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HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY RELATIVE PREINVEX FUNCTIONS

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Abstract: In this paper, we establish several new upper bounds of Hermite-Hadamard type integral inequalities for harmonically relative preinvex functions and their different types such as s-harmonic preinvex functions, s-harmonic Godunova-Levin functions and harmonic P-preinvex functions.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
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1. INTRODUCTION

The beginning of the theory of convexity can be traced back to the end of 19^{th} century [1, 3, 9]. The theory of convexity has been extended in numerous directions using advanced ideas and techniques. Several inequalities have been obtained for convex function but a very well-known is the Hermite-Hadamard inequality.
Hermite-Hadamard inequality was discovered by Ch. Hermite [8] in 1883 and rediscovered by J. Hadamard [6] in 1893. Hermite-Hadamard inequalities for convex functions and their

several forms exist in literature [1, 5, 9, 10], [11]-[24], [27]-[36].
The generalization of convexity is the invexity, many researchers have done work on it. Hanson [7] investigate and introduced the invex functions. Ben-Israel and Mond [4] worked on invex set and preinvex functions. Pini [35] investigated another class of generalized invex functions, named as preinvex functions. Mohan and Neogy [26] established some properties of generalized preinvex functions. Noor [33] introduced some Hermite-Hadamard type inequalities for preinvex functions. Various integral inequalities for preinvex functions have been established recently, see [30, 31]. Iscan [10] introduced the concept of harmonically convex functions.
Noor et. al. [29] investigate a class of preinvex functions with respect to an arbitrary function h, which are said to be relative preinvex functions. He also introduced the class of relative harmonic functions with respect to an arbitrary nonnegative function h and established an innovative class of convex function with respect to an arbitrary nonnegative function h, which is known as relative harmonic preinvex functions [30, 34]. We also obtain diverse classes of harmonic convex and harmonic preinvex functions such as Breckner type of s-harmonic preinvex functions, Godunova levin type of s-harmonic preinvex functions and harmonic P-preinvex functions [27, 28, 30, 32].

2. Notations and Preliminaries

Definition 2.1. [30]. *Let $h : [0,1] \subseteq I \longrightarrow \mathbb{R}$ be a non-negative function. A function $f : K = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is known as relative harmonic preinvex function with respect to an arbitrary function h and $\eta(.,.)$, if*

$$f\left(\frac{l_1(l_1+\eta(l_2,l_1))}{l_1+(1-t)\eta(l_2,l_1)}\right) \leq h(1-t)f(l_1) + h(t)f(l_2), \;\; t \in [0,1], \;\; \forall\, l_1, l_2 \in K.$$

Remark 2.2.

- *If $t = \frac{1}{2}$, then we get*

$$f\left(\frac{2l_1(l_1+\eta(l_2,l_1))}{2l_1+\eta(l_2,l_1)}\right) \leq h\left(\frac{1}{2}\right)[f(l_1)+f(l_1)], \;\; \forall\, l_1, l_2 \in K.$$

  which is known as Jensen type relative harmonic preinvex function.
- *If $h(t) = t$, $h(t) = t^s$, $h(t) = t^{-s}$, $h(t) = 1$, then relative harmonic preinvex functions reduces to harmonic preinvex functions, s-harmonic preinvex functions, s-harmonic Godunova-Levin functions and harmonic P-preinvex functions respectively.*

Definition 2.3. *Let $K \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(.,.) : K \times K \longrightarrow \mathbb{R}$. For any $l_1, l_2 \in K$ and any $t \in [0,1]$, we have*

$$\eta(l_2, l_2 + t\eta(l_1,l_2)) = -t\eta(l_1,l_2)$$
$$\eta(l_1, l_2 + t\eta(l_1,l_2)) = (1-t)\eta(l_1,l_2)$$

Note that for every $l_1, l_2 \in K$, $t_1, t_2 \in [0,1]$ and from condition C, we have

$$\eta(l_2 + t_2\eta(l_1,l_2), l_2 + t_1\eta(l_1,l_2)) = (t_2 - t_1)\eta(l_1,l_2)$$

This condition is automatically satisfied for the convex functions.

Theorem 2.4. [30]. *A function $f : K = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is relative harmonic preinvex function where $l_1, l_1 + \eta(l_2, l_1) \in K$ with $l_1 < l_1 + \eta(l_2, l_1)$. If $f \in L[l_1, l_1 + \eta(l_2, l_1)]$ and condition C holds,*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{2l_1(l_1+\eta(l_2,l_1))}{2l_1+\eta(l_2,l_1)}\right)\leq\frac{l_1(l_1+\eta(l_2,l_1))}{\eta(l_2,l_1)}\int_{l_1}^{l_1+\eta(l_2,l_1)}\frac{f(x)}{x^2}dx\leq[f(l_1)+f(l_2)]\int_0^1 h(t)dt. \quad (2.\ 1)$$

Remark 2.5. *If $h(t) = t^s$, $h(t) = t^{-s}$, $h(t) = 1$, then Theorem 2.4 becomes Hermite-Hadamard type integral inequalities for s-harmonic preinvex function, s-harmonic Godunova-Levin function and harmonic P-preinvex functions respectively.*

Definition 2.6. [34]. *let ${}_2F_1[r, s, t, x]$ denote the hypergeometric function given by*

$${}_2F_1[r, s, t, x] = \sum_{p=0}^{\infty} \frac{(r)_p(s)_p}{(t)_p}\frac{x^p}{p!}; \qquad |x| < 1$$

It is not defined if t equals a non-positive integer. Here $(v)_p$ is the Pochhammer symbol, which is given by

$$(v)_p = \begin{cases} 1, & p = 0 \\ v(v+1)...(v+p-1), & p > 0. \end{cases}$$

Definition 2.7. [2]. *In the product of two hypergeometric functions ${}_2F_1(\alpha; \beta; \gamma; x)$, ${}_2F_1(\alpha'; \beta'; \gamma'; y)$, we obtain a double series , resulting in four kinds of functions which are shown as follows:*

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}(\beta)_r(\beta')_s}{r!s!(\gamma)_{r+s}} x^r y^s, \qquad |x|, |y| < 1$$

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}(\beta)_r(\beta')_s}{r!s!(\gamma)_r(\gamma')_v} x^r y^s , \qquad |x| + |y| < 1$$

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r!s!(\gamma)_{r+s}} x^r y^s, \qquad |x|, |y| < 1$$

$$F_4(\alpha; \beta; \gamma, \gamma'; x, y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}(\beta)_{r+s}}{r!s!(\gamma)_r(\gamma')_s} x^r y^s, \qquad |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1.$$

Definition 2.8. [36]. *Given a generalized hypergeometric or hypergeometric function ${}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; x)$, the corresponding regularized hypergeometric function is given by*

$${}_p\tilde{F}_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; x) = \frac{{}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; x)}{(\Gamma(\beta_1)...\Gamma(\beta_q))},$$

$where\ \Gamma(x)\ is\ a\ gamma\ function.$

Definition 2.9. [34]. *The beta function is special function, also known as the Euler integral of the first kind is denoted as*

$$B(m, n) = \int_0^1 z^{m-1}(1-z)^{n-1}dz = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; \quad where\ m, n\ are\ real\ numbers.$$

Definition 2.10. [32]. *Let $f \in L[l_1, l_2]$. The Riemann-Liouville Integrals $J^{\beta}_{l_1+}f$ and $J^{\beta}_{l_2-}f$ of order $\beta > 0$ with $l_1 \geq 0$ are given by*

$$J^{\beta}_{l_1+}f(x) = \frac{1}{\Gamma(\beta)}\int_{l_1}^{x}(x-t)^{\beta-1}f(t)dt, \quad x > l_1$$

$$J^{\beta}_{l_2-}f(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{l_2}(t-x)^{\beta-1}f(t)dt, \quad l_2 > x$$

Here, $\Gamma(\beta) = \int_0^{+\infty} e^{-a}a^{\beta-1}da$.

- *If $\beta = 0$, then $J^{0}_{l_1+}f(x) = J^{0}_{l_2-}f(x) = f(x)$.*
- *If $\beta = 1$, then the fractional integral becomes the classical integral.*

3. Main Results

For establishing some new Hermite-Hadamard type inequalities connected with the right and left part of (2. 1) for functions whose derivatives are relative harmonically preinvex, we need the following lemma:

Lemma 3.1. *Assuming that $f : M = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[l_1, l_1 + \eta(l_2, l_1)]$ for $l_1, l_1 + \eta(l_2, l_1) \in M$ with $l_1 < l_1 + \eta(l_2, l_1)$, $\lambda, \alpha \in [0, 1]$, $g(x) = \frac{1}{x}$ and $\beta \in (0, 1]$ such that $(-1)^{\beta} \in \mathbb{R}$, then*

$$\begin{aligned}
&\Psi_f(\lambda, \alpha, \beta, l_1, l_1 + \eta(l_2, l_1)) \\
&:= -\left[f\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\bar{A}_{1-\alpha}}\right)[(1-\alpha)^{\beta} - (-1)^{\beta}\alpha^{\beta} - \lambda] + (1-\alpha)\lambda f\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\bar{A}_1}\right)\right. \\
&\times \alpha\lambda\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\bar{A}_o}\right) - \frac{\Gamma(\beta+1)\, l_1^{\beta}\{l_1 + \eta(l_2, l_1)\}^{\beta}}{\eta(l_2, l_1)^{\beta}} \\
&\left.\left\{J^{\beta}_{\frac{l_1\alpha + (1-\alpha)\{l_1 + \eta(l_2, l_1)\}}{l_1\{l_1 + \eta(l_2, l_1)\}}-} f\circ g\left(\frac{1}{l_1 + \eta(l_2, l_1)}\right) + (-1)^{\beta} J^{\beta}_{\frac{l_1\alpha + (1-\alpha)\{l_1 + \eta(l_2, l_1)\}}{l_1\{l_1 + \eta(l_2, l_1)\}}+} f\circ g\left(\frac{1}{l_1}\right)\right\}\right] \\
&= l_1\eta(l_2, l_1)\{l_1 + \eta(l_2, l_1)\}\left[\int_0^{1-\alpha}\frac{t^{\beta} - \alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\bar{A}_t}\right)dt\right. \\
&\left.+ \int_{1-\alpha}^{1}\frac{(t-1)^{\beta} - (\alpha-1)\lambda}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\bar{A}_t}\right)dt\right]
\end{aligned}$$

for $t \in [0, 1]$ and $\bar{A}_t = (1-t)l_1 + t(l_1 + \eta(l_2, l_1))$.

Proof. Let

$$\begin{aligned}
I_1 &= \int_0^{1-\alpha}\frac{t^{\beta} - \alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\bar{A}_t}\right)dt \\
&= -\frac{1}{l_1\{l_1 + \eta(l_2, l_1)\}\eta(l_2, l_1)}\left[\{(1-\alpha)^{\beta} - \alpha\lambda\}f\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\alpha l_1 + (1-\alpha)(l_1 + \eta(l_2, l_1))}\right)\right. \\
&\left.+ \alpha\lambda f\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{l_1}\right) - \beta\int_0^{1-\alpha} t^{\beta-1} f\left(\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{(1-t)l_1 + t(l_1 + \eta(l_2, l_1))}\right)dt\right]. \quad (3.\ 2)
\end{aligned}$$

Setting $x = \frac{(1-t)l_1+t(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}}$, so that $dx = \frac{\eta(l_2,l_1)}{l_1\{l_1+\eta(l_2,l_1)\}}dt$
For $0 \le t \le 1-\alpha$, we have $\frac{1}{l_1+\eta(l_2,l_1)} \le x \le \frac{l_1\alpha+(1-\alpha)(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}}$ and hence (3.2) becomes

$$\begin{aligned}I_1 &= -\frac{1}{l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}}\Bigg[\{(1-\alpha)^\beta-\alpha\lambda\}f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)+\alpha\lambda f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_o}\right)\\ &\quad -\frac{\beta l_1^\beta\{l_1+\eta(l_2,l_1)\}^\beta}{\{\eta(l_2,l_1)\}^\beta}\int_{\frac{1}{l_1+\eta(l_2,l_1)}}^{\frac{l_1\alpha+(1-\alpha)(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}}}(f\circ g)(x)\left[x-\frac{1}{l_1+\eta(l_2,l_1)}\right]^{\beta-1}dx\Bigg]\\ I_1 &= -\frac{1}{l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}}\Bigg[\{(1-\alpha)^\beta-\alpha\lambda\}f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)+\alpha\lambda f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_o}\right)\\ &\quad \frac{\beta l_1^\beta\{l_1+\eta(l_2,l_1)\}^\beta}{\{\eta(l_2,l_1)\}^\beta}\left\{\Gamma(\beta)J^\beta_{\frac{l_1\alpha+(1-\alpha)(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}}-}(f\circ g)\left(\frac{1}{l_1+\eta(l_2,l_1)}\right)\right\}\Bigg] \quad (3.\ 3)\end{aligned}$$

Let

$$\begin{aligned}I_2 &= \int_{1-\alpha}^1 \frac{(t-1)^\beta-(\alpha-1)\lambda}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)dt\\ &= -\frac{1}{l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}}\Bigg[-\{(-\alpha)^\beta+(1-\alpha)\lambda\}f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\alpha l_1+(1-\alpha)(l_1+\eta(l_2,l_1))}\right)\\ &\quad +(1-\alpha)\lambda f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{l_1+\eta(l_2,l_1)}\right)-\beta\int_{1-\alpha}^1(t-1)^{\beta-1}f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{(1-t)l_1+t(l_1+\eta(l_2,l_1))}\right)dt\Bigg] \quad (3.\ 4)\end{aligned}$$

For $1-\alpha \le t \le 1$, we have $\frac{l_1\alpha+(1-\alpha)(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}} \le x \le \frac{1}{l_1}$ and hence (3.4) becomes

$$\begin{aligned}I_2 &= -\frac{1}{l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}}\big[-\{(-\alpha)^\beta-(1-\alpha)\lambda)\\ &\quad f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)+(1-\alpha)\lambda f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_1}\right)\\ &\quad -\frac{\beta l_1^\beta\{l_1+\eta(l_2,l_1)\}^\beta}{\{\eta(l_2,l_1)\}^\beta}\int_{\frac{l_1\alpha+(1-\alpha)(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}}}^{\frac{1}{l_1}}(f\circ g)(x)\left[x-\frac{1}{l_1}\right]^{\beta-1}dx\Bigg]\\ I_2 &= -\frac{1}{l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}}\big[-\{(1-\alpha)^\beta-(1-\alpha)\lambda\}\\ &\quad f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)+(1-\alpha)\lambda f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_1}\right)-(-1)^{\beta-1}\\ &\quad \frac{\beta l_1^\beta\{l_1+\eta(l_2,l_1)\}^\beta}{\{\eta(l_2,l_1)\}^\beta}\left\{\Gamma(\beta)J^\beta_{\frac{l_1\alpha+(1-\alpha)(l_1+\eta(l_2,l_1))}{l_1\{l_1+\eta(l_2,l_1)\}}+}(f\circ g)\left(\frac{1}{l_1}\right)\right\}\Bigg] \quad (3.\ 5)\end{aligned}$$

Adding Equations (3.3) and (3.5), we have

$$
\begin{aligned}
&\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1)) \\
&:= -\Bigg[f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)[(1-\alpha)^\beta-(-1)^\beta\alpha^\beta-\lambda]+(1-\alpha)\lambda f\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_1}\right) \\
&\times\alpha\lambda\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_o}\right)-\frac{\Gamma(\beta+1)l_1^\beta\{l_1+\eta(l_2,l_1)\}^\beta}{\eta(l_2,l_1)^\beta}\Bigg\{J^\beta_{\frac{l_1\alpha+(1-\alpha)\{l_1+\eta(l_2,l_1)\}}{l_1\{l_1+\eta(l_2,l_1)\}}-}f\circ g\left(\frac{1}{l_1+\eta(l_2,l_1)}\right) \\
&+(-1)^\beta J^\beta_{\frac{l_1\alpha+(1-\alpha)\{l_1+\eta(l_2,l_1)\}}{l_1\{l_1+\eta(l_2,l_1)\}}+}f\circ g\left(\frac{1}{l_1}\right)\Bigg\}\Bigg] \\
&= l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left[\int_0^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)dt\right. \\
&\left.+\int_{1-\alpha}^1\frac{(t-1)^\beta-(\alpha-1)\lambda}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)dt\right]
\end{aligned}
$$

□

Remark 3.2. (a) *If $\lambda=0$, $\alpha=\frac{1}{2}$ and $\beta=1$, then Lemma 3.1 reduces to the following result*

$$
\begin{aligned}
&\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\int_{l_1}^{l_1+\eta(l_2,l_1)}\frac{f(z)}{z^2}dz-f\left(\frac{2l_1\{l_1+\eta(l_2,l_1)\}}{l_1+(l_1+\eta(l_2,l_1))}\right) \\
&= l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left[\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)dt\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\frac{(t-1)}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)dt\right]. \qquad (3.\ 6)
\end{aligned}
$$

(b) *If $\lambda=1$, $\alpha=\frac{1}{2}$ and $\beta=1$, then Lemma 3.1 reduces to the following result*

$$
\begin{aligned}
&\frac{f(l_1)+f(l_1+\eta(l_2,l_1))}{2}-\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\int_{l_1}^{l_1+\eta(l_2,l_1)}\frac{f(z)}{z^2}dz \\
&= l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left[\int_0^1\frac{(\frac{1}{2}-t)}{(\bar{A}_t)^2}f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)dt\right].
\end{aligned}
$$

Now we establish new integral inequalities of Hermite-Hadamard type for relative harmonically preinvex functions.

Theorem 3.3. *Assuming that $f: M=[l_1,l_1+\eta(l_2,l_1)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ is differentiable on the interior, M^o of M where $f'\in L[l_1+\eta(l_2,l_1)]$ for $l_1,l_1+\eta(l_2,l_1)\in M$ with $l_1<l_1+\eta(l_2,l_1)$, $\beta\in(0,1]$ and $\lambda,\alpha\in[0,1]$. If $|f'|^\mu$ is relative harmonically preinvex on M for $\mu>1$ with $\frac{1}{\gamma}+\frac{1}{\mu}=1$, we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[(k_1(\lambda,\alpha,\beta,l_1,l_2)+k_2(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(k_7(\lambda,\alpha,\beta,l_1,l_2,h)$$
$$+k_8(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_1)|^{\mu}+(k_9(\lambda,\alpha,\beta,l_1,l_2,h)+k_{10}(\lambda,\alpha,\beta,l_1,l_2,h))|f'(l_2)|^{\mu}\}^{\frac{1}{\mu}}$$
$$+(k_5(\lambda,\alpha,\beta,l_1,l_2)+k_6(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(k_{15}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{16}(\lambda,\alpha,\beta,l_1,l_2,h))$$
$$\times|f'(l_1)|^{\mu}+(k_{17}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{18}(\lambda,\alpha,\beta,l_1,l_2,h))|f'(l_2)|^{\mu}\}^{\frac{1}{\mu}}].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_2+\eta(l_2,l_1)\}[(k_1(\lambda,\alpha,\beta,l_1,l_2)+k_2(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(k_7(\lambda,\alpha,\beta,l_1,l_2,h)$$
$$+k_8(\lambda,\alpha,\beta,l_1,l_2,h))|f'(l_1)|^{\mu}+(k_9(\lambda,\alpha,\beta,l_1,l_2,h)+k_{10}(\lambda,\alpha,\beta,l_1,l_2,h))|f'(l_2)|^{\mu}\}^{\frac{1}{\mu}}$$
$$+(k_4(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(k_{13}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_1)|^{\mu}+k_{14}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_2)|^{\mu})\}^{\frac{1}{\mu}}].$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[(k_3(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(k_{11}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_1)|^{\mu}$$
$$+k_{12}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_1)|^{\mu})\}^{\frac{1}{\mu}}+(k_5(\lambda,\alpha,\beta,l_1,l_2)+k_6(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(k_{15}(\lambda,\alpha,\beta,l_1,l_2,h)$$
$$+k_{16}(\lambda,\alpha,\beta,l_1,l_2,h))|f'(l_1)|^{\mu}+(k_{17}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{18}(\lambda,\alpha,\beta,l_1,l_2,h))|f'(l_2)|^{\mu}\}^{\frac{1}{\mu}}].$$

Proof. By using Lemma 3.1 and power mean integral inequality, we have

$$|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))| \tag{3. 7}$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left[\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|dt\right.$$
$$\left.+\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|dt\right]$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left(\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\right.$$
$$\left.\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}+\left(\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}$$
$$\left(\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \tag{3. 8}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}dt = \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}dt$$
$$= k_1(\lambda,\alpha,\beta,l_1,l_2)+k_2(\lambda,\alpha,\beta,l_1,l_2) \tag{3. 9}$$

where

$$k_1(\lambda,\alpha,\beta,l_1,l_2):=\frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta+1}\beta_2F_1[1,1+\beta,2+\beta,-\frac{c\alpha\lambda^{\frac{1}{\beta}}}{l_1}]}{l_1^2(1+\beta)}+\frac{\alpha\lambda^{\frac{1}{\beta}}(-((\alpha\lambda^{\frac{1}{\beta}})^{\beta}-\alpha\lambda))}{l_1(l_1+c\alpha\lambda^{\frac{1}{\beta}})}$$

$$\begin{aligned}k_2(\lambda,\alpha,\beta,l_1,l_2):=&\frac{c(1-\alpha)^{1+\beta}+l_1\alpha\lambda}{l_1c(l_1+c-c\alpha)}-\frac{c(\alpha\lambda^{\frac{1}{\beta}})^{1+\beta}+l_1\alpha\lambda}{l_1c(l_1+c\alpha\lambda^{\frac{1}{\beta}})}\\&+\frac{(-l_1+c(\alpha-1))(1-\alpha)^{1+\beta}\beta_2F_1[1,1+\beta,2+\beta,\frac{c(\alpha-1)}{l_1}]}{l_1^2(l_1+c-c\alpha)(1+\beta)}\\&+\frac{(\alpha\lambda^{\frac{1}{\beta}})^{1+\beta}\beta_2F_1[1,1+\beta,2+\beta,-\frac{c\alpha\lambda^{\frac{1}{\beta}}}{l_1}]}{l_1^2(1+\beta)}\end{aligned}$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}dt=\int_0^{1-\alpha}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}dt=k_3(\lambda,\alpha,\beta,l_1,l_2). \quad (3.\ 10)$$

where

$$\begin{aligned}k_3(\lambda,\alpha,\beta,l_1,l_2):=&\frac{(-1+\alpha)(l_1(1+\beta)((1-\alpha)^{\beta}-\alpha\lambda))}{l_1^2(l_1+c-c\alpha)(1+\beta)}\\&-\frac{(-1+\alpha)(1-\alpha)^{\beta}(l_1+c-c\alpha)\beta_2F_1[1,1+\beta,2+\beta,\frac{c(-1+\alpha)}{l_1}])}{l_1^2(l_1+c-c\alpha)(1+\beta)}\end{aligned}$$

(b) (i) If $1+((\alpha-1)\lambda)^{\frac{1}{\beta}}\leq 1-\alpha$, then

$$\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}dt=k_4(\lambda,\alpha,\beta,l_1,l_2). \quad (3.\ 11)$$

where

$$k_4(\lambda,\alpha,\beta,l_1,l_2):=\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(1-t)l_1+t(l_1+c)}dt$$

(ii) If $1+((\alpha-1)\lambda)^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}dt=k_5(\lambda,\alpha,\beta,l_1,l_2)+k_6(\lambda,\alpha,\beta,l_1,l_2). \quad (3.\ 12)$$

where

$$k_5(\lambda,\alpha,\beta,l_1,l_2):=\int_{1-\alpha}^{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-(\alpha-1)\lambda)}{(1-t)l_1+t(l_1+c)}dt$$

$$k_6(\lambda,\alpha,\beta,l_1,l_2):=\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(1-t)l_1+t(l_1+c)}dt$$

Since $|f'|^{\mu}$ be relative harmonically preinvex on the interval $[l_1, l_1+\eta(l_2,l_1)]$ with respect to an arbitrary nonnegative function h and for $\mu>1$, as $t\in[0,1]$

$$\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{t(l_1+\eta(l_2,l_1))+(1-t)l_1}\right)\right|^{\mu}\leq h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}.$$

hence, by simple calculation, we obtain some inequalities

(c) (i) If $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha$, then

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\ \leq\ &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}[h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}]dt\\ &+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}[h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}]dt\\ =\ &[k_7(\lambda,\alpha,\beta,l_1,l_2,h)+k_8(\lambda,\alpha,\beta,l_1,l_2,h)]|f'(l_1)|^{\mu}\\ &+[k_9(\lambda,\alpha,\beta,l_1,l_2,h)+k_{10}(\lambda,\alpha,\beta,l_1,l_2,h)]|f'(l_2)|^{\mu}. \qquad (3.\ 13)\end{aligned}$$

where

$$k_7(\lambda,\alpha,\beta,l_1,l_2,h):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}h(t)dt$$

$$k_8(\lambda,\alpha,\beta,l_1,l_2,h):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(l_1(1-t)+(l_1+c)t)^2}h(t)dt$$

$$k_9(\lambda,\alpha,\beta,l_1,l_2,h):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}h(1-t)dt$$

$$k_{10}(\lambda,\alpha,\beta,l_1,l_2,h):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(l_1(1-t)+(l_1+c)t)^2}h(1-t)dt$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\ \leq\ &\int_0^{1-\alpha}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}[h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}]dt\\ =\ &k_{11}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_1)|^{\mu}+k_{12}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_2)|^{\mu}. \qquad (3.\ 14)\end{aligned}$$

where

$$k_{11}(\lambda,\alpha,\beta,l_1,l_2,h):=\int_0^{1-\alpha}\frac{-(t^{\beta}-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}h(t)dt$$

$$k_{12}(\lambda,\alpha,\beta,l_1,l_2,h):=\int_0^{1-\alpha}\frac{-(t^{\beta}-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}h(1-t)dt$$

(d) (i) If $1+((\alpha-1)\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu} dt \\
\leq\ & \int_{1-\alpha}^{1} \frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(\bar{A}_t)^2}[h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}]dt \\
=\ & k_{13}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_1)|^{\mu}+k_{14}(\lambda,\alpha,\beta,l_1,l_2,h)|f'(l_2)|^{\mu}. \qquad (3.\ 15)
\end{aligned}$$

where

$$\begin{aligned}
k_{13}(\lambda,\alpha,\beta,l_1,l_2,h) &:= \int_{1-\alpha}^{1} \frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(l_1(1-t)+(l_1+c)t)^2}h(t)dt \\
k_{14}(\lambda,\alpha,\beta,l_1,l_2,h) &:= \int_{1-\alpha}^{1} \frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(l_1(1-t)+(l_1+c)t)^2}h(1-t)dt
\end{aligned}$$

(ii) If $1+((\alpha-1)\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^{\beta}-(\alpha-1)\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu} dt \\
\leq\ & \int_{1-\alpha}^{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-(\alpha-1)\lambda)}{(\bar{A}_t)^2}[h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}]dt \\
&+\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(\bar{A}_t)^2}[h(t)|f'(l_1)|^{\mu}+h(1-t)|f'(l_2)|^{\mu}]dt \\
=\ & [k_{15}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{16}(\lambda,\alpha,\beta,l_1,l_2,h)]|f'(l_1)|^{\mu} \\
&+[k_{17}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{18}(\lambda,\alpha,\beta,l_1,l_2,h)]|f'(l_2)|^{\mu}. \qquad (3.\ 16)
\end{aligned}$$

where

$$\begin{aligned}
k_{15}(\lambda,\alpha,\beta,l_1,l_2,h) &:= \int_{1-\alpha}^{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-(\alpha-1)\lambda)}{(l_1(1-t)+(l_1+c)t)^2}h(t)dt \\
k_{16}(\lambda,\alpha,\beta,l_1,l_2,h) &:= \int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(l_1(1-t)+(l_1+c)t)^2}h(t)dt \\
k_{17}(\lambda,\alpha,\beta,l_1,l_2,h) &:= \int_{1-\alpha}^{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-(\alpha-1)\lambda)}{(l_1(1-t)+(l_1+c)t)^2}h(1-t)dt \\
k_{18}(\lambda,\alpha,\beta,l_1,l_2,h) &:= \int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(l_1(1-t)+(l_1+c)t)^2}h(1-t)dt
\end{aligned}$$

Where $c=\eta(l_1,l_2)$. By substituting (3.9) to (3.16) in equation (3.7) gives the required result.

□

If $\lambda=0$, $\alpha=\frac{1}{2}$ and $\beta=1$, then identity (3.7) reduces to the following result:

Corollary 3.4. *Assuming that* $f : M = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f' \in L_1[l_1, l_1 + \eta(l_2, l_1)]$ *for* $l_1, l_1+\eta(l_2, l_1) \in M$ *with* $l_1 < l_1+\eta(l_2, l_1)$. *If* $|f'|^\mu$ *is relative harmonically preinvex on* M *for* $\mu > 1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu} = 1$, *then*

$$\left|\frac{l_1\{l_1 + \eta(l_2, l_1)\}}{\eta(l_2, l_1)}\int_{l_1}^{l_1+\eta(l_2,l_1)} \frac{f(z)}{z^2}dz - f\left(\frac{2l_1\{l_1 + \eta(l_2, l_1)\}}{l_1 + (l_1 + \eta(l_2,l_1))}\right)\right|$$
$$\leq l_1\eta(l_2,l_1)\{l_1 + \eta(l_2,l_1)\}\,[s_5^{\frac{1}{\gamma}}(\lambda, l_1, l_2)\{s_1(\lambda,l_1,l_2,h)|f'(l_1)|^\mu + s_2(\lambda,l_1,l_2,h)|f'(l_2)|^\mu\}^{\frac{1}{\mu}}$$
$$+s_6^{\frac{1}{\gamma}}(\lambda, l_1, l_2)\{s_3(\lambda, l_1, l_2, h)|f'(l_1)|^\mu + s_4(\lambda, l_1, l_2, h)|f'(l_2)|^\mu\}^{\frac{1}{\mu}}].$$

where

$$s_1(\lambda, l_1, l_2, h) := \int_0^{\frac{1}{2}} \frac{th(t)}{(l_1(1-t) + (l_1 + c)t)^2}dt$$
$$s_2(\lambda, l_1, l_2, h) := \int_0^{\frac{1}{2}} \frac{th(1-t)}{(l_1(1-t) + (l_1 + c)t)^2}dt$$
$$s_3(\lambda, l_1, l_2, h) := \int_{\frac{1}{2}}^1 \frac{(1-t)h(t)}{(l_1(1-t) + (l_1 + c)t)^2}dt$$
$$s_4(\lambda, l_1, l_2, h) := \int_{\frac{1}{2}}^1 \frac{(1-t)h(1-t)}{(l_1(1-t) + (l_1 + c)t)^2}dt$$
$$s_5(\lambda, l_1, l_2) := \frac{-\frac{c}{2l_1+c} - \log(l_1) + \log(l_1 + \frac{c}{2})}{c^2}$$
$$s_6(\lambda, l_1, l_2) := \frac{\frac{c}{2l_1+c} + \log(l_1 + \frac{c}{2} - \log(l_1 + c))}{c^2};\ where\ \ c = \eta(l_2, l_1).$$

Corollary 3.5. *Assuming that* $f : M = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f' \in L_1[l_1, l_1 + \eta(l_2, l_1)]$ *for* $l_1, l_1+\eta(l_2, l_1) \in M$ *with* $l_1 < l_1 + \eta(l_2, l_1)$, $\beta \in (0, 1]$ *and* $\lambda, \alpha \in [0, 1]$. *If* $|f'|^\mu$ *is* s*-harmonic preinvex on* M *for* $\mu > 1$ *with* $\frac{1}{\gamma} + \frac{1}{\mu} = 1$, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1 - \alpha \leq 1 + ((\alpha - 1)\lambda)^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda, \alpha, \beta, l_1, l_1 + \eta(l_2, l_1))|$$
$$\leq l_1\eta(l_2, l_1)\{l_1+\eta(l_2, l_1)\}[(k_1(\lambda,\alpha,\beta,l_1,l_2)+k_2(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(\hat{k_7}(\lambda,\alpha,\beta,l_1,l_2,s)$$
$$+\hat{k_8}(\lambda,\alpha,\beta,l_1,l_2,s)|f'(l_1)|^\mu+(\hat{k_9}(\lambda,\alpha,\beta,l_1,l_2, s)+ \hat{k_{10}}(\lambda,\alpha,\beta,l_1,l_2,s))|f'(l_2)|^\mu\}^{\frac{1}{\mu}}$$
$$+(k_5(\lambda,\alpha,\beta,l_1,l_2)+ k_6(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(\hat{k_{15}}(\lambda,\alpha,\beta,l_1,l_2,s)+\hat{k_{16}}(\lambda,\alpha,\beta,l_1,l_2,s))$$
$$\times|f'(l_1)|^\mu+(\hat{k_{17}}(\lambda,\alpha,\beta,l_1,l_2,s) + \hat{k_{18}}(\lambda,\alpha,\beta,l_1,l_2,s))|f'(l_2)|^\mu\}^{\frac{1}{\mu}}].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1 + ((\alpha - 1)\lambda)^{\frac{1}{\beta}} \leq 1 - \alpha$, *then*

$$|\Psi_f(\lambda, \alpha, \beta,l_1,l_1 + \eta(l_2,l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2, l_1)\}[(k_1(\lambda,\alpha,\beta,l_1,l_2)+k_2(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(\hat{k_7}(\lambda,\alpha,\beta,l_1,l_2,s)$$
$$+\hat{k_8}(\lambda,\alpha,\beta,l_1,l_2,s))|f'(l_1)|^\mu+(\hat{k_9}(\lambda,\alpha,\beta,l_1,l_2,s)+\hat{k_{10}}(\lambda,\alpha,\beta,l_1,l_2,s))|f'(l_2)|^\mu\}^{\frac{1}{\mu}}$$
$$+(k_4(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(\hat{k_{13}}(\lambda,\alpha,\beta,l_1,l_2,s)|f'(l_1)|^\mu+\hat{k_{14}}(\lambda,\alpha,\beta,l_1,l_2,s)|f'(l_2)|^\mu)\}^{\frac{1}{\mu}}].$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, *then*

$$
\begin{aligned}
&|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))| \\
&\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[(k_3(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(\hat{k_{11}}(\lambda,\alpha,\beta,l_1,l_2,s)|f'(l_1)|^{\mu} \\
&+\hat{k_{12}}(\lambda,\alpha,\beta,l_1,l_2,s)|f'(l_2)|^{\mu})\}^{\frac{1}{\mu}}+(k_5(\lambda,\alpha,\beta,l_1,l_2)+k_6(\lambda,\alpha,\beta,l_1,l_2))^{\frac{1}{\gamma}}\{(\hat{k_{15}}(\lambda,\alpha,\beta,l_1,l_2,s) \\
&+\hat{k_{16}}(\lambda,\alpha,\beta,l_1,l_2,s))|f'(l_1)|^{\mu}+(\hat{k_{17}}(\lambda,\alpha,\beta,l_1,l_2,s)+\hat{k_{18}}(\lambda,\alpha,\beta,l_1,l_2,s))|f'(l_2)|^{\mu}\}^{\frac{1}{\mu}}].
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{k_7}(\lambda,\alpha,\beta,l_1,l_2,s) :=& \frac{(\alpha\lambda^{\frac{1}{\beta}})^{1+s}}{l_1^2(l_1+c\alpha\lambda^{\frac{1}{\beta}})}\Bigg(\frac{\alpha\lambda(l_1(1+s)-s(l_1+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1+s,2+s,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{l_1}])}{1+s} \\
&-\frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta}(l_1(1+s+\beta)-(l_1+c\alpha\lambda^{\frac{1}{\beta}})(s+\beta)_2F_1[1,1+s+\beta,2+s+\beta,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{l_1}])}{1+s+\beta}\Bigg) \\
\hat{k_8}(\lambda,\alpha,\beta,l_1,l_2,s) :=& \frac{\alpha\lambda}{c^2(1-s)}\Bigg(\frac{(\frac{1}{1-\alpha})^{-s}(c(-1+s+\alpha-s\alpha)-s(l_1+c-c\alpha)_2F_1[1,1-s,2-s,\frac{l_1}{c(-1+\alpha)}])}{(-1+\alpha)(l_1+c-c\alpha)} \\
&-\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1-s}(-c(-1+s)\alpha\lambda^{\frac{1}{\beta}}+s(l_1+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1-s,2-s,\frac{-l_1\alpha\lambda^{\frac{-1}{\beta}}}{c})])}{l_1+c\alpha\lambda^{\frac{1}{\beta}}}\Bigg) \\
&-\frac{1}{c^2(-1+s+\beta)}\Bigg(-\frac{(\frac{1}{1-\alpha})^{-s-\beta}(-c(-1+s+\beta))}{(l_1+c-c\alpha)} \\
&+\frac{(\frac{1}{1-\alpha})^{-s-\beta}(s+\beta)_2F_1[1,1-s-\beta,2-s-\beta,\frac{l_1}{c(-1+\alpha)}])}{(-1+\alpha)} \\
&+\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1-s-\beta}(-c\alpha\lambda^{\frac{1}{\beta}}(-1+s+\beta))}{l_1+c\alpha\lambda^{\frac{1}{\beta}}}+(\alpha\lambda^{\frac{-1}{\beta}})^{1-s-\beta}(s+\beta) \\
&\times {}_2F_1[1,1-s-\beta,2-s-\beta,-\frac{l_1\alpha\lambda^{\frac{-1}{\beta}}}{c}]\Bigg) \\
\hat{k_9}(\lambda,\alpha,\beta,l_1,l_2,s) :=& \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}(1-t)^s dt \\
\hat{k_{10}}(\lambda,\alpha,\beta,l_1,l_2,s) :=& \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(l_1(1-t)+(l_1+c)t)^2}(1-t)^s dt \\
\hat{k_{11}}(\lambda,\alpha,\beta,l_1,l_2,s) :=& \frac{(1-\alpha)^{1+s}}{l_1^2(l_1+c-c\alpha)}\Bigg(\frac{(\alpha\lambda(l_1(1+s)-s(l_1+c-c\alpha)_2F_1[1,1+s,2+s,\frac{c(-1+\alpha)}{l_1}]))}{1+s} \\
&-\frac{(1-\alpha)^{\beta}(l_1(1+s+\beta)-(l_1+c-c\alpha)(s+\beta)_2F_1[1,1+s+\beta,2+s+\beta,\frac{c(-1+\alpha)}{l_1}])}{1+s+\beta}\Bigg)
\end{aligned}
$$

$$\hat{k_{12}}(\lambda,\alpha,\beta,l_1,l_2,s):=\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}(1-t)^s dt$$

$$\begin{aligned}\hat{k_{13}}(\lambda,\alpha,\beta,l_1,l_2,s):=&-\frac{(-1)^\beta(1-\alpha)^s F_1[1+s,-\beta,2,2+s,1-\alpha,\frac{c(-1+\alpha)}{l_1}]}{l_1^2(1+s)}\\&+\frac{(-1)^\beta(1-\alpha)^s\alpha F_1[1+s,-\beta,2,2+s,1-\alpha,\frac{c(-1+\alpha)}{l_1}]}{l_1^2(1+s)}\\&-\frac{1}{(c^2(l_1+c)(-1+s)(-1+\alpha)(l_1+c-c\alpha))}\lambda(-c(-1+s)(-1+\alpha)\\&\times(l_1(-1+(1-\alpha)^s)+c(-1+(1-\alpha)^s+\alpha))-(l_1+c)s(-1+\alpha)\\&\times(l_1+c-c\alpha)_2F_1[1,1-s,2-s,-\frac{l_1}{c}]-(l_1+c)s(1-\alpha)^s(l_1+c-c\alpha)\\&\times{}_2F_1[1,1-s,2-s,\frac{l_1}{c(-1+\alpha)}])\\&+\frac{1}{(c^2(l_1+c)(-1+s)(-1+\alpha)(l_1+c-c\alpha))}\alpha\lambda(-c(-1+s)(-1+\alpha)\\&\times(l_1(-1+(1-\alpha)^s)+c(-1+(1-\alpha)^s+\alpha))-(l_1+c)s(-1+\alpha)\\&\times(l_1+c-c\alpha)_2F_1[1,1-s,2-s,-\frac{l_1}{c}]-(l_1+c)s(1-\alpha)^s(l_1+c-c\alpha)\\&\times{}_2F_1[1,1-s,2-s,\frac{l_1}{c(-1+\alpha)}])\\&+\frac{(-1)^\beta\Gamma(1+s)\Gamma(1+\beta)_2F_1[2,1+s,2+s+\beta,-\frac{c}{l_1}]}{l_1^2\Gamma(2+s+\beta)}\end{aligned}$$

$$\hat{k_{14}}(\lambda,\alpha,\beta,l_1,l_2,s):=\int_{1-\alpha}^1\frac{(t-1)^\beta-(\alpha-1)\lambda}{((1-t)l_1+t(l_1+c))^2}(1-t)^s dt$$

$$\begin{aligned}\hat{k_{15}}(\lambda,\alpha,\beta,l_1,l_2,s):=&\frac{-1}{l_1^2(1+s)}\alpha^{-\beta}(-((\alpha-1)\lambda)^{\frac{1}{\beta}})^{-\beta}\Big(-(1-\alpha)^{s+1}\\&\times(\alpha((-1+\alpha)\lambda)^{\frac{1}{\beta}})^\beta F_1[1+s,-\beta,2,2+s,1-\alpha,\frac{c(\alpha-1)}{l_1}]\\&+\alpha^\beta(((-1+\alpha)\lambda)^{\frac{1}{\beta}})^\beta(1+((-1+\alpha)\lambda^{\frac{1}{\beta}})^{1+s}\\&\times F_1[1+s,-\beta,2,2+s,1+((\alpha-1)\lambda)^{\frac{1}{\beta}},-\frac{c(1+((\alpha-1)\lambda))^{\frac{1}{\beta}}}{l_1}]\Big)\\&+\frac{(1-\alpha)}{c^2(s-1)}\lambda\Big(\frac{c(1-s)(1-\alpha)^s}{l_1+c-c\alpha}+\frac{(c+\frac{l_1}{1-\alpha})(1-\alpha)^s s_2F_1[1,1-s,2-s,\frac{l_1}{c(-1+\alpha)}]}{l_1+c-c\alpha}\\&+\Big(\frac{1}{1+((-1+\alpha)\lambda)^{\frac{1}{\beta}}}\Big)^{1-s}\Big(\frac{c(-1+s)(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}{l_1+c+c((-1+\alpha)\lambda)^{\frac{1}{\beta}}}\\&-\frac{s_2F_1\Big[1,1-s,2-s,-\frac{l_1}{c(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}\Big]}{l_1+c+c((-1+\alpha)\lambda)^{\frac{1}{\beta}}}\Big)\Big)\end{aligned}$$

$$\hat{k_{16}}(\lambda,\alpha,\beta,l_1,l_2,s):=\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^1\frac{(t-1)^\beta-(\alpha-1)\lambda}{((1-t)l_1+t(l_1+c))^2}t^s dt$$

$$\hat{k_{17}}(\lambda,\alpha,\beta,l_1,l_2,s):=\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^\beta-(\alpha-1)\lambda)}{((1-t)l_1+t(l_1+c))^2}(1-t)^s dt$$

$$\hat{k_{18}}(\lambda,\alpha,\beta,l_1,l_2,s):=\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-(\alpha-1)\lambda}{(1-t)l_1+t(l_1+c))^2}(1-t)^s dt; where c=\eta(l_2,l_1).$$

Theorem 3.6. *Assuming that* $f:M=[l_1,l_1+\eta(l_2,l_1)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[l_1,l_1+\eta(l_2,l_1)]$ *for* $l_1,l_1+\eta(l_2,l_1)\in M$ *with* $l_1<l_1+\eta(l_2,l_1),\beta\in(0,1]$ *and* $\lambda,\alpha\in[0,1]$. *If* $|f'|^{\mu}$ *is relative harmonically preinvex on* M *for* $\mu>1$, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha\leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}&|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|\\&\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,l_1,l_2,\gamma)+k_{20}(\lambda,\alpha,\beta,l_1,l_2,\gamma))^{\frac{1}{\gamma}}\Bigg((1-\alpha)\\&\times\left\{\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(l_2)|^{\mu}\right\}\int_0^1 h(t)dt\Bigg)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,l_1,l_2,\gamma)\\&+k_{24}(\lambda,\alpha,\beta,l_1,l_2,\gamma))^{\frac{1}{\gamma}}\left(\alpha\left\{|f'(l_1)|^{\mu}+\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\Bigg].\end{aligned}$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}\leq 1-\alpha$, *then*

$$\begin{aligned}&|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|\\&\leq l_1\eta(l_2,v_1)\{l_1+\eta(l_2,l_1)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,l_1,l_2,\gamma)+k_{20}(\lambda,\alpha,\beta,l_1,l_2,\gamma))^{\frac{1}{\gamma}}\Bigg((1-\alpha)\\&\times\left\{\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(l_2)|^{\mu}\right\}\int_0^1 h(t)dt\Bigg)^{\frac{1}{\mu}}+(k_{22}(\lambda,\alpha,\beta,l_1,l_2,\gamma))^{\frac{1}{\gamma}}\\&\times\left(\alpha\left\{|f'(l_1)|^{\mu}+\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\Bigg].\end{aligned}$$

(c) *If* $1-\alpha\leq(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}&|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|\\&\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\Bigg[(k_{21}(\lambda,\alpha,\beta,l_1,l_2,\gamma))^{\frac{1}{\gamma}}\Bigg((1-\alpha)\Bigg\{\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\\&+|f'(l_2)|^{\mu}\Bigg\}\int_0^1 h(t)dt\Bigg)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,l_1,l_2,\gamma)+k_{24}(\lambda,\alpha,\beta,l_1,l_2,\gamma))^{\frac{1}{\gamma}}\Bigg(\alpha\Bigg\{|f'(l_1)|^{\mu}\\&+\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\Bigg\}\int_0^1 h(t)dt\Bigg)^{\frac{1}{\mu}}\Bigg].\end{aligned}$$

Proof. By using Lemma 3.1 and Hölder's integral inequality, we have

$$|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))| \tag{3. 17}$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\Bigg[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|dt$$
$$+\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|dt\Bigg]$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left(\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{1-\alpha}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt\right)^{\frac{1}{\gamma}}\left(\int_{1-\alpha}^1\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}} \tag{3. 18}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt=k_{19}(\lambda,\alpha,\beta,l_1,l_2,\gamma)+k_{20}(\lambda,\alpha,\beta,l_1,l_2,\gamma). \tag{3. 19}$$

where

$$k_{19}(\lambda,\alpha,\beta,l_1,l_2,\gamma):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{(-t^\beta+\alpha\lambda)^\gamma}{((1-t)l_1+t(l_1+\eta(l_2,l_1)))^{2\gamma}}dt$$
$$k_{20}(\lambda,\alpha,\beta,l_1,l_2,\gamma):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{(t^\beta-\alpha\lambda)^\gamma}{((1-t)l_1+t(l_1+\eta(l_2,l_1)))^{2\gamma}}dt$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt=k_{21}(\lambda,\alpha,\beta,l_1,l_2,\gamma). \tag{3. 20}$$

where

$$k_{21}(\lambda,\alpha,\beta,l_1,l_2,\gamma):=\int_0^{1-\alpha}\frac{(-t^\beta+\alpha\lambda)^\gamma}{((1-t)l_1+t(l_1+\eta(l_2,l_1)))^{2\gamma}}dt$$

(b) (i) If $1+((\alpha-1)\lambda)^{\frac{1}{\beta}}\leq 1-\alpha$, then

$$\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt=k_{22}(\lambda,\alpha,\beta,l_1,l_2,\gamma). \tag{3. 21}$$

where

$$k_{22}(\lambda,\alpha,\beta,l_1,l_2,\gamma):=\int_{1-\alpha}^1\frac{((t-1)^\beta-(\alpha-1)\lambda)^\gamma}{((1-t)l_1+t(l_1+\eta(l_2,l_1)))^{2\gamma}}dt$$

(ii) If $1+((\alpha-1)\lambda)^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt=k_{23}(\lambda,\alpha,\beta,l_1,l_2,\gamma)+k_{24}(\lambda,\alpha,\beta,l_1,l_2,\gamma). \tag{3. 22}$$

where

$$k_{23}(\lambda,\alpha,\beta,l_1,l_2,\gamma):=\int_{1-\alpha}^{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}\frac{(-(t-1)^{\beta}+(\alpha-1)\lambda)^{\gamma}}{((1-t)l_1+t(l_1+\eta(l_2,l_1)))^{2\gamma}}dt$$
$$k_{24}(\lambda,\alpha,\beta,l_1,l_2,\gamma):=\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1}\frac{((t-1)^{\beta}-(\alpha-1)\lambda)^{\gamma}}{((1-t)l_1+t(l_1+\eta(l_2,l_1)))^{2\gamma}}dt$$

(c) Consider,

$$\int_0^{1-\alpha}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt \tag{3. 23}$$

Setting $x=\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}$, so that $dt=\frac{-l_1\{l_1+\eta(l_2,l_1)\}}{x^2\eta(l_2,l_1)}dx$
For $0\le t\le 1-\alpha$, we have $l_1+\eta(l_2,l_1)\le x\le\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}$ and hence (3.23) becomes

$$= \quad \frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\int_{\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}}^{l_1+\eta(l_2,l_1)}\frac{|f'(x)|^{\mu}}{x^2}dx \tag{3. 24}$$

Using Hermite-Hadamard's inequality for relative harmonic preinvex functions, we have

$$\begin{aligned}
&\int_0^{1-\alpha}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dx\\
\le\quad &\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\left(\frac{\{l_1+\eta(l_2,l_1)\}-\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}}{\{l_1+\eta(l_2,l_1)\}\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}}\right)\\
&\left[\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(l_1+\eta(l_2,l_1))|^{\mu}\right]\int_0^1 h(t)dt\\
\le\quad &(1-\alpha)\left[\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(l_2)|^{\mu}\right]\int_0^1 h(t)dt
\end{aligned}\tag{3. 25}$$

Above Inequality holds for $\alpha=1$.

(d) Consider,

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt \tag{3. 26}$$

Setting $x=\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}$, so that $dt=\frac{-l_1\{l_1+\eta(l_2,l_1)\}}{x^2\eta(l_2,l_1)}dx$
For $1-\alpha\le t\le 1$, we have $\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\le x\le l_1$ and hence (3.26) becomes

$$= \quad \frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\int_{l_1}^{\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}}\frac{|f'(x)|^{\mu}}{x^2}dx \tag{3. 27}$$

Using Hermite-Hadamard's inequality for relative harmonic preinvex functions, we have

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$
$$\leq \frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\left(\frac{\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}-l_1}{\frac{l_1{}^2\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}}\right)\left[|f'(l_1)|^{\mu}+\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right]\int_0^1 h(t)dt$$
$$\leq \alpha\left[|f'(l_1)|^{\mu}+\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right]\int_0^1 h(t)dt \tag{3. 28}$$

Above Inequality holds for $\alpha = 0$.

By substituting (3.19) to (3.22), (3.25) and (3.28) in equation (3.17) gives the required result. □

If $\lambda = 0$, $\alpha = \frac{1}{2}$ and $\beta = 1$, then identity (3.17) reduces to the following result:

Corollary 3.7. *Assuming that $f : M = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[l_1, l_1 + \eta(l_2, l_1)]$ for $l_1, l_1+\eta(l_2, l_1) \in M$ with $l_1 < l_1 + \eta(l_2, l_1)$. If $|f'|^{\mu}$ is harmonic P-preinvex on M for $\mu > 1$, then*

$$\left|\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\eta(l_2,l_1)}\int_{l_1}^{l_1+\eta(l_2,l_1)}\frac{f(z)}{z^2}dz - f\left(\frac{2l_1\{l_1+\eta(l_2,l_1)\}}{l_1+(l_1+\eta(l_2,l_1))}\right)\right|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{1-\frac{1}{\mu}}$$
$$[\{s_1^{**}(\lambda,l_1,l_2,0,\mu)+s_2^{**}(\lambda,l_1,l_2,0,\mu)\}(|f'(l_1)|^{\mu}+|f'(l_2)|^{\mu})]^{\frac{1}{\mu}}.$$

where

$$s_1^{**}(\lambda,l_1,l_2,0,\mu):=\frac{1}{l_1(2l_1+c)},\ s_2^{**}(\lambda,l_1,l_2,0,\mu):=\frac{1}{(l_1+c)(2l_1+c)};\ where\, c=\eta(l_2,l_1).$$

Theorem 3.8. *Assuming that $f : M = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[l_1, l_1 + \eta(l_2, l_1)]$ for $l_1, l_1+\eta(l_2, l_1) \in M$ with $l_1 < l_1 + \eta(l_2, l_1), \beta \in (0, 1]$ and $\lambda, \alpha \in [0, 1]$. If $|f'|$ is relative harmonically preinvex on M, we have*

(a) *If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, then*

$$|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[\{\{k_7(\lambda,\alpha,\beta,l_1,l_2,h)+k_8(\lambda,\alpha,\beta,l_1,l_2,h)\}+\{k_{15}(\lambda,\alpha,\beta,l_1,l_2,h)$$
$$+k_{16}(\lambda,\alpha,\beta,l_1,l_2,h)\}\}|f'(l_1)|+\{\{k_9(\lambda,\alpha,\beta,l_1,l_2,h)+k_{10}(\lambda,\alpha,\beta,l_1,l_2,h)\}$$
$$+\{k_{17}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{18}(\lambda,\alpha,\beta,l_1,l_2,h)\}\}|f'(l_2)|].$$

(b) *If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then*

$$|\Psi_f(\lambda,\beta,\alpha,l_1,l_1+\eta(l_2,l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[\{\{k_7(\lambda,\alpha,\beta,l_1,l_2,h)+k_8(\lambda,\alpha,\beta,l_1,l_2,h)\}+k_{13}(\lambda,\alpha,\beta,l_1,l_2,h)\}$$
$$\times|f'(l_1)|+\{\{k_9(\lambda,\alpha,\beta,l_1,l_2,h)+k_{10}(\lambda,\alpha,\beta,l_1,l_2,h)\}+k_{14}(\lambda,\alpha,\beta,l_1,l_2,h)\}|f'(l_2)|].$$

(c) *If* $1-\alpha \le (\alpha\lambda)^{\frac{1}{\beta}} \le 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$*, then*

$$\begin{aligned}&|\Psi_f(\lambda,\beta,\alpha,l_1,l_1+\eta(l_2,l_1))|\\ &\le l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[\{\{k_{15}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{16}(\lambda,\alpha,\beta,l_1,l_2,h)\}+k_{11}(\lambda,\alpha,\beta,l_1,l_2,h)\}\\ &\times|f'(l_1)|+\{\{k_{17}(\lambda,\alpha,\beta,l_1,l_2,h)+k_{18}(\lambda,\alpha,\beta,l_1,l_2,h)\}+k_{12}(\lambda,\alpha,\beta,l_1,l_2,h)\}|f'(l_2)|].\end{aligned}$$

Where the values of $k_7(\lambda,\alpha,\beta,l_1,l_2,h)$ *to* $k_{18}(\lambda,\alpha,\beta,l_1,l_2,h)$ *are defined in Theorem* 3.3.

Proof. By using Lemma 3.1, we have

$$\begin{aligned}&|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|\\ &\le l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|dt\right.\\ &\left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{l_1\{l_1+\eta(l_2,l_1)\}}{\bar{A}_t}\right)\right|dt\right].\end{aligned}$$

Since $|f'|$ be relative harmonically preinvex on the interval $[l_1,l_1+\eta(l_2,l_1)]$ with respect to an arbitrary nonnegative function h and $t\in[0,1]$, we have

$$\begin{aligned}&|\Psi_f(\lambda,\alpha,\beta,l_1,l_1+\eta(l_2,l_1))|\\ &\le l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}\left[\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(t)dt|f'(l_1)|\right.\right.\\ &\left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|}{(\bar{A}_t)^2}h(t)dt|f'(l_2)|\right\}+\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(1-t)dt|f'(l_1)|\right.\\ &\left.\left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-(\alpha-1)\lambda|}{(\bar{A}_t)^2}h(1-t)dt|f'(l_2)|\right\}\right] \qquad (3.\ 29)\end{aligned}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}}\le 1-\alpha$, then

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(t)dt\\ &= k_7(\lambda,\alpha,\beta,l_1,l_2,h)+k_8(\lambda,\alpha,\beta,l_1,l_2,h). \qquad (3.\ 30)\end{aligned}$$

and

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(1-t)dt\\ &= k_9(\lambda,\alpha,\beta,l_1,l_2,h)+k_{10}(\lambda,\alpha,\beta,l_1,l_2,h). \qquad (3.\ 31)\end{aligned}$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}}\ge 1-\alpha$, then

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(t)dt\\ &= k_{11}(\lambda,\alpha,\beta,l_1,l_2,h). \qquad (3.\ 32)\end{aligned}$$

and

$$\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} h(1-t)dt$$
$$= \quad k_{12}(\lambda, \alpha, \beta, l_1, l_2, h). \tag{3. 33}$$

(b) (i) If $1 + ((\alpha-1)\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - (\alpha-1)\lambda|}{(\bar{A}_t)^2} h(t)dt$$
$$= \quad k_{13}(\lambda, \alpha, \beta, l_1, l_2, h). \tag{3. 34}$$

and

$$\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - (\alpha-1)\lambda|}{(\bar{A}_t)^2} h(1-t)dt$$
$$= \quad k_{14}(\lambda, \alpha, \beta, l_1, l_2, h). \tag{3. 35}$$

(ii) If $1 + ((\alpha-1)\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - (\alpha-1)\lambda|}{(\bar{A}_t)^2} h(t)dt$$
$$= \quad k_{15}(\lambda, \alpha, \beta, l_1, l_2, h) + k_{16}(\lambda, \alpha, \beta, l_1, l_2, h). \tag{3. 36}$$

and

$$\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - (\alpha-1)\lambda|}{(\bar{A}_t)^2} h(1-t)dt$$
$$= \quad k_{17}(\lambda, \alpha, \beta, l_1, l_2, h) + k_{18}(\lambda, \alpha, \beta, l_1, l_2, h). \tag{3. 37}$$

By substituting (3.30) to (3.37) in equation (3.29) gives the required result. □

Corollary 3.9. *Assuming that* $f : M = [l_1, l_1 + \eta(l_2, l_1)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ *is differentiable on the interior ,* M^o *of* M *where* $f' \in L_1[l_1, l_1 + \eta(l_2, l_1)]$ *for* $l_1, l_1+\eta(l_2, l_1) \in M$ *with* $l_1 < l_1 + \eta(l_2, l_1), \beta \in (0, 1]$ *and* $\lambda, \alpha \in [0, 1]$. *If* $|f'|$ *is* s*-harmonic Godunova-Levin preinvex on* M*, we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1 + ((\alpha-1)\lambda)^{\frac{1}{\beta}}$*, then*

$$|\Psi_f(\lambda, \alpha, \beta, l_1, l_1 + \eta(l_2, l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[\{\{k_7^*(\lambda,\alpha,\beta,l_1,l_2,-s)+k_8^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}+\{k_{15}^*(\lambda,\alpha,\beta,l_1,l_2,-s)$$
$$+k_{16}^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}\}|f'(l_1)| + \{\{k_9^*(\lambda,\alpha,\beta,l_1,l_2,-s)+k_{10}^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}$$
$$+\{k_{17}^*(\lambda,\alpha,\beta,l_1,l_2,-s)+k_{18}^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}\}|f'(l_2)|].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1 + ((\alpha-1)\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$*, then*

$$|\Psi_f(\lambda, \beta, \alpha, l_1, l_1 + \eta(l_2, l_1))|$$
$$\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[\{\{k_7^*(\lambda,\alpha,\beta,l_1,l_2,-s)+k_8^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}+k_{13}^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}$$
$$\times|f'(l_1)|+\{\{k_9^*(\lambda,\alpha,\beta,l_1,l_2,-s)+k_{10}^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}+k_{14}^*(\lambda,\alpha,\beta,l_1,l_2,-s)\}|f'(l_2)|].$$

(c) *If* $1-\alpha\leq(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+((\alpha-1)\lambda)^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}&|\Psi_f(\lambda,\beta,\alpha,l_1,l_1+\eta(l_2,l_1))|\\ &\leq l_1\eta(l_2,l_1)\{l_1+\eta(l_2,l_1)\}[\{\{k^*_{15}(\lambda,\alpha,\beta,l_1,l_2,-s)+k^*_{16}(\lambda,\alpha,\beta,l_1,l_2,-s)\}+k^*_{11}(\lambda,\alpha,\beta,l_1,l_2,-s)\}\\ &\times|f'(l_1)|+\{\{k^*_{17}(\lambda,\alpha,\beta,l_1,l_2,-s)+k^*_{18}(\lambda,\alpha,\beta,l_1,l_2,-s)\}+k^*_{12}(\lambda,\alpha,\beta,l_1,l_2,-s)\}|f'(l_2)|].\end{aligned}$$

where

$$\begin{aligned}k^*_7(\lambda,\alpha,\beta,l_1,l_2,-s):=&\frac{(\alpha\lambda^{\frac{1}{\beta}})^{1-s}}{l_1^2(l_1+c\alpha\lambda^{\frac{1}{\beta}})}\Bigg(\frac{\alpha\lambda(l_1(1-s)+s(l_1+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1-s,2-s,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{l_1}])}{1-s}\\ &+\frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta}(l_1+c\alpha\lambda^{\frac{1}{\beta}})(-s+\beta)_2F_1[1,1-s+\beta,2-s+\beta,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{l_1}]}{1-s+\beta}\\ &+\frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta}l_1(-1+s-\beta)}{1-s+\beta}\Bigg)\end{aligned}$$

$$\begin{aligned}k^*_8(\lambda,\alpha,\beta,l_1,l_2,-s):=&\frac{\alpha\lambda}{c^2(1+s)}\Bigg(\frac{c(-1-s+\alpha+s\alpha)-s(l_1+c-c\alpha)_2F_1[1,1+s,2+s,\frac{l_1}{c(-1+\alpha)}]}{(1-\alpha)^s(-1+\alpha)(l_1+c-c\alpha)}\\ &-\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1+s}(c(1+s)\alpha\lambda^{\frac{1}{\beta}}-s(l_1+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1+s,2+s,\frac{-l_1\alpha\lambda^{\frac{-1}{\beta}}}{c})])}{l_1+c\alpha\lambda^{\frac{1}{\beta}}}\Bigg)\\ &+\frac{1}{c^2(1+s-\beta)}\Bigg(-\frac{(\frac{1}{1-\alpha})^{s-\beta}(c(1+s-\beta))}{(l_1+c-c\alpha)}+\frac{(\frac{1}{1-\alpha})^{s-\beta}(s-\beta)}{(-1+\alpha)}\\ &\times{}_2F_1[1,1+s-\beta,2+s-\beta,\frac{l_1}{c(-1+\alpha)}]+\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1+s-\beta}(-c\alpha\lambda^{\frac{1}{\beta}}(-1+s+\beta))}{l_1+c\alpha\lambda^{\frac{1}{\beta}}}\\ &+(\alpha\lambda^{\frac{-1}{\beta}})^{1+s-\beta}(-s+\beta)_2F_1[1,1+s-\beta,2+s-\beta,-\frac{l_1\alpha\lambda^{\frac{-1}{\beta}}}{c}]\Bigg)\end{aligned}$$

$$k^*_9(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^\beta-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}(1-t)^{-s}dt$$

$$k^*_{10}(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(l_1(1-t)+(l_1+c)t)^2}(1-t)^{-s}dt$$

$$\begin{aligned}k^*_{11}(\lambda,\alpha,\beta,l_1,l_2,-s):=&\frac{(1-\alpha)^{-s}}{l_1^2(l_1-c\alpha+c)}\Bigg((1-\alpha)\alpha\lambda l_1-(1-\alpha)^{1+\beta}l_1\\ &+\frac{(-1+\alpha)\alpha\lambda s(l_1+c-c\alpha)_2F_1[1,1-s,2-s,\frac{c(-1+\alpha)}{l_1}])}{s-1}\\ &+\frac{(1-\alpha)^{1+\beta}(l_1+c-c\alpha)(s-\beta)_2F_1[1,1-s+\beta,2-s+\beta,\frac{c(-1+\alpha)}{l_1}]}{-1+s-\beta}\Bigg)\end{aligned}$$

$$k_{12}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_{0}^{1-\alpha}\frac{-(t^{\beta}-\alpha\lambda)}{(l_1(1-t)+(l_1+c)t)^2}(1-t)^{-s}dt$$

$$k_{13}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\frac{(-1)^{\beta}(1-\alpha)^{-s}F_1[1-s,-\beta,2,2-s,1-\alpha,\frac{c(-1+\alpha)}{l_1}]}{l_1^2(-1+s)}$$
$$-\frac{(-1)^{\beta}(1-\alpha)^{-s}\alpha F_1[1-s,-\beta,2,2-s,1-\alpha,\frac{c(-1+\alpha)}{l_1}]}{l_1^2(-1+s)}$$
$$+\frac{1}{c^2(1+s)}(1-\alpha)\lambda\Bigg(\frac{-c(1+s)+(l_1+c)s_2F_1[1,1+s,2+s,-\frac{l_1}{c}]}{l_1+c}$$
$$+\frac{(1-\alpha)^{-s-1}(-c(s+1)(\alpha-1)-s(l_1+c-c\alpha)_2F_1[1,1+s,2+s,\frac{l_1}{c(-1+\alpha)}])}{l_1+c-c\alpha}\Bigg)$$
$$+\frac{(-1)^{\beta}\Gamma(1-s)\Gamma(1+\beta)_2F_1[2,1-s,2-s+\beta,-\frac{c}{l_1}]}{l_1^2\Gamma(2-s+\beta)}$$

$$k_{14}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-(\alpha-1)\lambda}{((1-t)l_1+t(l_1+c))^2}(1-t)^{-s}dt$$

$$k_{15}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\frac{-1}{l_1^2(-1+s)}\alpha^{-\beta}(-((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{-\beta}((-(-1+\alpha)$$
$$\times(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}}))^{-s})\Bigg(-(-1+\alpha)(\alpha((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{\beta}$$
$$\times F_1[1-s,-\beta,2,2-s,1-\alpha,\frac{c}{-1+\alpha}l_1]$$
$$-(1-\alpha)^{s}\alpha^{\beta}(((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{\beta}(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})$$
$$\times F_1[1-s,-\beta,2,2-s,1+((-1+\alpha)\lambda)^{\frac{1}{\beta}},\frac{-c(1+((-1+\alpha)\lambda))^{\frac{1}{\beta}}}{l_1}]\Bigg)+\frac{\lambda(\alpha-1)}{c^2(1+s)}$$
$$\times\Bigg((1-\alpha)^{-1-s}\Bigg(\frac{-c(1+s)(-1+\alpha)}{l_1+c-c\alpha}-s_2F_1[1,1+s,2+s,\frac{l_1}{(c(-1+\alpha))}]\Bigg)$$
$$+\Bigg(\frac{1}{(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}\Bigg)^{1+s}\Bigg(\frac{-c(1+s)(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}{l_1+c+c((-1+\alpha)\lambda)^{\frac{1}{\beta}}}$$
$$+s_2F_1[1,1+s,2+s,-\frac{l_1}{c(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}]\Bigg)\Bigg)$$

$$k_{16}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-(\alpha-1)\lambda}{((1-t)l_1+t(l_1+c))^2}t^{-s}dt$$

$$k_{17}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_{1-\alpha}^{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-(\alpha-1)\lambda)}{((1-t)l_1+t(l_1+c))^2}(1-t)^{-s}dt$$

$$k_{18}^{*}(\lambda,\alpha,\beta,l_1,l_2,-s):=\int_{1+((\alpha-1)\lambda)^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-(\alpha-1)\lambda}{((1-t)l_1+t(l_1+c))^2}(1-t)^{-s}dt;\ where\ c=\eta(l_2,l_1)$$

Remark 3.10. *If $\beta = 1$, $h(t) = t$ and $\eta(l_2, l_1) = l_2 - l_1$, then our results coincide with the results for harmonically convex functions* [34].

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REFERENCES

[1] M.U. Awan, M.A. Noor, M.V. Mihai, K.I. Noor, Conformable fractional Hermite-Hadamard inequalities via preinvex functions, Tblisi Mathematical Journal. **10,** No. 4 (2017) 129-141.

[2] W. N. Bailey, A Reducible Case of the Fourth Type of Appell's Hypergeometric Functions of Two Variables. Quart. J. Math. (Oxford) 4 (1933).

[3] A. Barbagallo, M.A. Ragusa, On Lagrange duality theory for dynamics vaccination games, Ricerche di Matematica. **67,** No. 2 (2018) 969-982.

[4] A. Ben-Isreal, B. Mond, what is invexity? J. Aust. Math. Soc., Ser. B. **28,** No. 1 (1986) 1-9.

[5] F. Chen, S. Wu, Fejer and Hermite-Hadamard type inqequalities for harmonically convex functions, J. Appl . Math. (2014) Article Id:386806.

[6] J. Hadamard, *Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considerée par Riemann*, J. Math. Pures et Appl. 58 (1893) 171-215.

[7] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80, (1981) 545-550.

[8] Ch. Hermite, *Sur deux limites d' une intégrale définie*, Mathesis. 3, (1883) 82.

[9] S. Hussain, S. Rafeeq, Some new Hermite-Hadamard type integral inequalities for functions whose nth derivatived are logarithmically relative h-preinvex, Miskolc Mathematical Notes. 18, No. 2 (2017) 837-849.

[10] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat. 43, No. 6 (2014) 935-942.

[11] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some weighted integral inequalities for differentiable h-preinvex functions*, Georgian Math. J., **25** (3) (2018) 441-450.

[12] M. A. Latif, S. S. Dragomir and E. Momoniat, *Fejér type inequalities for harmonically-convex functions with applications*, Journal of Applied Analysis and Computation **7**, No. 3 (2017) 795-813.

[13] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejér type inequalities for harmonically-convex functions with applications to special means*, International Journal of Analysis and Applications **13**, No. 1 (2017) 1-14.

[14] M. A. Latif, *New Hermite Hadamard type integral inequalities for GA-convex functions with applications*, Analysis **34**, No. 4 (2014) 379–389.

[15] M. A. Latif, S. S. Dragomir and E. Momoniat, *Fejér type integral inequalities related with geometrically-arithmetically-convex functions with applications*, Journal of Applied Analysis and Computation **7**, No. 3 (2017) 795-813.

[16] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejér type integral inequalities related with geometrically-arithmetically-convex functions with applications*, Filomat **32**, No. 6 (2018), 21932206.

[17] M. A. Latif, *New Fejer and Hermite-Hadamard type inequalities for differentiable p-convex mappings*, Punjab Univ. j. math. **51**, No. 2 (2019) 39-59.

[18] M. A. Latif and W. Irshad, *Some Fejer and Hermite-Hadamard type inequalities considering ϵ-convex and(σ, ϵ)-convex functions*, Punjab Univ. j. math. **50**, No. 3 (2018) 13-24.

[19] M. A. Latif, *Estimates of Hermite-Hadamard inequality for twice differentiable harmonically-convex functions with applications*, Punjab Univ. j. math. **50**, No. 1 (2018) 1-13.

[20] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some ϕ-analogues of Hermite-Hadamard inequality for s-convex functions in the second sense and related estimates*, Punjab Univ. j. math. **48**, No. 2 (2016) 147-166.

[21] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions*, Punjab Univ. j. math. **47**, No. 1 (2015) 57-72.

[22] M. A. Latif and S. Hussain, *New HermiteHadamard type inequalities for harmonically-convex functions*, Punjab Univ. j. math. **51**, No. 6 (2019) 1-17.

[23] M. A. Latif and S. S. Dragomir, New Inequalities of Hermite-Hadamard and Fejer type via preinvexity, Journal of Computational Analysis and Application. 19, No. 4 (2015) 725-739.
[24] M. A. Latif, S. S. Dragomir and E. Momoniat, Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions, Punjab Univ. J. Math. Vol. 47, No. 1 (2015) 57-72.
[25] J. Y. Li, On Hadamard-type inequalities for s-preinvex functions, J. Chongqing Norm. Univ. (Natural Science) China. 27, (2010) 5-8.
[26] S. R. Mohan, S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189, (1995) 901-908.
[27] M. A. Noor, K. I. Noor, M. U. Awan, Integral inequalities for harmonically s-Godunova-Levin functions. FACTA Universitatis (NIS)-Mathematics-Informatics. 29, No. 4 (2014) 415-424.
[28] M. A. Noor, K. I. Noor, M. U. Awan, Hermite-Hadamard inequalities for s-Godunova-Levin preinvex functions, J. Adv. Math. Stud. 7, No. 2 (2014) 12-19.
[29] M. A. Noor, K. I. Noor, M. U. Awan and Jueyou Li, On Hermite-Hadamard inequalities for h-preinvex functions, Filomat. 28, No. 7 (2014) 1463-1474.
[30] M. A. Noor, K. I. Noor, S. Iftikhar,Integral inequalities for differentiable relative harmonic preinvex functions TWMS J. Pure Appl. Math. 7, No. 1 (2016) 3-19.
[31] M. A. Noor, K. I. Noor, S. Iftikhar, Harmoic Beta-Preinvex Functions and Inequalities, Int. J. Anal. Appl. 13, No. 2 (2017) 144-160.
[32] M. A. Noor, K. I. Noor, S. Iftikhar, Fractional Ostrowski inequalities for harmonic h-preinvex functions, FACTA Universitatis (NIS), Mathematics - Informatics. 31, No. 2 (2016) 417-445.
[33] M. A. Noor, K. I. Noor, S. Iftikhar, Hermite-Hadamard inequalities for harmonic preinvex functions, Saussurea. 6, No. 2 (2016) 34-53.
[34] J. Park, Hermite-Hadamard-like and Simpson-like type inequalities for harmonically convex functions, Int. J. Math. Anal. 27, No. 8 (2014) 1321-1337.
[35] R. Pini, Invexity and generalized convexity, Optimization. 22, (1991) 513525.
[36] E. W. Weisstein, Regularized Hypergeometric Function. http://mathworld.wolfram.com/ RegularizedHypergeometricFunction.html, (2003).