ of K.*

Example shows that converse of Corollary 5.5 is not true in general.

Example 5.6. *Consider Example 1. Let Γ be a $fsst$of K as follows:*

$$\Gamma = \frac{0.5}{\perp} + \frac{0.65}{e} + \frac{0.7}{f} + \frac{0.65}{k} + \frac{0.75}{h} + \frac{0.95}{\top}.$$

Then Γ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-$FFR$ of K but it is not an $(\in_{0.3}, \in_{0.3} \vee q_{0.9})$-$FFR$ of K.

For any $\Gamma \in (K)$, where (K) denotes the set of all $fsst$, of K, we define

$$\Gamma_v = \{y \in K \mid y_v \in_\gamma \Gamma\} \text{ for all } v \in (\gamma, 1];$$

$$\Gamma_v^\delta = \{y \in K \mid y_v q_\delta \Gamma\} \text{ for all } v \in (\gamma, 1];$$

and

$$[\Gamma]_v^\delta = \{y \in K \mid y_v(\in_\gamma \vee q_\delta)\Gamma\} \text{ for all } v \in (\gamma, 1].$$

It follows that $[\Gamma]_v^\delta = \Gamma_v \cup \Gamma_v^\delta$.

Now, we characterize $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K by their level sets.

Theorem 5.7. *Let $\Gamma \in K$. Then*

(1) *Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of K if and only if $\emptyset \neq \Gamma_v$ is filter of K for all $v \in (\gamma, \delta]$.*

(2) *If $2\delta = 1 + \gamma$, then Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ if and only if $\Gamma_v^\delta (\neq \emptyset)$ is a filter of K for all $v \in (\delta, 1]$.*

(3) *If $2\delta = 1 + \gamma$, then Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ if and only if $[\Gamma]_v^\delta$ $(\neq \emptyset)$ is a filter of K for all $v \in (\gamma, 1]$.*

Proof. (**1**). Let Γ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K. Suppose $z, w \in K$ with $w \leq z$ and $v \in (\gamma, \delta]$ be such that $w \in \Gamma_v$. Then $w_v \in_\gamma \Gamma$ and since Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K, so $z_v(\in_\gamma \vee q_\delta)\Gamma$. If $z_v \in_\gamma \Gamma$, then $z \in \Gamma_v$ and if $z_v q_\delta \Gamma$, then $\Gamma(z) > 2\delta - v \geq v > \gamma$, that is, $z \in \Gamma_v$. Now we have to show that $z \otimes w \in \Gamma_v$ for all $z, w \in \Gamma_v$. Let $z, w \in K$ be such that $z, w \in \Gamma_v$ for some $v \in (\gamma, \delta]$. Then $w_v \in_\gamma \Gamma$ and $z_v \in_\gamma \Gamma$, and since Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of K, therefore $(w \otimes z)_v(\in_\gamma \vee q_\delta)\Gamma$. If $(w \vee z)_v \in_\gamma \Gamma$, then $(w \otimes z) \in \Gamma_v$ and if $(w \otimes z)_v q_\delta \Gamma$, then $\Gamma(w \otimes z) > 2\delta - v \geq v > \gamma$, that is, $w \otimes z \in \Gamma_v$. Thus Γ_v is filter of K.

Conversely, suppose that $\emptyset \neq \Gamma_v$ is a filter of K for all $v \in (\gamma, \delta]$. Let $z, w \in K$ with $z \leq w$ and $Max(\Gamma(w), \gamma) < Min(\Gamma(z), \delta)$. Then there exist $v \in (\gamma, \delta]$ such that $Max(\Gamma(w), \gamma) < v \leq Min(\Gamma(z), \delta)$. This shows that $z_v \in_\gamma \Gamma$; that is $z \in \Gamma_v$ but $w \notin \Gamma_v$, a contradiction. Thus, $Max(\Gamma(w), \gamma) \geq Min(\Gamma(z), \delta)$ with $z \leq w$. Let $z, w \in K$ and $Max(\Gamma(z \otimes w), \gamma) < Min(\Gamma(z), \Gamma(w), \delta)$. Then $Max(\Gamma(z \otimes w), \gamma) < v \leq Min(\Gamma(z), \Gamma(w), \delta)$ for some $v \in (\gamma, \delta]$. This implies that $z \in \Gamma_v$ and $w \in \Gamma_v$ but

$(z \otimes w) \notin \Gamma_v$, a contradiction. Therefore, $Max(\Gamma(z \otimes w), \gamma) \geq Min(\Gamma(z), \Gamma(w), \delta)$. Consequently, Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K by Theorem 5.4.

(**2**). Let Γ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K. Let $z, w \in K$ with $w \leq z$ be such that $w \in \Gamma_v^\delta$. Then $w_v q_\delta \Gamma$, that is $\Gamma(w) + v > 2\delta \Rightarrow \Gamma(w) > 2\delta - v \geq 2\delta - 1 = \gamma$. Thus, $\Gamma(w) > \gamma$. By hypothesis, we have

$$\begin{aligned} Max(\Gamma(z), \gamma) \geq & \quad Min(\Gamma(w), \delta) \\ \Rightarrow \Gamma(z) > & \quad Min(2\delta - v, \delta) \end{aligned}$$

Since $v \in (\delta, 1]$, $\delta < v \leq 1 \Rightarrow 2\delta - v < \delta < v$. Thus, $\Gamma(z) > 2\delta - v \Rightarrow \Gamma(z) + v > 2\delta$. Hence, $z \in \Gamma_v^\delta$.

Now we have to show that $z \otimes w \in \Gamma_v^\delta$ for all $z, w \in \Gamma_v^\delta$. Let $z, w \in K$ be such that $z, w \in \Gamma_v^\delta$. Then $w_v q_\delta \Gamma$ and $z_v q_\delta \Gamma$, that is $\Gamma(w) + v > 2\delta \Rightarrow \Gamma(w) > 2\delta - v \geq 2\delta - 1 = \gamma$ and similarly $\Gamma(z) > \gamma$. By assumption, we have

$$\begin{aligned} Max(\Gamma(z \otimes w), \gamma) \geq & \quad Min(\Gamma(w), \Gamma(z), \delta) \\ \Rightarrow \Gamma(z \otimes w) > & \quad Min(2\delta - v, 2\delta - v, \delta) \end{aligned}$$

Since $v \in (\delta, 1]$, $\delta < v \leq 1 \Rightarrow 2\delta - v < \delta < v$. So, $\Gamma(z \otimes w) > 2\delta - v \Rightarrow \Gamma(z \otimes w) + v > 2\delta$. Hence, $z \otimes w \in \Gamma_v^\delta$.

Conversely, suppose that $\emptyset \neq \Gamma_v^\delta$ is a filter of K for all $v \in (\delta, 1]$. We show that Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$. Let $z, w \in K$ with $z \leq w$ be such that $z_p q_\delta \Gamma$. Let $Max(\Gamma(w), \gamma) < Min(\Gamma(z), \delta)$. Then

$$\begin{aligned} 2\delta - inf(\Gamma(z), \delta) < & \quad 2\delta - sup(\Gamma(w), \gamma) \\ \Rightarrow Max(2\delta - \Gamma(z), \delta) < & \quad Min(2\delta - \Gamma(w), 2\delta - \gamma). \end{aligned}$$

Take $p \in (\delta, 1]$ such that $Max(2\delta - \Gamma(z), \delta) < p \leq Min(2\delta - \Gamma(w), 2\delta - \gamma)$. Then $2\delta - \Gamma(z) < p$ and $2\delta - \Gamma(w) \geq p \Rightarrow \Gamma(z) + p > 2\delta$ and $\Gamma(w) + p \leq 2\delta$. This shows that $z_p q_\delta \Gamma$; that is $z \in \Gamma_v^\delta$ but $w \notin \Gamma_v^\delta$, a contradiction. Hence, $Max(\Gamma(w), \gamma) \geq Min(\Gamma(z), \delta)$ with $z \leq w$. Let $z, w \in K$ and $Max(\Gamma(z \otimes w), \gamma) < Min(\Gamma(z), \Gamma(w), \delta)$. Then $2\delta - Min(\Gamma(z), \Gamma(w), \delta) < 2\delta - Max(\Gamma(z \otimes w), \gamma) \Rightarrow Max(2\delta - \Gamma(z), 2\delta - \Gamma(w), \delta) < Min(2\delta - \Gamma(z \otimes w), 2\delta - \gamma)$. There exist $u \in (\delta, 1]$ such that $Max(2\delta - \Gamma(z), 2\delta - \Gamma(w), \delta) < u \leq Min(2\delta - \Gamma(z \otimes w), 2\delta - \gamma)$. Then $2\delta - \Gamma(z) < u$, $2\delta - \Gamma(w) < u$ and $2\delta - \Gamma(z \otimes w) \geq u \Rightarrow \Gamma(z) + u > 2\delta$, $\Gamma(w) + u > 2\delta$ but $\Gamma(z \otimes w) + u \leq 2\delta$. Thus, $z \in \Gamma_v^\delta$ and $w \in \Gamma_v^\delta$ but $(z \otimes w) \notin \Gamma_v^\delta$, a contradiction. Therefore $Max(\Gamma(z \otimes w), \gamma) \geq Min(\Gamma(z), \Gamma(w), \delta)$. Consequently, Γ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K by Theorem 5.4.

(**3**). The proof of part 3 is a a routine verification and similar to the proof of parts 1 and 2. Hence omit here. □

6. Generalized Approximation of Fuzzy Filters

In this section, approximations of generalized fuzzy filters with respect to congruence relations are presented here.

Theorem 6.1. *Let Γ be a FFR of a quantale K and Ω be a CCR. Then $\underline{\Omega}(\Gamma)$ is a FFR of K.*

Proof. Let Γ be a FFR of K. Then $\Gamma(a \otimes c) \geq Min\{\Gamma(a), \Gamma(c)\}$ and if $a \leq c$ then $\Gamma(a) \leq \Gamma(c)$ for all $a, c \in K$. Since Ω is a CCR, so $[z]_\Omega \vee [w]_\Omega = [z \vee w]_\Omega$ and

$[z]_\Omega \otimes [w]_\Omega = [z \otimes w]_\Omega$ for all $z, w \in K$. Since Γ is a FFR and let $c \leq d$. Then $\Gamma(c) \leq \Gamma(d)$. This shows that $\underline{\Omega}(\Gamma)(c) \leq \underline{\Omega}(\Gamma)(d)$.

Consider,

$$\begin{aligned}\underline{\Omega}(\Gamma)(p \otimes q) &= \bigwedge_{u\in[p\otimes q]_\Omega} \Gamma(u) \\ &= \bigwedge_{u\in[p]_\Omega\otimes[q]_\Omega} \Gamma(u).\end{aligned}$$

Now since $u \in [p]_\Omega \otimes [q]_\Omega$ so there exist $r \in [p]_\Omega$ and $s \in [q]_\Omega$ such that $u = r \otimes s$.

Thus,

$$\begin{aligned}\underline{\Omega}(\Gamma)(p \otimes q) &= \bigwedge_{r\otimes s \in [p]_\Omega\otimes[q]_\Omega} \Gamma(r \otimes s) \\ &\geq \bigwedge_{r\otimes s \in [p]_\Omega\otimes[q]_\Omega} Min\{\Gamma(r), \Gamma(s)\} \\ &= \bigwedge_{r \in [p]_\Omega,\, s\in[q]_\Omega} Min\{\Gamma(r), \Gamma(s)\} \\ &= Min\left\{\bigwedge_{r\in[p]_\Omega} \Gamma(r), \bigwedge_{s\in[q]_\Omega} \Gamma(s)\right\} \\ &= Min\left\{\underline{\Omega}(\Gamma)(p), \underline{\Omega}(\Gamma)(q)\right\}.\end{aligned}$$

Hence, $\underline{\Omega}(\Gamma)(p\otimes q) \geq Min\left\{\underline{\Omega}(\Gamma)(p), \underline{\Omega}(\Gamma)(q)\right\}$ for all $p, q \in K$. Thus $\underline{\Omega}(\Gamma)$ is a FFR of K. □

$Table.2$

$\otimes$	$\bot$	i	j	$\top$
$\bot$	$\bot$	i	j	$\top$
i	$\bot$	i	j	$\top$
j	$\bot$	i	j	$\top$
$\top$	$\bot$	i	j	$\top$

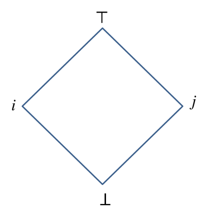

FIGURE 2

The next Example shows that lower approximation of FFR of K is not necessary FFR by using CR.

Example 6.2. *Let K be a complete lattice shown in Fig.2 and operation $\otimes$ on K is shown in Table 2. Let Ω be an ER on K with the Ω-equivalence classes being $\{i, \top\}$, $\{\bot\}$, $\{j\}$. Then obviously, Ω is a CR on K but not CCR. Let $\Gamma = \frac{0.5}{\bot} + \frac{0.6}{i} + \frac{0.8}{j} + \frac{0.8}{\top}$. Then Γ is FFR of K. The lower approximation of Γ of K is $\underline{\Omega}\ (\Gamma) = \frac{0.5}{\bot} + \frac{0.6}{i} + \frac{0.8}{j} + \frac{0.6}{\top}$. It is observed that for $\top \geq j$, we have $\underline{\Omega}\ (\Gamma)(\top) \ngeq \underline{\Omega}\ (\Gamma)(j)$. Thus $\underline{\Omega}\ (\Gamma)$ is not FFR of K while using CR.*

Theorem 6.3. *Let Ω be a CR and Γ be a FFR of a quantale K. Then $\overline{\Omega}(\Gamma)$ is a FFR of K.*

Proof. Let Ω be a CR in K. Then obviously $[p]_\Omega \otimes [q]_\Omega \subseteq [p \otimes q]_\Omega$ for all $p, q \in K$. Also Γ is a FFR and if $p \leq q$ then $\Gamma(p) \leq \Gamma(q)$ and $\Gamma(p \otimes q) \geq Min\{\Gamma(p), \Gamma(q)\}$ for all $p, q \in K$. Since $\Gamma(p) \leq \Gamma(q) \Rightarrow \overline{\Omega}(\Gamma)(p) \leq \overline{\Omega}(\Gamma)(q)$

Consider,

$$\begin{aligned}
Min\left\{\overline{\Omega}(\Gamma)(p), \overline{\Omega}(\Gamma)(q)\right\} &= Min\left\{\bigvee_{r\in[p]_\Omega}\Gamma(r), \bigvee_{s\in[q]_\Omega}\Gamma(s)\right\} \\
&= \bigvee_{r \in [p]_\Omega,\, s\in[q]_\Omega} Min\{\Gamma(r), \Gamma(s)\} \\
&= \bigvee_{r\otimes s \in [p]_\Omega\otimes[q]_\Omega} Min\{\Gamma(r), \Gamma(s)\} \\
&\leq \bigvee_{r\otimes s \in [p]_\Omega\otimes[q]_\Omega} \Gamma(r \otimes s) \\
&\leq \bigvee_{r\otimes s \in[p\otimes q]_\Omega} \Gamma(r \otimes s) \\
&= \overline{\Omega}(\Gamma)(p \otimes q)
\end{aligned}$$

Hence, $\overline{\Omega}(\Gamma)(p \otimes q) \geq Min\left\{\overline{\Omega}(\Gamma)(p), \overline{\Omega}(\Gamma)(q)\right\}$ for all $p, q \in K$. Thus $\overline{\Omega}(\Gamma)$ is a FFR of K. □

Now approximations are applied to $(\in, \in \vee q)$-FFR of quantales.

Theorem 6.4. *Let Ω be a CCR and Γ be an $(\in, \in \vee q)$-FFR of quantale K. Then $\underline{\Omega}(\Gamma)$ is an $(\in, \in \vee q)$-FFR of K.*

Proof. Since Ω be a CCR, we have $[k]_\Omega \otimes [w]_\Omega = [k \otimes w]_\Omega$.

Consider,

$$\begin{aligned}
\underline{\Omega}(\Gamma)(k \otimes w) &= \bigwedge_{u \in [k\otimes w]_\Omega} \Gamma(u) \\
&= \bigwedge_{u \in [k]_\Omega\otimes[w]_\Omega} \Gamma(u)
\end{aligned}$$

As $u \in [k]_\Omega \otimes [w]_\Omega$, therefore there are $p \in [k]_\Omega$ and $q \in [w]_\Omega$ such that $u = p \otimes q$.

Hence,

$$\begin{aligned}
\underline{\Omega}(\Gamma)(k \otimes w) &= \bigwedge_{p\otimes q\in[k]_\Omega\otimes[w]_\Omega} \Gamma(p \otimes q) \\
&\geq \bigwedge_{p \in [k]_\Omega,\, q \in [w]_\Omega} Min[\Gamma(p), \Gamma(q), 0.5] \text{ by Theorem 4.5} \\
&= Min\left\{\left(\bigwedge_{p \in [k]_\Omega} \Gamma(p)\right), \left(\bigwedge_{q \in [w]_\Omega} \Gamma(q)\right), 0.5\right\} \\
&= Min\{\underline{\Omega}(\Gamma)(k), \underline{\Omega}(\Gamma)(w), 0.5\}
\end{aligned}$$

Hence $\underline{\Omega}(\Gamma)(k \otimes w) \geq Min\{\underline{\Omega}(\Gamma)(k), \underline{\Omega}(\Gamma)(w), 0.5\}\ \forall\, k, w \in K$..

Let $w \leq k$. Then $\Gamma(w) \leq \Gamma(k)$. This shows that $\underline{\Omega}(\Gamma)(w) \leq \underline{\Omega}(\Gamma)(k)$ and obviously $Min\{\underline{\Omega}(\Gamma)(w), 0.5\} \leq \underline{\Omega}(\Gamma)(k)$.

Hence $\underline{\Omega}(\Gamma)(k) \geq Min\{\underline{\Omega}(\Gamma)(w), 0.5\}$. Thus $\underline{\Omega}(\Gamma)$ is an $(\in, \in \vee q)$-FFR of K. □

$Table.3$

$\otimes$	$\perp$	i	j	k	$\top$
$\perp$	$\perp$	i	j	k	$\top$
i	$\perp$	i	j	k	$\top$
j	$\perp$	i	j	k	$\top$
k	$\perp$	i	j	k	$\top$
$\top$	$\perp$	i	j	k	$\top$

FIGURE 3

The next Example shows that lower approximation of $(\in, \in \vee q)$-FFR of K is not necessary $(\in, \in \vee q)$-FFR by using CR.

Example 6.5. *Let K be a complete lattice shown in Fig.3 and operation $\otimes$ on K is shown in Table 3. Then $(K, \otimes)$ be a quantale. Let Ω be an ER on K with Ω-equivalence classes are $\{\perp\}$, $\{i, k\}$, $\{j\}$, $\{\top\}$. Then Ω is CR on K and it is not CCR. Let Γ be fsst of K define by $\Gamma = \frac{0.4}{\perp} + \frac{0.4}{i} + \frac{0.5}{j} + \frac{0.6}{k} + \frac{0.7}{\top}$. Then Γ is a $(\in, \in \vee q)$-FFR. The lower approximation of $(\in, \in \vee q)$-FFR of K is given as $\underline{\Omega}(\Gamma) = \frac{0.4}{\perp} + \frac{0.4}{i} + \frac{0.5}{j} + \frac{0.4}{k} + \frac{0.7}{\top}$. It is obvious that $\underline{\Omega}(\Gamma)$ is not $(\in, \in \vee q)$-FFR of K because for $k \geq j$ the condition $\underline{\Omega}(\Gamma)(k) \ngeq Min\{\underline{\Omega}(\Gamma)(j), 0.5\}$ is not satisfied.*

The next Theorem follows from Theorem 6.4.

Theorem 6.6. *Let Ω be a CR and Γ be an $(\in, \in \vee q)$-FFR of a quantale K. Then $\overline{\Omega}(\Gamma)$ is an $(\in, \in \vee q)$-FFR of K.*

Theorem 6.7. *Let Ω be a CCR and Γ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K. Then $\underline{\Omega}(\Gamma)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K.*

Proof. Let $k, y \in K$ and $\gamma, \delta \in (0, 1]$ such that $\gamma < \delta$. Since Ω is a CCR, so we have $[k \otimes y]_\Omega = [k]_\Omega \otimes [y]_\Omega$. Consider

$$\begin{aligned} Max\{\underline{\Omega}(\Gamma)(k \otimes y), \gamma\} &= Max\left\{\bigwedge_{u \in [k \otimes y]_\Omega} \Gamma(u), \gamma\right\} \\ &= \bigwedge_{u \in [k \otimes y]_\Omega} Max\{\Gamma(u), \gamma\} \\ &= \bigwedge_{u \in [k]_\Omega \otimes [y]_\Omega} Max\{\Gamma(u), \gamma\}. \end{aligned}$$

Since $u \in [k]_\Omega \otimes [y]_\Omega$, there exist $p \in [k]_\Omega$ and $q \in [y]_\Omega$ such that $u = p \otimes q$. So,

$$\begin{aligned} Max\{\underline{\Omega}(\Gamma)(k \otimes y), \gamma\} &= \bigwedge_{p \otimes q \in [k]_\Omega \otimes [y]_\Omega} Max\{\Gamma(p \otimes q), \gamma\} \\ &\geq \bigwedge_{p \otimes q \in [k]_\Omega \otimes [y]_\Omega} Min\{\Gamma(p), \Gamma(q), \delta\} \\ &= \bigwedge_{p \in [k]_\Omega,\, q \in [y]_\Omega} Min\{\Gamma(p), \Gamma(q), \delta\} \\ &= Min\{\bigwedge_{p \in [k]_\Omega} \Gamma(p), \bigwedge_{q \in [y]_\Omega} \Gamma(q), \delta\} \\ &= Min\{\underline{\Omega}(\Gamma)(k), \underline{\Omega}(\Gamma)(y), \delta\}. \end{aligned}$$

Thus, we have $Max\,\{\underline{\Omega}(\Gamma)(k \otimes y), \gamma\} \geq Min\{\underline{\Omega}(\Gamma)(k), \underline{\Omega}(\Gamma)(y), \delta\}$.

Furthermore, let $w \leq k$. Then $w \vee k = k$. Since Ω is a CCR, so $[k]_\Omega = [w \vee k]_\Omega = [k]_\Omega \vee [w]_\Omega$.

Consider,

$$\begin{aligned} Max\{\underline{\Omega}(\Gamma)(k), \gamma\} &= Max\{\bigwedge_{u \in [k]_\Omega} \Gamma(u), \gamma\} \\ &= \bigwedge_{u \in [k]_\Omega \vee [w]_\Omega} Max\{\Gamma(u), \gamma\}. \end{aligned}$$

Since $u \in [k]_\Omega \vee [w]_\Omega$ so there exist $a \in [k]_\Omega$ and $b \in [w]_\Omega$ such that $u = a \vee b$. As $a \vee b \geq b$. We have,

$$\begin{aligned} Max\{\underline{\Omega}(\Gamma)(k), \gamma\} &= \bigwedge_{a \vee b \in [k]_\Omega \vee [w]_\Omega} Max\{\Gamma(a \vee b), \gamma\} \\ &= \bigwedge_{a \in [k]_\Omega,\, b \in [w]_\Omega} Max\{\Gamma(a \vee b), \gamma\} \\ &\geq \bigwedge_{a \in [k]_\Omega,\, b \in [w]_\Omega} Min\{\Gamma(b), \delta\} \\ &= Min\{\bigwedge_{b \in \Omega(w)} \Gamma(b), \delta\} \\ &= Min\{\underline{\Omega}(\Gamma)(w), \delta\}. \end{aligned}$$

Thus, we have $Max\{\underline{\Omega}(\Gamma)(k), \gamma\} \geq Min\{\underline{\Omega}(\Gamma)(w), \delta\}$. Therefore $\underline{\Omega}(\Gamma)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K. □

$Table.4$

$\otimes$	$\bot$	i	j	l	k	$\top$
$\bot$	$\bot$	i	j	l	k	$\top$
i	$\bot$	i	j	l	k	$\top$
j	$\bot$	i	j	l	k	$\top$
k	$\bot$	i	j	l	k	$\top$
l	$\bot$	i	j	l	k	$\top$
$\top$	$\bot$	i	j	l	k	$\top$

FIGURE 4

The following Example shows that lower approximation of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K is not necessary $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ by using CR.

Example 6.8. *Let K be a complete lattice shown in $Fig.4$ with $\otimes$ be a binary operation shown in Table 4. Then $(K, \otimes)$ be a quantale. Let Ω be an ER on K with the Ω-equivalence classes being $\{\bot\}$, $\{i\}$, $\{j, l\}$, $\{k\}$, $\{\top\}$. Clearly, Ω is a CR but not a CCR. Let Γ be a fsst of K defined by $\Gamma = \frac{0.3}{\bot} + \frac{0.3}{i} + \frac{0.4}{j} + \frac{0.5}{k} + \frac{0.6}{l} + \frac{0.7}{\top}$. Then Γ is an $(\in_{0.2}, \in_{0.2} \vee q_{0.7})$-$FFR$. Since $\underline{\Omega}(\Gamma) = \frac{0.3}{\bot} + \frac{0.3}{i} + \frac{0,4}{j} + \frac{0.5}{k} + \frac{0.4}{l} + \frac{0.7}{\top}$. It is clear that $\underline{\Omega}(\Gamma)$ is not an $(\in_{0,2}, \in_{0.2} \vee q_{0.7})$-$FFR$ while taking CR. Since for $l \geq k$ with $\gamma = 0.2$ and $\delta = 0.7$, the condition $Max(\underline{\Omega}(\Gamma)(l), 0.2) \ngeq Min(\underline{\Omega}(\Gamma)(k), 0.7)$ with $l \geq k$ is not satisfied.*

Theorem 6.9. *Let Γ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of a quantale K and Ω be a CR. Then $\overline{\Omega}(\Gamma)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K.*

Proof. Let Γ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$. Let $k, w \in K$ and $\gamma, \delta \in (0, 1]$ be such that $\gamma < \delta$. Let $k \leq w$. Then $k \vee w = w$.

Consider,

$$\begin{aligned} Min\{\overline{\Omega}(\Gamma)(k), \delta\} &= Min\{\bigvee_{x \in [k]_\Omega} \Gamma(x), \delta\} \\ &= \bigvee_{x \in [k]_\Omega} Min\{\Gamma(x), \delta\}. \end{aligned}$$

Since Ω is a CR, so $[k]_\Omega \vee [w]_\Omega \subseteq [k \vee w]_\Omega = [w]_\Omega$ for $x \in [k]_\Omega$ and $y \in [w]_\Omega$. As $x \vee y \geq x$. We have,

$$\begin{aligned} Min\{\overline{\Omega}(\Gamma)(k), \delta\} &= \bigvee_{x \in [k]_\Omega} Min\{\Gamma(x), \delta\} \\ &\leq \bigvee_{x \in [k]_\Omega,\, y \in [w]_\Omega} Max\{\Gamma(x \vee y), \gamma\} \\ &= \bigvee_{x \vee y \in [k]_\Omega \vee [w]_\Omega} Max\{\Gamma(x \vee y), \gamma\} \\ &\leq \bigvee_{x \vee y \in [k \vee w]_\Omega} Max\{\Gamma(x \vee y), \gamma\} \\ &= \bigvee_{u \in [k \vee w]_\Omega} Max\{\Gamma(u), \gamma\} \\ &= \bigvee_{u \in [w]_\Omega} Max\{\Gamma(u), \gamma\} \\ &= Max\{\bigvee_{u \in [w]_\Omega} \Gamma(u), \gamma\} \\ &= Max\{\overline{\Omega}(\Gamma)(w), \gamma\}. \end{aligned}$$

Thus, we have $Max\{\overline{\Omega}(\Gamma)(w), \gamma\} \geq Min\{\overline{\Omega}(\Gamma)(k), \delta\}$.

Further, consider

$$\begin{aligned} Min\{\overline{\Omega}(\Gamma)(k), \overline{\Omega}(\Gamma)(w)), \delta\} &= Min\{\bigvee_{a \in [k]_\Omega} \Gamma(a), \bigvee_{b \in [w]_\Omega} \Gamma(b), \delta\} \\ &= \bigvee_{a \in [k]_\Omega, b \in [w]_\Omega} Min\{\Gamma(a), \Gamma(b), \delta\} \end{aligned}$$

Since Ω is a CR, so $[k]_\Omega \otimes [w]_\Omega \subseteq [k \otimes w]_\Omega$, we have,

$$\begin{aligned} Min\{\overline{\Omega}(\Gamma)(k), \overline{\Omega}(\Gamma)(w)), \delta\} &= \bigvee_{a \in [k]_\Omega, b \in [w]_\Omega} Min\{\Gamma(a), \Gamma(b), \delta\} \\ &\leq \bigvee_{a \in [k]_\Omega, b \in [w]_\Omega} Max\{\Gamma(a \otimes b), \gamma\} \\ &= \bigvee_{a \otimes b \in [k]_\Omega \otimes [w]_\Omega} Max\{\Gamma(a \otimes b), \gamma\} \\ &\leq \bigvee_{a \otimes b \in [k \otimes w]_\Omega} Max\{\Gamma(a \otimes b), \gamma\} \\ &= Max\{\bigvee_{u \in [k \otimes w]_\Omega} \Gamma(u), \gamma\} \\ &= Max\{\overline{\Omega}(\Gamma)(k \otimes w), \gamma\} \end{aligned}$$

Thus, we have $Max\left\{\overline{\Omega}(\Gamma)(k \otimes w), \gamma\right\} \geq Min\{\overline{\Omega}(\Gamma)(k), \overline{\Omega}(\Gamma)(w), \delta\}$. Therefore $\overline{\Omega}(\Gamma)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ of K. □

7. Conclusion

In the present paper, we substitute a universe set by a quantale, and introduce the characterizations of quantale by (α, β)-fuzzy filter by using fuzzy points. It is additionally demonstrated that by utilizing an (α, β)-fuzzy map, the inverse image of an (α, β)-fuzzy filter under quantale homomorphism is an (α, β)-fuzzy filter. It is also investigated that homomorphic image of an $(\in, \in \vee q)$-fuzzy filter under quantale homomorphism is an $(\in, \in \vee q)$-fuzzy filter. In the last section, more general form of $(\in, \in \vee q)$-fuzzy filter are introduced. The relationship between ordinary filters and fuzzy filters of the type $(\in_\gamma, \in_\gamma \vee q_\delta)$ is also constructed. Keen observation is carried out while finding lower approximation of fuzzy filters and of the type $(\in, \in \vee q)$-FFR and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$ with the help of complete congruence. It is observed that complete congruence is compulsory to find out lower approximations fuzzy filters of the type $(\in, \in \vee q)$-FFR and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFR$.

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