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Integral Inequalities for Exponentially Geometrically Convex Functions via Fractional Operators

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Abstract. We establish several basic inequalities for exponentially geometrically convex function in fractional integrals versions, also we define a new identity for Riemann-Liouville fractional integrals and use it to obtain new estimates of general Hermite-Hadamard type inequalities utilizing geometrically convex function. Several special cases are also discussed.

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1. Introduction

Let $\mathcal{F} : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\mathcal{F}\left(\frac{\nu+\omega}{2}\right) \leq \frac{1}{\omega-\nu}\int_{\nu}^{\omega} \mathcal{F}(x)\mathrm{d}x \leq \frac{\mathcal{F}(\nu)+\mathcal{F}(\omega)}{2}.$$

The above inequality is known as Hermite-Hadamard's inequality. Equality holds in either side only for affine functions. It gives us an estimate of the (integral) mean value of a continuous convex functions. This result of Hermite and Hadamard is very simple in nature but

very powerful. Interestingly both sides of the above integral inequality characterize convex functions. For some interesting details and applications of Hermite-Hadamard's inequality, we refer readers to [2, 5, 6, 7, 8, 10, 11, 13, 14, 19, 20, 23, 24, 38, 39, 40, 41, 42, 43, 55, 56, 57, 58, 59]. Theory of convexity play a vital role in the development of theory of inequalities. Other than Hermite-Hadamard's inequality there are many famous results in the theory of inequalities which can be obtained using the functions with convexity property. Many researchers have used different novel and innovative ideas in obtaining new generalizations of classical inequalities [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]. Sarikaya et al. [57] used elegantly the concepts of fractional calculus and obtained a fractional refinement of Hermite-Hadamard's inequality. This idea compelled many researchers to use fractional calculus concepts in theory of inequalities and gradually many new fractional analogues of classical inequalities have been obtained in the literature. For details, see [3, 7, 9, 12, 16, 17, 18, 21, 22, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 55, 56, 57, 58].
We set forth some terminologies, definitions, and essential details that will be used throughout the remaining part of the paper.

In [19], Niculescu mentioned the following considerable definitions:

Definition 1.1. *(*[19]*) The class of all $GA-$convex functions is constituted by all functions $\mathcal{F}:K\to\mathbb{R}$(acting on subintervals of $(0,\infty)$) such that*

$$\mathcal{F}\big(\nu^{1-\xi}\omega^{\xi}\big)\leq(1-\xi)\mathcal{F}(\nu)+\xi\mathcal{F}(\omega),\quad\forall\nu,\omega\in K,\ \xi\in[0,1].$$

Definition 1.2. *(*[19] *) The $GG-$convex functions are those functions $\mathcal{F}:K\to J$ (acting on subintervals of $(0,\infty)$) such that*

$$\mathcal{F}\big(\nu^{1-\xi}\omega^{\xi}\big)\leq\big(\mathcal{F}(\nu)\big)^{1-\xi}\big(\mathcal{F}(\omega)\big)^{\xi},\quad\forall\nu,\omega\in K,\ \xi\in[0,1].$$

The class of exponentially convex functions was introduces by Antczak [4], Dragomir and Gomm [8].

Definition 1.3. *(*[4, 8]*). A positive real-valued function $\mathcal{F}:K\subseteq\mathbb{R}\longrightarrow(0,\infty)$ is said to be exponentially convex on K if the inequality*

$$e^{\mathcal{F}(\xi\nu+(1-\xi)\omega)}\leq\xi e^{\mathcal{F}(\nu)}+(1-\xi)e^{\mathcal{F}(\omega)},\quad\nu,\omega\in K,\ \xi\in[0,1].\tag{1. 1}$$

Exponentially convex functions can be used in different fields such as statistical learning, sequential prediction and stochastic optimization, [1, 4, 25].
Rashid et al. [50] proposed a new class of exponentially convex function, which is known as exponentially GA-convex function stated as follows.

Definition 1.4. *(*[50]*) A function $\mathcal{F}:K\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be exponentially GA-convex function, if*

$$e^{\mathcal{F}\left(\nu^{1-\xi}\omega^{\xi}\right)}\leq(1-\xi)e^{\mathcal{F}(\nu)}+\xi e^{\mathcal{F}(\omega)},\ \forall\nu,\omega\in K,\ \ \xi\in[0,1].\tag{1. 2}$$

Also note that for $\xi=\frac{1}{2}$ in Definition 1.4, we have Jensen type exponentially $GA-$convex functions.

$$e^{\mathcal{F}\left(\sqrt{\nu\omega}\right)}\leq\frac{1}{2}[e^{\mathcal{F}(\nu)}+e^{\mathcal{F}(\omega)}],\ \forall\nu,\omega\in K.$$

Remark 1.5. *If* $\log \mathcal{F}$ *is geometrically convex, then under the assumption of Definition 1.4, one can say that* $\mathcal{F}$ *is exponentially geometrically convex function.*

In this paper, utilizing the concept of exponentially geometrically convex function, Hermite-Hadamard's inequalities in fractional integral forms is established. Also, a new identity for RiemannLiouville fractional integrals is defined.
In the following, we introducethe hypergeometric function, beta function and gamma function which will be used in obtaining some integrals.

Definition 1.6. *([15]). The integral representation of the hypergeometric functions is as follows:*

$$ {}_2F_1[\nu,\kappa,\delta;z] = \frac{1}{\beta(\omega,\kappa-\omega)} \int\limits_0^1 \xi^{\omega-1}(1-\xi)^{\kappa-\omega-1}(1-z\xi)^{-\nu} d\xi, $$

where $|z|<1, \quad \kappa>\omega>0.$

$$ \begin{aligned} \Gamma(x) &= \int\limits_0^\infty e^{-\xi}\xi^{x-1}d\xi, \\ \beta(x,y) &= \int\limits_0^1 \xi^{x-1}(1-\xi)^{y-1}d\xi = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0. \end{aligned} $$

We now give the definition of the Riemann-Liouville fractional integral, which is mainly due to [26].

Definition 1.7. *(See [26]) Let* $\mathcal{F} \in L[\nu,\omega]$. *The Riemann-Liouville integrals* $J^\alpha_{\nu+}\mathcal{F}$ *and* $J^\alpha_{\omega-}\mathcal{F}$ *of order* $\alpha>0$ *are defined by*

$$ J^\alpha_{\nu+}\mathcal{F}(\xi) = \frac{1}{\Gamma(\alpha)} \int_\nu^\xi (\xi-x)^{\alpha-1}\mathcal{F}(x)dx, \qquad \xi>\nu $$

and

$$ J^\alpha_{\omega-}\mathcal{F}(\xi) = \frac{1}{\Gamma(\alpha)} \int_\xi^\omega (x-\xi)^{\alpha-1}\mathcal{F}(x)dx, \qquad \xi<\omega, $$

respectively where $\Gamma(\alpha)=\int_0^\infty e^{-\xi}\xi^{\alpha-1}d\xi$. *where* $\Gamma(.)$ *is known as Gamma function. Here* $J^0_{\nu+}\mathcal{F}(\xi)=J^0_{\omega-}\mathcal{F}(\xi)=\mathcal{F}(\xi)$.

2. MAIN RESULTS

We denote $I=[\nu,\omega]$, unless otherwise specified.

Theorem 2.1. *Let* $\mathcal{F}: I \subseteq (0,\infty) \to \mathbb{R}$ *be a function such that* $\mathcal{F}\in L[\nu,\omega]$, *where* $\nu,\omega\in I$ *with* $\nu<\omega$. *If* $\mathcal{F}$ *is an exponentially geometrically convex function on* $[\nu,\omega]$ *with* $\alpha>0$, *then*

$$ e^{\mathcal{F}(\sqrt{\nu\omega})} \le \frac{\Gamma(\alpha+1)}{2(\ln\frac{\omega}{\nu})^\alpha}\left\{J^\alpha_{\nu_+}e^{\mathcal{F}(\omega)} + J^\alpha_{\omega_-}e^{\mathcal{F}(\nu)}\right\} \le \frac{e^{\mathcal{F}(\nu)}+e^{\mathcal{F}(\omega)}}{2}. \tag{2. 3} $$

Proof. Since $\mathcal{F}$ is exponentially geometrically convex function. For $\xi = \frac{1}{2}$ in inequality (1. 2), then

$$e^{\mathcal{F}(\sqrt{xy})} \leq \frac{e^{\mathcal{F}(x)} + e^{\mathcal{F}(y)}}{2}, \quad , x, y \in I.$$

Choosing $x = \nu^{\xi}\omega^{1-\xi}$ and $y = \nu^{1-\xi}\omega^{\xi}$, we get

$$e^{\mathcal{F}(\sqrt{\nu\omega})} \leq \frac{e^{\mathcal{F}(\nu^t\omega^{1-t})} + e^{\mathcal{F}(\nu^{1-t}\omega^t)}}{2}. \tag{2. 4}$$

Multiplying both sides of (2. 4) by $\xi^{\alpha-1}$, then integrating the resulting inequality with respect to ξ over $[0, 1]$, we obtain

$$\begin{aligned} e^{\mathcal{F}(\sqrt{\nu\omega})} &\leq \frac{\alpha}{2}\Big[\int_0^1 \big(e^{\mathcal{F}(\nu^t\omega^{1-t})} + e^{\mathcal{F}(\nu^{1-\xi}\omega^{\xi})}\big)d\xi\Big] \\ &= \frac{\alpha}{2}\Big[\int_0^1 \Big(\frac{\ln\omega - \ln u}{\ln\omega - \ln\nu}\Big)^{\alpha-1}\frac{e^{\mathcal{F}(u)}}{u\ln\frac{\omega}{\nu}}du + \int_0^1 \Big(\frac{\ln u - \ln\nu}{\ln\omega - \ln\nu}\Big)^{\alpha-1}\frac{e^{\mathcal{F}(u)}}{u\ln\frac{\omega}{\nu}}du\Big] \\ &= \frac{\alpha\Gamma(\alpha)}{2\big(\ln\frac{\omega}{\nu}\big)^{\alpha}}\big\{J^{\alpha}_{\nu_+}e^{\mathcal{F}(\omega)} + J^{\alpha}_{\omega_-}e^{\mathcal{F}(\nu)}\big\} \\ &= \frac{\Gamma(\alpha+1)}{2\big(\ln\frac{\omega}{\nu}\big)^{\alpha}}\big\{J^{\alpha}_{\nu_+}e^{\mathcal{F}(\omega)} + J^{\alpha}_{\omega_-}e^{\mathcal{F}(\nu)}\big\}, \end{aligned}$$

which proves the left part of (2. 3).

For the proof of the second inequality in (2. 3), we first note that if $\mathcal{F}$ is exponentially geometrically convex function, then for $t \in [0, 1]$, we have

$$e^{\mathcal{F}(\nu^{\xi}\omega^{1-\xi})} \leq \xi e^{\mathcal{F}(\nu)} + (1-\xi)e^{\mathcal{F}(\omega)}$$

and

$$e^{\mathcal{F}(\nu^{1-\xi}\omega^{\xi})} \leq (1-\xi)e^{\mathcal{F}(\nu)} + \xi e^{\mathcal{F}(\omega)}.$$

By adding these inequalities, we have

$$e^{\mathcal{F}(\nu^{\xi}\omega^{1-\xi})}e^{\mathcal{F}(\nu^{1-\xi}\omega^{\xi})} \leq e^{\mathcal{F}(\nu)} + e^{\mathcal{F}(\omega)}. \tag{2. 5}$$

Then multiplying both sides of (2. 5) by $\xi^{\alpha-1}$, and integrating the resulting inequality with respect to ξ over $[0, 1]$, we obtain

$$\int_0^1 \xi^{\alpha-1}\big(e^{\mathcal{F}(\nu^{\xi}\omega^{1-\xi})} + e^{\mathcal{F}(\nu^{1-\xi}\omega^{\xi})}\big)d\xi \leq [e^{\mathcal{F}(\nu)} + e^{\mathcal{F}(\omega)}]\int_0^1 \xi^{\alpha-1}d\xi.$$

That is

$$\frac{\Gamma(\alpha+1)}{2\big(\ln\frac{\omega}{\nu}\big)^{\alpha}}\big\{J^{\alpha}_{\nu_+}e^{\mathcal{F}(\omega)} + J^{\alpha}_{\omega_-}e^{\mathcal{F}(\nu)}\big\} \leq e^{\mathcal{F}(\nu)} + e^{\mathcal{F}(\omega)}.$$

The proof is completed. □

Before we introduce our main results, we commence with the following lemma.

Lemma 2.2. *Let* $\mathcal{F} : I \subset (0,\infty) \to \mathbb{R}$ *be an exponentially convex function on* I° *such that such that* $\mathcal{F}' \in L[\nu,\omega]$, *where* $\nu,\omega \in I$ *with* $\nu < \omega$. *If* $\mathcal{F}$ *is a geometrically convex function on* $[\nu,\omega]$, $\lambda \in [0,1]$ *and* $\alpha > 0$, *then*

$$\begin{aligned}\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega) \quad &= \nu\left(\ln\frac{x}{\nu}\right)\int_0^1 \left(\xi^\alpha-\lambda\right)\left(\frac{x}{\nu}\right)^\xi e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\mathcal{F}'((x^\xi\nu^{1-\xi}))d\xi\\ &-\omega\left(\ln\frac{x}{\omega}\right)\int_0^1 \left(\xi^\alpha-\lambda\right)\left(\frac{x}{\omega}\right)^\xi e^{\mathcal{F}(x^\xi\omega^{1-\xi})}\mathcal{F}'(x^\xi\omega^{1-\xi})d\xi,\end{aligned} \tag{2. 6}$$

where

$$\begin{aligned}\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega) \quad &= (1-\lambda)\ \ln(\frac{x}{\nu})^\alpha+\ln(\frac{x}{\omega})^\alpha\ e^{\mathcal{F}(x)}+\lambda\ e^{\mathcal{F}(\nu)}\ln(\frac{x}{\nu})^\alpha+e^{\mathcal{F}(\omega)}\ln(\frac{x}{\omega})^\alpha\\ &\Gamma(\alpha+1)[J^\alpha_{\nu_+}e^{\mathcal{F}(\omega)}+J^\alpha_{\omega_-}e^{\mathcal{F}(\nu)}].\end{aligned}$$

Proof. By integration by parts and twice changing the variable, for $x \neq \nu$, we can state that

$$\begin{aligned}&\nu\left(\ln\frac{x}{\nu}\right)\int_0^1 \left(\xi^\alpha-\lambda\right)\left(\frac{x}{\nu}\right)^\xi e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\mathcal{F}'(x^\xi\nu^{1-\xi})d\xi\\ &=\int_0^1 \left(\xi^\alpha-\lambda\right)\left(\frac{x}{\nu}\right)^\xi de^{\mathcal{F}(x^\xi\nu^{1-\xi})}d\xi\\ &=\left(\xi^\alpha-\lambda\right)e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\Big|_0^1-\frac{\alpha}{(\ln\frac{x}{\nu})^\alpha}\int_\nu^x \left(\ln\frac{x}{\nu}\right)^{\alpha-1}\frac{e^{\mathcal{F}(u)}}{u}du\\ &=(1-\lambda)e^{\mathcal{F}(x)}+\lambda e^{\mathcal{F}(\nu)}-\frac{\Gamma(\alpha+1)}{(\ln\frac{x}{\nu})^\alpha}J^\alpha_{x_-}e^{\mathcal{F}(\nu)},\end{aligned} \tag{2. 7}$$

and for $x \neq \omega$, similarly, we get

$$\begin{aligned}
&-\omega\bigg(\ln\frac{\omega}{x}\bigg)\int_0^1\big(\xi^{\alpha}-\lambda\big)\bigg(\frac{x}{\omega}\bigg)^{\xi}e^{\mathcal{F}(x^t\omega^{1-xi})}\mathcal{F}'(x^{\xi}\omega^{1-\xi})d\xi\\
&=\int_0^1\big(\xi^{\alpha}-\lambda\big)\bigg(\frac{x}{\omega}\bigg)^{\xi}de^{\mathcal{F}(x^{\xi}\omega^{1-\xi})}d\xi\\
&=\big(\xi^{\alpha}-\lambda\big)e^{\mathcal{F}(x^{\xi}\omega^{1-\xi})}\big|_0^1-\frac{\alpha}{(\ln\frac{\omega}{x})^{\alpha}}\int_x^{\omega}\big(\ln\frac{x}{\omega}\big)^{\alpha-1}\frac{e^{\mathcal{F}(u)}}{u}du\\
&=(1-\lambda)e^{\mathcal{F}(x)}+\lambda e^{\mathcal{F}(\omega)}-\frac{\Gamma(\alpha+1)}{(\ln\frac{b}{x})^{\alpha}}J^{\alpha}_{x_+}e^{\mathcal{F}(\omega)}. \qquad (2.\ 8)
\end{aligned}$$

Multiplying both sides of (2. 7) and (2. 8) by $\big(\ln\frac{x}{\nu}\big)^{\alpha}$ and $\big(\ln\frac{\omega}{x}\big)^{\alpha}$, respectively, and adding the resulting inequalities, we obtain the desired result.

For $x=\nu$ and $x=\omega$, the identities

$$\Phi_{\mathcal{F}}(\nu,\lambda,\alpha,\nu,\omega)=\omega\bigg(\ln\frac{\omega}{\nu}\bigg)^{\alpha+1}\int_0^1\big(\xi^{\alpha}-\lambda\big)\bigg(\frac{\nu}{\omega}\bigg)^{\xi}e^{\mathcal{F}(\nu^{\xi}\omega^{1-\xi})}\mathcal{F}'(\nu^{\xi}\omega^{1-\xi})d\xi$$

and

$$\Phi_{\mathcal{F}}(\omega,\lambda,\alpha,\nu,\omega)=\nu\bigg(\ln\frac{\omega}{\nu}\bigg)^{\alpha+1}\int_0^1\big(\xi^{\alpha}-\lambda\big)\bigg(\frac{\omega}{\nu}\bigg)^{\xi}e^{\mathcal{F}(\omega^{\xi}\nu^{1-\xi})}\mathcal{F}'(\omega^{\xi}\nu^{1-\xi})d\xi.$$

□

Theorem 2.3. *Let $\mathcal{F}: I\subset(0,\infty)\to\mathbb{R}$ be an exponentially convex function on I° such that such that $\mathcal{F}'\in L[\nu,\omega]$, where $\nu,\omega\in I^{\circ}$ with $\nu<\omega$. If $|\mathcal{F}|^q$ is geometrically convex function on $[a,\omega]$ for some fixed $q\geq 1$, $\lambda\in[0,1]$ and $\alpha>0$, then*

$$\begin{aligned}
&\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega)\ \leq C_1^{1-\frac{1}{q}}(\alpha,\lambda)\ \nu\ \ \ln\frac{x}{\nu}^{\ \alpha+1}\ \ C_2(x,\alpha,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q\\
&+C_3(x,\alpha,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q+C_4(x,\alpha,\lambda,q)\triangle_1(x,\nu)^{\ \frac{1}{q}}+\omega\ \ \ln\frac{\omega}{x}^{\ \alpha+1}\\
&C_5(x,\alpha,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+C_6(x,\alpha,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7(x,\alpha,\lambda,q)\triangle_2(x,\omega)^{\ \frac{1}{q}}\ \ ,
\end{aligned}$$

where

$$\triangle_1(\nu,x)=\left(|e^{\mathcal{F}(x)}\mathcal{F}'(\nu)|^q+|e^{\mathcal{F}(\nu)}\mathcal{F}'(x)|^q\right),\quad \triangle_2(x,\omega)=\left(|e^{\mathcal{F}(x)}\mathcal{F}'(\omega)|^q+|e^{\mathcal{F}(\omega)}\mathcal{F}'(x)|^q\right),$$

$$C_1(\alpha,\lambda)=\int_0^1|\xi^\alpha-\lambda|d\xi,\qquad C_2(x,\alpha,\lambda,q)=\int_0^1\xi^2|\xi^\alpha-\lambda|\left(\frac{x}{\nu}\right)^{q\xi}d\xi,$$

$$C_3(x,\alpha,\lambda,q)=\int_0^1(\xi-1)^2|\xi^\alpha-\lambda|\left(\frac{x}{\nu}\right)^{q\xi}d\xi,\quad C_5(x,\alpha,\lambda,q)=\int_0^1\xi^2|\xi^\alpha-\lambda|\left(\frac{x}{\omega}\right)^{q\xi}d\xi,$$

$$C_4(x,\alpha,\lambda,q)=\int_0^1\xi(\xi-1)|\xi^\alpha-\lambda|\left(\frac{x}{\nu}\right)^{q\xi}d\xi,\quad C_6(x,\alpha,\lambda,q)=\int_0^1(\xi-1)^2|\xi^\alpha-\lambda|\left(\frac{x}{\omega}\right)^{q\xi}d\xi,$$

$$C_7(x,\alpha,\lambda,q)=\int_0^1\xi(\xi-1)|\xi^\alpha-\lambda|\left(\frac{x}{\omega}\right)^{q\xi}d\xi.$$

Proof. Since $|\mathcal{F}'|^q$ is an exponentially geometrically convex on $[\nu,\omega]$, for all $\xi\in[0,1]$. Then we have

$$\begin{aligned}\left|e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\mathcal{F}'(x^\xi\nu^{1-\xi})\right|^q &\le \left(\xi|e^{\mathcal{F}(x)}|^q+(1-\xi)|e^{\mathcal{F}(\nu)}|^q\right)\left(\xi|\mathcal{F}'(x)|^q+(1-\xi)|\mathcal{F}'(\nu)|^q\right)\\ &=\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q\\ &\quad+\xi(1-\xi)\left(|e^{\mathcal{F}(x)}\mathcal{F}'(\nu)|^q+|e^{\mathcal{F}(\nu)}\mathcal{F}'(x)|^q\right)\\ &=\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q+\xi(1-\xi)\triangle_1(\nu,x)\end{aligned}$$

and

$$\begin{aligned}\left|e^{\mathcal{F}(x^\xi\omega^{1-\xi})}\mathcal{F}'(x^\xi\omega^{1-\xi})\right|^q &\le \left(\xi|e^{\mathcal{F}(x)}|^q+(1-\xi)|e^{\mathcal{F}(\omega)}|^q\right)\left(\xi|\mathcal{F}'(x)|^q+(1-\xi)|\mathcal{F}'(\omega)|^q\right)\\ &=\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+\xi(1-\xi)\triangle_2(x,\omega).\end{aligned}$$

Hence, using Lemma 2.1 and the power mean inequality, we get

$$\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega)$$
$$\leq \nu \left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(\int_0^1 \left|\xi^\alpha-\lambda\right| d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\xi^\alpha-\lambda\right|\left(\frac{x}{\nu}\right)^{q\xi}\left|e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\mathcal{F}'((x^\xi\nu^{1-\xi}))\right|^q d\xi\right)^{\frac{1}{q}}$$
$$+\omega \left(\ln\frac{x}{\omega}\right)^{\alpha+1}\left(\int_0^1 \left|\xi^\alpha-\lambda\right| d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\xi^\alpha-\lambda\right|\left(\frac{x}{\omega}\right)^{q\xi}\left|e^{\mathcal{F}(x^\xi\omega^{1-\xi})}\mathcal{F}'(x^\xi\omega^{1-\xi})\right|^q d\xi\right)$$
$$\leq \left(\int_0^1|\xi^\alpha-\lambda|d\xi\right)^{1-\frac{1}{q}}\times\left(\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(\int_0^1|t^\alpha-\lambda|\left(\frac{x}{\nu}\right)^{q\xi}\left(\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2\right.\right.\right.$$
$$\left.\left.|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q+\xi(1-\xi)\triangle_1(x,\nu)\right)d\xi\right)^{\frac{1}{q}}+\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(\int_0^1|\xi^\alpha-\lambda|\left(\frac{x}{\nu}\right)^{q\xi}\right.$$
$$\left.\left(\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+\xi(1-\xi)\triangle_1(x,\omega)\right)\right)^{\frac{1}{q}}$$
$$\leq C_1^{1-\frac{1}{q}}(\alpha,\lambda)\left(\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(C_2^{\frac{1}{q}}(x,\alpha,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+C_3^{\frac{1}{q}}(x,\alpha,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q\right.\right.$$
$$\left.+C_4^{\frac{1}{q}}(x,\alpha,\lambda,q)\triangle_1(x,\nu,)\right)^{\frac{1}{q}}+\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(C_5^{\frac{1}{q}}(x,\alpha,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q\right.$$
$$\left.\left.+C_6^{\frac{1}{q}}(x,\alpha,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(x,\alpha,\lambda,q)\triangle_2(x,\omega)\right)^{\frac{1}{q}}\right),$$

which completes the proof. □

Corollary 2.4. *Under the assumptions of Theorem 2.3 with $q=1$, inequality (2. 9) reduces to the following inequality:*

$$\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega)\;\leq\;\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(C_2(x,\alpha,\lambda,1)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|+C_3(x,\alpha,\lambda,1)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|\right.$$
$$\left.+C_4(x,\alpha,\lambda,1)\triangle_1(x,\nu)\right)+\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(C_5(x,\alpha,\lambda,1)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|\right.$$
$$\left.+C_6(x,\alpha,\lambda,1)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|+C_7(x,\alpha,\lambda,1)\triangle_2(x,\omega)\right).$$

Corollary 2.5. *Let under the assumptions of Theorem 2.3 hold. If $|e^{\mathcal{F}(x)}\mathcal{F}^{'}(x)|\leq M$ for all $x\in[\nu,\omega]$ and $\lambda=0$, then we get the following Ostrowski-type inequality for fractional* *integrals from inequality* (2. 8)*:*

$$\left|\left[\left(\ln\frac{x}{\nu}\right)^{\alpha}+\left(\ln\frac{x}{\omega}\right)^{\alpha}\right]e^{\mathcal{F}(x)}-\Gamma(\alpha+1)[J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)}+J^{\alpha}_{\sqrt{\nu\omega}_+}e^{\mathcal{F}(\omega)}]\right|$$
$$\leq\frac{1}{(\alpha+1)^{1-\frac{1}{q}}}\left[\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left\{C_2^{\frac{1}{q}}(x,1,\lambda,q)M^q+C_3^{\frac{1}{q}}(x,1,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q\right.\right.$$
$$\left.+C_4^{\frac{1}{q}}(x,1,\lambda,q)\triangle_1(x,\nu)\right\}^{\frac{1}{q}}+\omega\left(\ln\frac{b}{x}\right)^{\alpha+1}\left\{C_5^{\frac{1}{q}}(x,1,\lambda,q)M^q\right.$$
$$\left.\left.+C_6^{\frac{1}{q}}(x,1,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(x,1,\lambda,q)\triangle_2(x,\omega)\right\}^{\frac{1}{q}}\right].$$

Corollary 2.6. *Under the assumptions of Theorem 2.3 with $\alpha=1$, inequality* (2. 9) *reduces to the following inequality:*

$$ln\frac{\omega}{\nu}^{\;-1}\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega)$$
$$\leq\ (1-\lambda)e^{\mathcal{F}(x)}+\lambda\ \frac{e^{\mathcal{F}(\nu)}\ln\frac{x}{\nu}+e^{\mathcal{F}(\omega)}\ln\frac{\omega}{x}}{\ln\frac{\omega}{\nu}}-\frac{1}{\ln\frac{\omega}{\nu}}\int_{\nu}^{\omega}\frac{e^{\mathcal{F}(u)}}{u}du$$
$$\leq\ \ ln\frac{\omega}{\nu}^{\;-1}\ \frac{2\lambda^2-2\lambda+1}{2}^{\;1-\frac{1}{q}}\ \nu\ \ \ln\frac{x}{\nu}^{\;2}\ C_2^{\frac{1}{q}}(x,1,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q$$
$$+C_3^{\frac{1}{q}}(x,1,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q+C_4^{\frac{1}{q}}(x,1,\lambda,q)\triangle_1(x,\nu)^{\;\frac{1}{q}}+\omega\ \ \ln\frac{\omega}{x}^{\;2}\ C_5^{\frac{1}{q}}(x,1,\lambda,q)$$
$$|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+C_6^{\frac{1}{q}}(x,1,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(x,1,\lambda,q)\triangle_2(x,\omega)^{\;\frac{1}{q}}\ ,$$

especially for $x=\sqrt{\nu\omega}$, we get

$$(1-\lambda)e^{\mathcal{F}(\sqrt{\nu\omega})}+\lambda\ \frac{e^{\mathcal{F}(\nu)}+e^{\mathcal{F}(\omega)}}{2}\ -\frac{1}{\ln\frac{\omega}{\nu}}\int_{\nu}^{\omega}\frac{e^{\mathcal{F}(u)}}{u}du\ \leq\ ln\frac{\omega}{\nu}^{\;-1}\ \frac{2\lambda^2-2\lambda+1}{2}^{\;1-\frac{1}{q}}$$
$$\nu\ \ \ln\sqrt{\frac{\omega}{\nu}}^{\;\alpha+1}\ C_2^{\frac{1}{q}}(\sqrt{\omega\nu},1,\lambda,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q+C_3^{\frac{1}{q}}(\sqrt{\nu\omega},1,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q$$
$$+C_4^{\frac{1}{q}}(\sqrt{\nu\omega},1,\lambda,q)\triangle_1(\nu\omega,\nu)^{\;\frac{1}{q}}+\omega\ \ \ln\sqrt{\frac{\omega}{\nu}}^{\;\alpha+1}\ C_5^{\frac{1}{q}}(\sqrt{\nu\omega},1,\lambda,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$
$$+C_6^{\frac{1}{q}}(\sqrt{\nu\omega},1,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(\sqrt{ab},1,\lambda,q)\triangle_2(\nu\omega,\omega)^{\;\frac{1}{q}}\ .$$

Corollary 2.7. *In Theorem 2.3:*
I. *If we take $x=\sqrt{\nu\omega},\ \lambda=0$, then we get the following mid point-type inequality for fractional integrals:*

$$e^{\mathcal{F}(\sqrt{\nu\omega})}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{\omega}{\nu})^{\alpha}}[J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)}+J^{\alpha}_{\sqrt{\nu\omega}_+}e^{\mathcal{F}(\nu)}]\ \leq\ \ln\frac{\omega}{\nu}^{-1}\ \frac{1}{\alpha+1}^{1-\frac{1}{q}}$$
$$\nu\ \ln\sqrt{\frac{\omega}{\nu}}^{\alpha+1}\ C_2^{\frac{1}{q}}(\sqrt{\omega\nu},\alpha,0,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q+C_3^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,0,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q$$
$$+C_4^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,0,q)\triangle_1(\nu\omega,\nu)^{\frac{1}{q}}+\omega\ \ln\sqrt{\frac{\omega}{\nu}}^{\alpha+1}\ C_5^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,0,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$
$$+C_6^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,0,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,0,q)\triangle_2(\nu\omega,\omega)^{\frac{1}{q}}\ .$$

The above inequality with $\alpha=1$ yields,

$$e^{\mathcal{F}(\sqrt{\nu\omega})}-\frac{1}{\ln\frac{\omega}{\nu}}\int_{\nu}^{\omega}\frac{e^{\mathcal{F}(u)}}{u}du]\ \leq\ \frac{\ln\frac{\omega}{\nu})^{-1}}{2^{1-\frac{1}{q}}}$$
$$\nu\ \ln\sqrt{\frac{\omega}{\nu}}^{2}\ C_2^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,0,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q+C_3^{\frac{1}{q}}(\sqrt{\nu\omega},1,0,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q$$
$$+C_4^{\frac{1}{q}}(\sqrt{\nu\omega},1,0,q)\triangle_1(\nu,\nu\omega)^{\frac{1}{q}}+\omega\ \ln\sqrt{\frac{\omega}{\nu}}^{2}\ C_5^{\frac{1}{q}}(\sqrt{\nu\omega},1,0,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$
$$+C_6^{\frac{1}{q}}(\sqrt{\nu\omega},1,0,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(\sqrt{\nu\omega},1,0,q)\triangle_2(\nu\omega,\omega)^{\frac{1}{q}}\ .$$

II. *If we take $x=\sqrt{\nu\omega},\ \lambda=1$, then we get the following Trapezoid-type inequality for fractional integrals:*

$$\frac{e^{\mathcal{F}(\nu)}+e^{\mathcal{F}(\omega)}}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{\omega}{\nu})^{\alpha}}[J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)}+J^{\alpha}_{\sqrt{\nu\omega}_+}e^{\mathcal{F}(\nu)}]\ \leq\ \ln\frac{\omega}{\nu}^{-1}\ \frac{\alpha}{\alpha+1}^{1-\frac{1}{q}}$$
$$\nu\ \ln\sqrt{\frac{\omega}{\nu}}^{\alpha+1}\ C_2^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,1,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q+C_3^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,1,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q$$
$$+C_4^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,1,q)\triangle_1(\nu\omega,\nu)^{\frac{1}{q}}+\omega\ \ln\sqrt{\frac{\omega}{\nu}}^{\alpha+1}\ C_5^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,1,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$
$$+C_6^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,1,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+C_7^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,1,q)\triangle_2(\nu\omega,\omega)^{\frac{1}{q}}\ .$$

III. *If we take* $x = \sqrt{\nu\omega},\ \lambda = \frac{1}{2}$, *then we get the following Trapezoidal formula for two points-type inequality for fractional integrals:*

$$\frac{1}{4}[e^{\mathcal{F}(\nu)} + 2e^{\mathcal{F}(\sqrt{\nu\omega})} + e^{\mathcal{F}(\omega)}] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{\omega}{\nu})^{\alpha}}[J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)} + J^{\alpha}_{\sqrt{\nu\omega}_+}e^{\mathcal{F}(\nu)}]$$

$$\leq (ln\frac{\omega}{\nu})^{-1} \quad \frac{2\alpha + 2^{\frac{1}{\alpha}}(1-\alpha)}{(\alpha+1)^{1+\frac{1}{\alpha}}}^{\ 1-\frac{1}{q}} \quad \nu \quad \ln\sqrt{\frac{\omega}{\nu}}^{\ \alpha+1} \quad C_2^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{2},q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$

$$+ C_3^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{2},q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q + C_4^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{2},q)\triangle_1(\nu,\nu\omega)^{\ \frac{1}{q}} + \omega \quad \ln\sqrt{\frac{\omega}{\nu}}^{\ \alpha+1}$$

$$C_5^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{2},q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q + C_6^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{2},q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q$$

$$+ C_7^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{2},q)\triangle_2(\nu\omega,\omega)^{\ \frac{1}{q}} \quad .$$

IV. *If we take* $x = \sqrt{\nu\omega},\ \lambda = \frac{1}{3}$, *then we get the following Simpson-type inequality for fractional integrals:*

$$\frac{1}{6}[e^{\mathcal{F}(\nu)} + 4e^{\mathcal{F}(\sqrt{\nu\omega})} + e^{\mathcal{F}(\omega)}] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{\omega}{\nu})^{\alpha}}[J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)} + J^{\alpha}_{\sqrt{\nu\omega}_+}e^{\mathcal{F}(\nu)}]$$

$$\leq (ln\frac{\omega}{\nu})^{-1} \quad \frac{2\alpha + (2-\alpha)3^{\frac{1}{\alpha}}}{(\alpha+1)3^{1+\frac{1}{\alpha}}}^{\ 1-\frac{1}{q}} \quad \nu \quad \ln\sqrt{\frac{\omega}{\nu}}^{\ \alpha+1} \quad C_2^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{3},q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$

$$+ C_3^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{3},q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q + C_4^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{3},q)\triangle_1(\nu,\nu\omega)^{\ \frac{1}{q}} + \omega \quad \ln\sqrt{\frac{\omega}{\nu}}^{\ \alpha+1} \quad C_5^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{3},q)$$

$$|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q + C_6^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{3},q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q + C_7^{\frac{1}{q}}(\sqrt{\nu\omega},\alpha,\frac{1}{3},q)\triangle_2(\omega,\nu\omega)^{\ \frac{1}{q}} \quad .$$

Theorem 2.8. *Let* $\mathcal{F} : I \subset (0,\infty) \to \mathbb{R}$ *be an exponentially convex function on* I° *such that* $\mathcal{F}' \in L[\nu,\omega]$, *where* $\nu,\omega \in I^{\circ}$ *with* $\nu < \omega$. *If* $|\mathcal{F}|^q$ *is geometrically convex function on* $[\nu,\omega]$ *for some fixed* $q > 1$, $\lambda \in [0,1]$, $\alpha > 0$ *and* $p^{-1} + q^{-1} = 1$, *then*

$$\Phi_{\mathcal{F}}(x,\lambda,\alpha,\nu,\omega)$$

$$\leq \quad H_1(\alpha,\lambda,p)^{\ \frac{1}{p}} \times \quad \nu \quad \ln\frac{x}{\nu}^{\ \alpha+1} \quad H_2(x,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q + H_3(x,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q$$

$$+H_4(x,\lambda,q)\triangle_1(x,\nu)]^{\ \frac{1}{q}} + \quad \omega \quad \ln\frac{\omega}{x}^{\ \alpha+1} \quad H_5(x,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q$$

$$+H_6(x,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q + H_7(x,\lambda,q)\triangle_2(x,\omega)]^{\ \frac{1}{q}} \quad , \tag{2. 9}$$

where

$$\begin{aligned}
\triangle_1(x,\nu) &= |e^{\mathcal{F}(x)}\mathcal{F}'(\nu)|^q + |e^{\mathcal{F}(\nu)}\mathcal{F}'(x)|^q \ , \\
\triangle_2(x,\omega) &= |e^{\mathcal{F}(x)}\mathcal{F}'(\omega)|^q + |e^{\mathcal{F}(\omega)}\mathcal{F}'(x)|^q \ , \\
H_1(\alpha,\lambda,p) &= \int_0^1 |\xi^{\alpha} - \lambda|^p d\xi
\end{aligned}$$

$$= \begin{cases} \frac{1}{\alpha+p}, & \lambda = 0, \\ \frac{\lambda^{\frac{p\alpha+1}{\alpha}}}{\alpha}(\beta(\frac{1}{\alpha}, p+1)) + \frac{(1-\lambda)^{p+1}}{p+1} \times_2 F_1(\frac{1}{\alpha}+p+1, p+1, p+2; 1-\lambda), & 0 < \lambda < 1, \\ \frac{1}{\alpha}\beta(p+1, \frac{1}{\alpha}), & \lambda = 1 \end{cases}$$

$$\begin{aligned} H_2(x,\lambda,q) &= \int_0^1 \xi^2 \left(\frac{x}{\nu}\right)^{q\xi} d\xi = \frac{\frac{x}{\nu}\{(q\ln\frac{x}{\nu})^2+2\} - 2(q\ln\frac{x}{\nu}) + 1}{(q\ln\frac{x}{\nu})^3}, \\ H_3(x,\lambda,q) &= \int_0^1 (\xi-1)^2 \left(\frac{x}{\nu}\right)^{q\xi} d\xi = \frac{2\{(\frac{x}{\nu})^q - q\ln\frac{x}{\nu}\} - (q\ln\frac{x}{\nu})^2}{(q\ln\frac{x}{\nu})^3}, \\ H_4(x,\lambda,q) &= \int_0^1 \xi(\xi-1) \left(\frac{x}{\nu}\right)^{q\xi} d\xi = \frac{2(\frac{x}{\nu})^q\{q\ln\frac{x}{\nu} - 1\} + 2}{(q\ln\frac{x}{\nu})^3}, \\ H_5(x,\lambda,q) &= \int_0^1 \xi^2 \left(\frac{x}{\omega}\right)^{q\xi} d\xi = \frac{\frac{x}{\omega}\{(q\ln\frac{x}{\omega})^2+2\} - 2(q\ln\frac{x}{\omega}) + 1}{(q\ln\frac{x}{\omega})^3}, \\ H_6(x,\lambda,q) &= \int_0^1 (\xi-1)^2 \left(\frac{x}{\omega}\right)^{q\xi} d\xi = \frac{2\{(\frac{x}{\omega})^q - q\ln\frac{x}{\omega}\} - (q\ln\frac{x}{\omega})^2}{(q\ln\frac{x}{\omega})^3}, \\ H_7(x,\lambda,q) &= \int_0^1 \xi(\xi-1) \left(\frac{x}{\omega}\right)^{q\xi} d\xi = \frac{2(\frac{x}{\omega})^q\{q\ln\frac{x}{\omega} - 1\} + 2}{(q\ln\frac{x}{\omega})^3}. \end{aligned}$$

Proof. Since $|\mathcal{F}'|^q$ is exponentially geometrically convex function on $[\nu,\omega]$, for all $\xi \in [0,1]$. Then we have

$$\begin{aligned} \left|e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\mathcal{F}'(x^\xi\nu^{1-\xi})\right|^q &\le \left(\xi|e^{\mathcal{F}(x)}|^q + (1-\xi)|e^{\mathcal{F}(\nu)}|^q\right)\left(\xi|\mathcal{F}'(x)|^q + (1-\xi)|\mathcal{F}'(\nu)|^q\right) \\ &= \xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q + (1-\xi)^2|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q \\ &\quad + \xi(1-\xi)\left(|e^{\mathcal{F}(x)}\mathcal{F}'(\nu)|^q + |e^{\mathcal{F}(\nu)}\mathcal{F}'(x)|^q\right) \\ &= \xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q + (1-\xi)^2|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q + t(1-t)\triangle_1(\nu,\omega) \end{aligned}$$

and

$$\begin{aligned} \left|e^{\mathcal{F}(x^\xi\omega^{1-\xi})}\mathcal{F}'(x^\xi\omega^{1-\xi})\right|^q &\le \left(\xi|e^{\mathcal{F}(x)}|^q + (1-\xi)|e^{\mathcal{F}(\omega)}|^q\right)\left(\xi|\mathcal{F}'(x)|^q + (1-\xi)|\mathcal{F}'(\omega)|^q\right) \\ &= \xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q + (1-\xi)^2|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q + \xi(1-\xi)\triangle_2(\nu,\omega). \end{aligned}$$

Hence, using Lemma 2.1, the Hölder inequality and geometrically convexity of $|\mathcal{F}'|^q$, we get

$$
\begin{aligned}
&|\Phi(x,\lambda,\alpha,\nu,\omega)| \\
&\leq \nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(\int_0^1|\xi^\alpha-\lambda|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{x}{\nu}\right)^{q\xi}\left|e^{\mathcal{F}(x^\xi\nu^{1-\xi})}\mathcal{F}'(x^\xi\nu^{1-\xi})\right|d\xi\right)^{\frac{1}{q}} \\
&\quad+\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(\int_0^1|\xi^\alpha-\lambda|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{x}{\omega}\right)^{q\xi}\left|e^{\mathcal{F}(x^\xi\omega^{1-\xi})}\mathcal{F}'(x^\xi\omega^{1-\xi})\right|d\xi\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1|\xi^\alpha-\lambda|^p d\xi\right)^{\frac{1}{p}} \\
&\times\left[\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(\int_0^1\left(\frac{x}{\nu}\right)^{q\xi}[\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q+\xi(1-\xi)\triangle_1(\nu,\omega)]d\xi\right)^{\frac{1}{q}}\right. \\
&+\left.\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(\int_0^1\left(\frac{x}{\omega}\right)^{q\xi}[\xi^2|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+(1-\xi)^2|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+\xi(1-\xi)\triangle_2(\nu,\omega)]d\xi\right)^{\frac{1}{q}}\right] \\
&\leq \left(H_1(\alpha,\lambda,p)\right)^{\frac{1}{p}} \\
&\times\left[\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(H_2(x,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+H_3(x,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q+H_4(x,\lambda,q)\triangle_1(\nu,\omega)\right)^{\frac{1}{q}}\right. \\
&+\left.\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(H_5(x,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^q+H_6(x,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+H_7(x,\lambda,q)\triangle_2(\nu,\omega)\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

which is the required result. □

Corollary 2.9. *Let under the assumptions of Theorem 2.8 hold. If* $|e^{\mathcal{F}(x)}\mathcal{F}'(x)|\leq M$ *for all* $x\in[\nu,\omega]$ *and* $\lambda=0$*, then we get the following Ostrowski-type inequality for fractional integrals from inequality* (2. 9)*:*

$$
\begin{aligned}
&\left|\left[\left(\ln\frac{x}{\nu}\right)^{\alpha}+\left(\ln\frac{x}{\omega}\right)^{\alpha}\right]e^{\mathcal{F}(x)}-\Gamma(\alpha+1)[J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)}+J^{\alpha}_{\sqrt{\nu\omega}_+}e^{\mathcal{F}(\omega)}]\right| \\
&\leq\frac{1}{(p\alpha+1)^{\frac{1}{p}}}\left[\nu\left(\ln\frac{x}{\nu}\right)^{\alpha+1}\left(H_2^{\frac{1}{q}}(x,0,q)M^q+H_3^{\frac{1}{q}}(x,0,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q\right.\right. \\
&\left.+H_4^{\frac{1}{q}}(x,0,q)\triangle_1(x,\nu)\right)^{\frac{1}{q}}+\omega\left(\ln\frac{\omega}{x}\right)^{\alpha+1}\left(H_5^{\frac{1}{q}}(x,0,q)M^q\right. \\
&\left.\left.+H_6^{\frac{1}{q}}(x,0,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q+H_7^{\frac{1}{q}}(x,0,q)\triangle_2(x,\omega)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 2.10. *Under the assumptions of Theorem 2.8 with* $\alpha=1$*, inequality* (2. 9) *reduces to the following inequality:*

$$\left|(1-\lambda)e^{\mathcal{F}(x)}+\lambda\left[\frac{e^{\mathcal{F}(\nu)}\ln\frac{x}{\nu}+e^{\mathcal{F}(\omega)}\ln\frac{x}{\omega}}{\ln\frac{\omega}{\nu}}-\frac{1}{\ln\frac{\omega}{\nu}}\int\limits_{\nu}^{\omega}\frac{e^{\mathcal{F}(u)}}{u}du\right]\right|$$
$$\leq\left(\ln\frac{\omega}{\nu}\right)^{-1}\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\times\left[\left\{\nu\left(\ln\frac{x}{\nu}\right)^{2}\left(H_2(x,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^{q}\right.\right.\right.$$
$$\left.\left.+H_3(x,\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^{q}+H_4(x,\lambda,q)\triangle_1(x,\nu)]\right)\right\}^{\frac{1}{q}}+\left\{b\left(\ln\frac{b}{x}\right)^{2}\right.$$
$$\left.\left.\left(H_5(x,\lambda,q)|e^{\mathcal{F}(x)}\mathcal{F}'(x)|^{q}+H_6(x,\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^{q}+H_7(x,\lambda,q)\triangle_2(x,\omega)]\right)\right\}^{\frac{1}{q}}\right],$$

specially for $x=\sqrt{\nu\omega}$, we get

$$\left|(1-\lambda)e^{\mathcal{F}(\sqrt{\nu\omega})}+\lambda\left[\frac{e^{\mathcal{F}(\nu)}\ln\frac{x}{\nu}+e^{\mathcal{F}(\omega)}\ln\frac{x}{\omega}}{\ln\frac{\omega}{\nu}}-\frac{1}{\ln\frac{\omega}{\nu}}\int\limits_{\nu}^{\omega}\frac{e^{\mathcal{F}(u)}}{u}du\right]\right|\leq\left(\ln\left(\frac{\omega}{\nu}\right)\right)^{-1}$$
$$\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left[\left\{\nu\left(\ln\sqrt{\frac{\omega}{\nu}}\right)^{2}\left(H_2(\sqrt{\nu\omega},\lambda,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^{q}\right.\right.\right.$$
$$\left.\left.+H_3(\sqrt{\nu\omega},\lambda,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^{q}+H_4(\sqrt{\nu\omega},\lambda,q)\triangle_1(\nu\omega,\nu)]\right)\right\}^{\frac{1}{q}}+\left\{\omega\left(\ln\sqrt{\frac{\omega}{\nu}}\right)^{2}\right.$$
$$\left.\left.\left(H_5(\sqrt{\nu\omega},\lambda,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^{q}+H_6(\sqrt{\nu\omega},\lambda,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^{q}+H_7(\sqrt{\nu\omega},\lambda,q)\triangle_2(\omega,\nu\omega)]\right)\right\}^{\frac{1}{q}}\right].$$

Corollary 2.11. *In Theorem 2.8:*

I. *If we take $x=\sqrt{\nu\omega}$, $\lambda=0$, then we get the following midpoint-type inequality for fractional integrals:*

$$e^{\mathcal{F}(\sqrt{\nu\omega})}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\ln(\frac{\omega}{\nu})^{\alpha}}\ J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)}+J^{\alpha}_{\sqrt{\nu\omega}_-}e^{\mathcal{F}(\nu)}\quad\leq\quad\ln\ \frac{\omega}{\nu}\quad{}^{-1}\quad\frac{1}{\alpha(p+1)}\quad{}^{\frac{1}{p}}$$
$$\nu\quad\ln\sqrt{\frac{\omega}{\nu}}\quad{}^{\alpha+1}\quad H_2(\sqrt{\nu\omega},0,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^{q}+H_3(\sqrt{\nu\omega},0,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^{q}$$
$$+H_4(\sqrt{\nu\omega},0,q)\triangle_1(\nu,\nu\omega)]\quad{}^{\frac{1}{q}}\quad+\quad\omega\quad\ln\sqrt{\frac{\omega}{\nu}}\quad{}^{\alpha+1}\quad H_5(\sqrt{\nu\omega},0,q)$$
$$|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^{q}+H_6(\sqrt{\nu\omega},0,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^{q}+H_7(\sqrt{\nu\omega},0,q)\triangle_2(\omega,\nu\omega)]\quad{}^{\frac{1}{q}}\quad.$$

II. *If we take* $x = \sqrt{\nu\omega},\ \lambda = 1$, *then we get the following Trapezoidal-type inequality for fractional integrals:*

$$\frac{e^{\mathcal{F}(\nu)} + e^{\mathcal{F}(\omega)}}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\ln(\frac{\omega}{\nu})^{\alpha}} \ J^{\alpha}_{\sqrt{\nu\omega}\_} e^{\mathcal{F}(\nu)} + J^{\alpha}_{\sqrt{\nu\omega}\_} e^{\mathcal{F}(\nu)} \quad \leq \quad \ln\frac{\omega}{\nu} \quad ^{-1} \quad \frac{\beta(p+1,\frac{1}{\alpha})}{\alpha} \quad ^{\frac{1}{p}}$$

$$\nu \quad \ln\sqrt{\frac{\omega}{\nu}} \quad ^{\alpha+1} \quad H_2(\sqrt{\nu\omega},1,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q + H_3(\sqrt{\nu\omega},1,q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q$$

$$+ H_4(\sqrt{\nu\omega},1,q)\triangle_1(\nu,\nu\omega)] \quad ^{\frac{1}{q}} + \quad \omega \quad \ln\sqrt{\frac{\omega}{\nu}} \quad ^{\alpha+1} \quad H_5(\sqrt{\nu\omega},1,q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$

$$+ H_6(\sqrt{\nu\omega},1,q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q + H_7(\sqrt{\nu\omega},1,q)\triangle_2(\omega,\nu\omega)] \quad ^{\frac{1}{q}} \quad .$$

III.*If we take* $x = \sqrt{\nu\omega},\ \lambda = \frac{1}{3}$, *then we get the following Simpson's-type inequality for fractional integrals:*

$$\frac{1}{6} \ e^{\mathcal{F}(\nu)} + 4e^{\mathcal{F}(\sqrt{\nu\omega})} + e^{\mathcal{F}(\omega)} \quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\ln(\frac{\omega}{\nu})^{\alpha}} \ J^{\alpha}_{\sqrt{\nu\omega}\_} e^{\mathcal{F}(\nu)} + J^{\alpha}_{\sqrt{\nu\omega}\_} e^{\mathcal{F}(\nu)} \quad \leq \quad \ln\frac{\omega}{\nu} \quad ^{-1}$$

$$\frac{(1+2^{p+1})}{3^{p+1}(p+1)} \quad ^{\frac{1}{p}} \quad \nu \quad \ln\sqrt{\frac{\omega}{\nu}} \quad ^{\alpha+1} \quad H_2(\sqrt{\nu\omega},\frac{1}{3},q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q$$

$$+ H_3(\sqrt{\nu\omega},\frac{1}{3},q)|e^{\mathcal{F}(\nu)}\mathcal{F}'(\nu)|^q + H_4(\sqrt{\nu\omega},\frac{1}{3},q)\triangle_1(\nu,\nu\omega)] \quad ^{\frac{1}{q}} + \quad \omega \quad \ln\sqrt{\frac{\omega}{\nu}} \quad ^{\alpha+1}$$

$$H_5(\sqrt{\nu\omega},\frac{1}{3},q)|e^{\mathcal{F}(\sqrt{\nu\omega})}\mathcal{F}'(\sqrt{\nu\omega})|^q + H_6(\sqrt{\nu\omega},\frac{1}{3},q)|e^{\mathcal{F}(\omega)}\mathcal{F}'(\omega)|^q + H_7(\sqrt{\nu\omega},\frac{1}{3},q)\triangle_2(\omega,\nu\omega)] \quad ^{\frac{1}{q}} \quad .$$

3. CONCLUSION

We have introduced and investigated a new class of convex functions that is the exponentially geometrically convex functions. Some basic inequalities related in fractional integral forms are established. Moreover, a new identity for Hadamard fractional integrals have been defined. By using this identity, we have obtained a generalizations of Hadamard, Ostrowski, Trapezoidal and Simpson type inequalities for exponentially geometrically convex functions via Riemann-Liouville fractional integrals.

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