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Some Smarandache Curves Constructed from a Spacelike Salkowski Curve with Timelike Principal Normal

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Abstract. In this article, we investigate the regular Smarandache curves constructed from the Frenet vectors of spacelike Salkowski curve with a timelike principal normal. In the first part of the study, literature research was conducted. In the second part, general information about the curve and spacelike Salkowski curve in Minkowski space are given. In the last part, the Frenet apparatus of the Smarandache curves are calculated. We draw a graphic of the obtained Smarandache curves and some related results about Smarandache curves are given.

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1. Introduction

Curves are one of the most important research topics in differential geometry. Research in recent years has shown that the issue of curves has an important place in other disciplines. Curves have become indispensable for research topics, especially in areas such as engineering and science. For example, as in studies [6, 8], curve theory was used in subjects such as waves in physics and cell modeling in biology. In this study, the Smarandache curve, which is a special curve in curve theory, has been studied. In 1909, the Salkowski curve is defined by E. Salkowski as a family of curves whose curvature is constant but the torsion is not constant [11]. In literature, these curves are known as Salkowski curves. The equation of the Salkowski curve was given by J. Monterde [9]. In Minkowski space, M. Turgut and S. Ylmaz described the Smarandache curves [14]. Later, according to the Darboux frame, Bishop frame and Sabban frame, some features of the Smarandache curves were investigated by [3, 4, 15, 5]. The definition of timelike and spacelike Salkowski curve is given by

A. T. Ali [1, 2]. S. enyurt and K. Eren also studied the Smarandache curves obtained from the Frenet vectors of the spacelike Salkowski and anti-Salkowski curve [12, 13].

In this study, Smarandache curves are defined by a unit vector that is obtained from the linear combination of Frenet vectors of spacelike Salkowski curve. The Frenet apparatus of each curve are calculated and the graph of Smarandache curves is given.

2. Preliminaries

Lorentzian inner product in the Minkowski space R_1^3 is defined by

$$\langle , \rangle_L = du_1^2 + du_2^2 - du_3^2$$

where $u = (u_1, u_2, u_3) \in R_1^3$. The vector product of $u = (u_1, u_2, u_3)$ and v $= (v_1, v_2, v_3)$ in R_1^3 is given by

$$u \wedge v = - \begin{vmatrix} i & j & -k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

For $u \in R_1^3$, if $\langle u, u \rangle_L > 0$ or $u = 0$, then u is spacelike vector, if $\langle u, u \rangle_L < 0$, then u is timelike vector, if $\langle u, u \rangle_L = 0$, $u \neq 0$, then u is lightlike (or null) vector. The norm of $u \in R_1^3$ is $\|u\| = \sqrt{|\langle u, u \rangle_L|}$. In [10], the Frenet vectors and the curvatures of the non-unit speed spacelike curve with a timelike principal normal γ are

$$\begin{aligned} &T(y) = \tfrac{\gamma'(y)}{\|\gamma'(y)\|}, \ \ B(t) = \tfrac{\gamma'(y)\wedge\gamma''(y)}{\|\gamma'(y)\wedge\gamma''(y)\|}, \ \ N(y) = B(y) \wedge T(y), \\ &\kappa(y) = \frac{\|\gamma'(y)\wedge\gamma''(y)\|}{\|\gamma'(y)\|^3}, \ \ \tau(y) = \frac{\langle \gamma'(y)\wedge\gamma''(y), \gamma'''(y) \rangle}{\|\gamma'(y)\wedge\gamma''(y)\|^2}, \end{aligned} \tag{2. 1}$$

$$\begin{aligned} T' &= \kappa N, \\ N' &= \kappa T + \tau B, \\ B' &= \tau B. \end{aligned} \tag{2. 2}$$

Definition 2.1. Let γ and $\tilde{\gamma}$ be any two curves with Frenet frame $\{T, N, B\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively. If $\left\langle T, \tilde{T} \right\rangle = 0$, the curve $\tilde{\gamma}$ is called involute of the curve γ and the curve γ is called evolute of the curve $\tilde{\gamma}$ [7].

Definition 2.2. For $m > 1$ or $m < -1$ and an arbitrary $m \in R$, let's define the space curve as follows

$$\gamma_m(y) = \frac{n}{4m} \left(\begin{array}{l} 2\sin(y) - \sin\left((1-2n)y\right) \frac{1+n}{1-2n} - \sin\left((1+2n)y\right) \frac{1-n}{1+2n}, \\ 2\cos(y) - \cos\left((1-2n)y\right) \frac{1+n}{1-2n} - \cos\left((1+2n)y\right) \frac{1-n}{1+2n}, \\ \cos\left(2ny\right) \frac{1}{m} \end{array} \right), \tag{2. 3}$$

where $n = \frac{m}{\sqrt{m^2-1}}$. This curve is called the spacelike Salkowski curve. (Figure 1). The first curvature and the second curvature of Salkowski curve are $\kappa(y) = 1$ and $\tau(y) = \cot(ny)$, respectively.

In the work [2], the Frenet frame of spacelike Salkowski curve is given as following:

$$\begin{aligned} T(y) &= \begin{pmatrix} \sin(ny)\cos(y) - n\sin(y)\cos(ny), \\ -\sin(y)\sin(ny) - n\cos(y)\cos(ny), -\cos(ny)\,\frac{n}{m} \end{pmatrix}, \\ N(y) &= (\sin(y),\ \cos(y),\ m)\,\frac{n}{m}, \\ B(y) &= \begin{pmatrix} -\cos(y)\cos(ny) - n\sin(y)\sin(ny), \\ \sin(y)\cos(ny) - n\cos(y)\sin(ny),\ -\sin(ny)\,\frac{n}{m} \end{pmatrix}. \end{aligned} \tag{2. 4}$$

After this definition, the equation (2.2) is

$$T' = N, N' = T + \tau B, B' = \tau B. \tag{2. 5}$$

3. Smarandache Curves Constructed from a Spacelike Salkowski Curve

In this chapter, we describe the Smarandache curves constructed from a spacelike Salkowski curve and we calculate Frenet apparatus of the Smarandache curves.

Definition 3.1. Let γ_m be a non planar spacelike Salkowski curve. The Smarandache curve constructed from a spacelike Salkowski curve γ_m is defined by frame vectors as follows (Figure 2):

$$\beta_1(y) = T(y) + N(y). \tag{3. 6}$$

Substituting T and N vectors into the equation (2. 4), we get the curve $\beta_1(y)$ as follow:

$$\beta_1(y) = \begin{pmatrix} \sin(ny)\cos(y) - \sin(y)\cos(ny)n + \sin(y)\frac{n}{m}, \\ -\sin(y)\sin(ny) - \cos(y)\cos(ny)n + \cos(y)\frac{n}{m}, \\ -\cos(ny)\,\frac{n}{m} + n \end{pmatrix}. \tag{3. 7}$$

Theorem 3.2. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and β_1 be a Smarandache curve constructed from the curve γ_m, then Frenet aparatus of the curve β_1 is given by the followings:*

$$\begin{aligned} T_{TN} &= \frac{T+N+\tau B}{|\tau|}, \\ N_{TN} &= \frac{(\tau'-\tau)T + (\tau'-\tau^3-\tau)N - \tau^2 B}{\tau\sqrt{|\tau(\tau^3-2\tau'+\tau)|}}, \\ B_{TN} &= \frac{(\tau^3-\tau')T - \tau' N - \tau^2 B}{|\tau|\sqrt{|\tau(\tau^3-2\tau'+\tau)|}}, \end{aligned}$$

where $\tau \neq 0, \tau^3 - 2\tau' + \tau \neq 0$ and $\kappa = 1$.

Proof. Considering (2. 5), the derivate of the equation (3. 6) is

$$\beta_1{}'(y) = T + N + \tau B. \tag{3. 8}$$

The norm of this equation is

$$\left\|\beta_1{}'(y)\right\| = |\tau|. \tag{3. 9}$$

From the equations (3. 8) and (3. 9), the tangent vector T_{TN} of the Smarandache curve β_1 is obtained by

$$T_{TN} = \frac{T + N + \tau B}{|\tau|}, \tau \neq 0. \tag{3. 10}$$

If we take derivate the equation (3. 8), we get

$$\beta_1{}''(y) = T + \left(\tau^2 + 1\right) N + (\tau + \tau')\, B. \tag{3. 11}$$

From the equations (3. 8) and (3. 11), we find

$$\beta_1{}'(y) \wedge \beta_1{}''(y) = \left(\tau^3 - \tau'\right) T - \tau' N - \tau^2 B. \tag{3. 12}$$

The norm of the equation (3. 12) is

$$\left\|\beta_1{}'(y) \wedge \beta_1{}''(y)\right\| = |\tau| \sqrt{|\tau\left(\tau^3 - 2\tau' + \tau\right)|}. \tag{3. 13}$$

From the equations (3. 12) and (3. 13), the binormal vector B_{TN} of the Smarandache curve β_1 is given

$$B_{TN} = \frac{\left(\tau^3 - \tau'\right) T - \tau' N - \tau^2 B}{|\tau| \sqrt{|\tau\left(\tau^3 - 2\tau' + \tau\right)|}}, \tau \neq 0, \tau^3 - 2\tau' + \tau \neq 0. \tag{3. 14}$$

From the equations (3. 10) and (3. 14), the principal normal vector N_{TN} of the Smarandache curve β_1 is found

$$N_{TN} = \frac{\left(\tau' - \tau\right) T + \left(\tau' - \tau^3 - \tau\right) N - \tau^2 B}{\tau \sqrt{|\tau\left(\tau^3 - 2\tau' + \tau\right)|}}, \tau \neq 0, \tau^3 - 2\tau' + \tau \neq 0. \tag{3. 15}$$

Theorem 3.3. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the first curvature and the second curvature of the Smarandache curve β_1 constructed from a spacelike Salkowski curve γ_m are*

$$\kappa_{TN} = \frac{\sqrt{|\tau\left(\tau^3 - 2\tau' + \tau\right)|}}{\tau^2} \text{ and } \tau_{TN} = \frac{3\tau'^2 - \tau'\tau - \tau''\tau}{\tau\left|\tau\left(\tau^3 - 2\tau' + \tau\right)\right|}, \tag{3. 16}$$

where $\tau \neq 0, \tau^3 - 2\tau' + \tau \neq 0$ and $\kappa = 1$.

Proof. Considering the equations (2. 1), (3. 9) and (3. 13), the curvature κ_{TN} of the Smarandache curve β_1 is found

$$\kappa_{TN} = \frac{\sqrt{|\tau\left(\tau^3 - 2\tau' + \tau\right)|}}{\tau^2}, \tau \neq 0.$$

If we take derivate the equation (3. 11), we get

$$\beta_1{}'''(y) = \left(1 + \tau^2\right) T + \left(\tau^2 + 3\tau\tau' + 1\right) N + \left(\tau^3 + \tau + \tau' + \tau''\right) B. \tag{3. 17}$$

Considering the equations (3. 8), (3. 11), (3. 13) and (3. 17), the torsion τ_{TN} of the Smarandache curve β_1 is obtained by

$$\tau_{TN} = \frac{3\tau'^2 - \tau'\tau - \tau''\tau}{\tau\left|\tau\left(\tau^3 - 2\tau' + \tau\right)\right|}, \ \tau \neq 0, \ \tau^3 - 2\tau' + \tau \neq 0.$$

Definition 3.4. Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the Smarandache curve constructed from a spacelike Salkowski curve γ_m is defined by frame vectors as follows (Figure 3):

$$\beta_2(y) = \frac{T(y) + B(y)}{\sqrt{2}}. \tag{3. 18}$$

Substituting T and B vectors into the equation (2. 4), we get the curve β_2 as following:

$$\beta_2(y) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(y)\left(\sin(ny) - \cos(ny)\right) \\ -n\sin(y)\left(\cos(ny) + \sin(ny)\right), \\ -\sin(y)\left(\sin(ny) - \cos(ny)\right) \\ -n\cos(y)\left(\cos(ny) + \sin(ny)\right), \\ -\left(\cos\left(ny\right) + \sin\left(ny\right)\right)\frac{n}{m} \end{pmatrix}. \tag{3. 19}$$

Theorem 3.5. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and β_2 be a the Smarandache curve constructed from the curve γ_m, then Frenet aparatus of the curve β_2 is given by the followings:*

$$T_{TB} = \pm N,\ N_{TB} = \pm\frac{T + \tau B}{\sqrt{\tau^2 + 1}},\ B_{TB} = \frac{-\tau T + B}{\sqrt{\tau^2 + 1}},$$

where $\kappa = 1$.

Proof. Considering (2. 5) in derivate of the equation (3. 18), we get

$$\beta_2{}'(y) = \frac{(1 + \tau)\,N}{\sqrt{2}}. \tag{3. 20}$$

The norm of this equation is

$$\left\|\beta_2{}'(y)\right\| = \frac{|1 + \tau|}{\sqrt{2}}. \tag{3. 21}$$

From the equations (3. 20) and (3. 21), the tangent vector T_{TB} of the Smarandache curve β_2 is found as

$$T_{TB} = \pm N. \tag{3. 22}$$

If we take derivate the equation (3. 20), we find

$$\beta_2{}''(y) = \frac{(1 + \tau)\,T + \tau' N + \tau\,(1 + \tau)\,B}{\sqrt{2}}. \tag{3. 23}$$

From the equations (3. 20) and (3. 23), we get

$$\beta_2{}'(y) \wedge \beta_2{}''(y) = \frac{(1 + \tau)^2\,(-\tau T + B)}{2}. \tag{3. 24}$$

The norm of the equation (3. 24) is obtained by

$$\left\|\beta_2{}'(y) \wedge \beta_2{}''(y)\right\| = \frac{(\tau + 1)^2\sqrt{\tau^2 + 1}}{2}. \tag{3. 25}$$

From the equations (3. 24) and (3. 25), the binormal vector B_{TB} of the Smarandache curve β_2 is found as

$$B_{TB} = \frac{-\tau T + B}{\sqrt{\tau^2 + 1}} \tag{3. 26}$$

and from the equations (3. 22) and (3. 26), the principal normal vector N_{TB} of the curve β_2 is obtained by

$$N_{TB} = \pm\frac{T + \tau B}{\sqrt{\tau^2 + 1}}. \tag{3. 27}$$

Theorem 3.6. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the curvature and the torsion of the Smarandache curve β_2 constructed from a spacelike Salkowski curve γ_m are*

$$\kappa_{TB} = \frac{\sqrt{2}\sqrt{\tau^2+1}}{|\tau+1|} \text{ and } \tau_{TB} = \frac{\sqrt{2}\tau'}{(\tau+1)(\tau^2+1)}, \tag{3. 28}$$

where $\tau \neq -1$ and $\kappa = 1$.

Proof. Considering the equations (3. 21), (3. 25) and (2. 1), the curvature κ_{TN} of the Smarandache curve β_2 is

$$\kappa_{TB} = \frac{\sqrt{2}\sqrt{\tau^2+1}}{|\tau+1|}, \tau \neq -1.$$

If we take derivate the equation (3. 23), we found

$$\beta_2{}'''(y) = \frac{2\tau' T + \left(\tau^3+\tau^2+\tau+1+\tau''\right) N + \tau'(1+3\tau) B}{\sqrt{2}}. \tag{3. 29}$$

From the equations (3. 20), (3. 23), (3. 25) and (3. 29), the torsion of the Smarandache curve β_2 is obtained by

$$\tau_{TB} = \frac{\sqrt{2}\tau'}{(\tau+1)(\tau^2+1)}, \tau \neq -1.$$

Definition 3.7. Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the Smarandache curve constructed from a spacelike Salkowski curve γ_m is defined by frame vectors as follows (Figure 4):

$$\beta_3(y) = N(y) + B(y). \tag{3. 30}$$

Substituting N and B vectors into the equation (2. 4), we get the curve β_3 as following:

$$\beta_3(y) = \begin{pmatrix} -\cos(y)\cos(ny) - \sin(y)\sin(ny)n + \sin(y)\frac{n}{m}, \\ \sin(y)\cos(ny) - \sin(ny)\cos(y)n + \cos(y)\frac{n}{m}, \\ n - \sin(ny)\frac{n}{m} \end{pmatrix}. \tag{3. 31}$$

Theorem 3.8. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and β_3 be a Smarandache curve constructed from the curve γ_m, then Frenet aparatus of the curve β_3 is given by the followings:*

$$\begin{aligned} T_{NB} &= T + \tau N + \tau B, \\ N_{NB} &= -\frac{\tau T + \left(\tau^2+\tau'+1\right)N + \left(\tau^2+\tau'\right)B}{\sqrt{|\tau^2+2\tau'+1|}}, \\ B_{NB} &= \frac{\tau T - \tau' N - \left(1+\tau'\right)B}{\sqrt{|\tau^2+2\tau'+1|}}, \end{aligned}$$

where $\tau^2 + 2\tau' + 1 \neq 0$ and $\kappa = 1$.

Proof. If we take derivate the equation (3. 30), we obtain

$$\beta_3{}'(y) = T + \tau N + \tau B. \tag{3. 32}$$

The norm of this equation is found as

$$\left\|\beta_3{}'(y)\right\| = 1. \tag{3. 33}$$

From the equations (3. 32) and (3. 33), the tangent vector T_{NB} of the Smarandache curve β_3 is

$$T_{NB} = T + \tau N + \tau B. \tag{3. 34}$$

If we take derivate the equation (3. 32), we have

$$\beta_3{}''(y) = \tau T + \left(\tau^2 + \tau' + 1\right) N + \left(\tau' + \tau^2\right) B. \tag{3. 35}$$

From the equations (3. 32) and (3. 35), we found

$$\beta_3{}'(y) \wedge \beta_3{}''(y) = \tau T - \tau' N - (1 + \tau') B. \tag{3. 36}$$

The norm of the equation (3. 36) is

$$\left\|\beta_3{}'(y) \wedge \beta_3{}''(y)\right\| = \sqrt{|\tau^2 + 2\tau' + 1|}. \tag{3. 37}$$

From the equations (3. 36) and (3. 37), the binormal vector B_{NB} of the Smarandache curve β_3 is found by

$$B_{NB} = \frac{\tau T - \tau' N - (\tau' + 1) B}{\sqrt{|\tau^2 + 2\tau' + 1|}}, \tau^2 + 2\tau' + 1 \neq 0. \tag{3. 38}$$

From the equations (3. 34) and (3. 38), the principal normal vector N_{NB} of the curve β_3 is obtained

$$N_{NB} = -\frac{\tau T + \left(\tau^2 + \tau' + 1\right) N + \left(\tau^2 + \tau'\right) B}{\sqrt{|\tau^2 + 2\tau' + 1|}}, \tau^2 + 2\tau' + 1 \neq 0. \tag{3. 39}$$

Theorem 3.9. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the curvature and the torsion of the Smarandache curve β_3 constructed from a spacelike Salkowski curve γ_m are*

$$\begin{aligned} \kappa_{NB} &= \sqrt{|\tau^2 + 2\tau' + 1|}, \\ \tau_{NB} &= \frac{\tau\left(\tau^2+1+2\tau'\right)-\tau'\left(\tau^3+2+3\tau\tau'+\tau''\right)-\left(1+\tau'\right)\left(\tau^2+1+2\tau\tau'+\tau''+\tau'\right)}{|\tau^2+2\tau'+1|}, \end{aligned} \tag{3. 40}$$

where $\tau^2 + 2\tau' + 1 \neq 0$ and $\kappa = 1$.

Proof. Considering the equations (2. 1), (3. 33) and (3. 37), the curvature of the curve β_3 is found by

$$\kappa_{NB} = \sqrt{|\tau^2 + 2\tau' + 1|}.$$

If we take derivate the equation (3. 35), we get

$$\begin{aligned} \gamma'''{}_{NB}(y) = &\left(\tau^2 + 1 + 2\tau'\right) T + \left(\tau^3 + \tau + 3\tau\tau' + \tau''\right) N \\ &+ \left(\tau^3 + 1 + 2\tau\tau' + \tau'' + \tau'\right) B. \end{aligned} \tag{3. 41}$$

From the equations (3. 32), (3. 35), (3. 37) and (3. 42), the torsion of the curve β_3 is obtained by

$$\begin{aligned} \tau_{NB} &= \frac{\tau\left(\tau^2+1+2\tau'\right)-\tau'\left(\tau^3+2+3\tau\tau'+\tau''\right)-\left(1+\tau'\right)\left(\tau^2+1+2\tau\tau'+\tau''+\tau'\right)}{|\tau^2+2\tau'+1|}, \\ &\tau^2 + 2\tau' + 1 \neq 0. \end{aligned} \tag{3. 42}$$

Definition 3.10. Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the Smarandache curve constructed from a spacelike Salkowski curve γ_m is defined by frame vectors as follows (Figure 5):

$$\beta_4(y) = T(y) + N(y) + B(y). \tag{3. 43}$$

Substituting T, N and B vectors into the equation (2. 4), we get the curve β_4 as following:

$$\beta_4(y)=\left(\begin{array}{l}\cos(y)\left(\sin(ny)-\cos(ny)\right)-n\sin(y)\left(\cos(ny)+\sin(ny)\right)\\+\sin(y)\frac{n}{m},\\\sin(y)\left(\cos(ny)-\sin(ny)\right)-n\cos(y)\left(\cos(ny)+\sin(ny)\right)\\+\cos(y)\frac{n}{m},\\n-\frac{n}{m}\left(\cos\left(ny\right)+\sin\left(ny\right)\right)\end{array}\right). \quad (3.\ 44)$$

Theorem 3.11. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T,N,B,\kappa,\tau\}$ and β_4 be a the Smarandache curve constructed from the curve γ_m, then Frenet aparatus of the curve β_4 is given by the followings:*

$$\begin{aligned}T_{TNB}&=\tfrac{T+(1+\tau)N+\tau B}{\sqrt{2|\tau|}},\\N_{TNB}&=\tfrac{(\tau+1)T+\left(\tau^2+1\right)N+\tau(\tau+1)B}{\sqrt{2|\tau|(\tau^2+1)}},\\B_{TNB}&=\tfrac{-\tau T+B}{\sqrt{\tau^2+1}},\end{aligned}$$

where $\tau\neq0$ and $\kappa=1$.

Proof. If we take derivate the equation (3. 43), we get

$$\beta_4{}'(y)=T+(\tau+1)\,N+\tau B. \quad (3.\ 45)$$

The norm of this equation is

$$\left\|\beta_4{}'(y)\right\|=\sqrt{2\,|\tau|}. \quad (3.\ 46)$$

From the equations (3. 45) and (3. 46), the tangent vector T_{TNB} of the Smarandache curve β_4 is found

$$T_{TNB}=\frac{T+(1+\tau)\,N+\tau B}{\sqrt{2\,|\tau|}},\tau\neq0. \quad (3.\ 47)$$

The derivate of the equation (3. 45) is

$$\beta_4{}''(y)=(\tau+1)\,T+\left(\tau^2+\tau'+1\right)N+\tau\,(\tau+1)\,B. \quad (3.\ 48)$$

From the equations (3. 45) and (3. 48), it is found

$$\beta_4{}'(y)\wedge\beta_4{}''(y)=(2\tau-\tau')\,(-\tau T+B)\,. \quad (3.\ 49)$$

The norm of this equation (3. 49) is

$$\left\|\beta_4{}'(y)\wedge\beta_4{}''(y)\right\|=(2\tau-\tau')\sqrt{\tau^2+1}. \quad (3.\ 50)$$

From the equations (3. 49) and (3. 50), the binormal vector B_{TNB} of the curve β_4 is found by

$$B_{TNB}=\frac{-\tau T+B}{\sqrt{\tau^2+1}}. \quad (3.\ 51)$$

From the equations (3. 47) and (3. 51), the principal normal vector N_{TNB} of the curve β_4 is obtained by

$$N_{TNB}=\frac{(\tau+1)\,T+\left(\tau^2+1\right)N+\tau\,(\tau+1)\,B}{\sqrt{2\,|\tau|\,(\tau^2+1)}},\tau\neq0. \quad (3.\ 52)$$

Theorem 3.12. *Let γ_m be a non planar spacelike Salkowski curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, then the curvature and the torsion of the Smarandache curve β_4 constructed from a spacelike Salkowski curve γ_m are*

$$\kappa_{TNB} = \frac{(2\tau - \tau')\sqrt{\tau^2+1}}{|2\tau|^{\frac{3}{2}}} \text{ and } \tau_{TNB} = \frac{\tau'(\tau+1)}{(2\tau-\tau')(\tau^2+1)}, \tag{3. 53}$$

where $\tau \neq 0, 2\tau - \tau' \neq 0$ and $\kappa = 1$.

Proof. Considering the equations (2. 1), (3. 46) and (3. 50) the curvature of the Smarandache curve β_4 is found

$$\kappa_{TNB} = \frac{(2\tau - \tau')\sqrt{\tau^2+1}}{|2\tau|^{\frac{3}{2}}}, \tau \neq 0.$$

The derivate of the equation (3. 48) is

$$\begin{aligned} \beta_4'''(y) &= \left(\tau^2 + 2\tau' + 1\right) T + \left(\tau^3 + \tau^2 + \tau + 3\tau\tau' + \tau'' + 1\right) N \\ &+ \left(\tau^3 + \tau + 3\tau\tau' + \tau'\right) B. \end{aligned} \tag{3. 54}$$

From the equations (3. 45), (3. 48), (3. 50) and (3. 54), the torsion of the curve β_4 is obtained

$$\tau_{TNB} = \frac{\tau'(\tau+1)}{(2\tau-\tau')(\tau^2+1)}.$$

Corollary 3.13. *The Smarandache curves β_1 and β_3 constructed from a spacelike Salkowski curve γ_m are spacelike curves with a timelike principal normal.*

Proof. Considering Theorems 3.2 and Theorem 3.8, proof is easily seen.

Corollary 3.14. *The Smarandache curves β_2 and β_4 constructed from a spacelike Salkowski curve γ_m are timelike curves.*

Proof. Considering Theorem 3.5 and Theorem 3.11, proof is easily seen.

Corollary 3.15. *The Smarandache curve β_2 is evolute of spacelike Salkowski curve with a timelike principal normal γ_m.*

Proof. Considering the equations (2. 4) and (3. 22), we get $\langle T, T_{TB}\rangle = \langle T, N\rangle = 0$. In that case, we call that the Smarandache curve β_2 is evolute of spacelike Salkowski curve γ_m.

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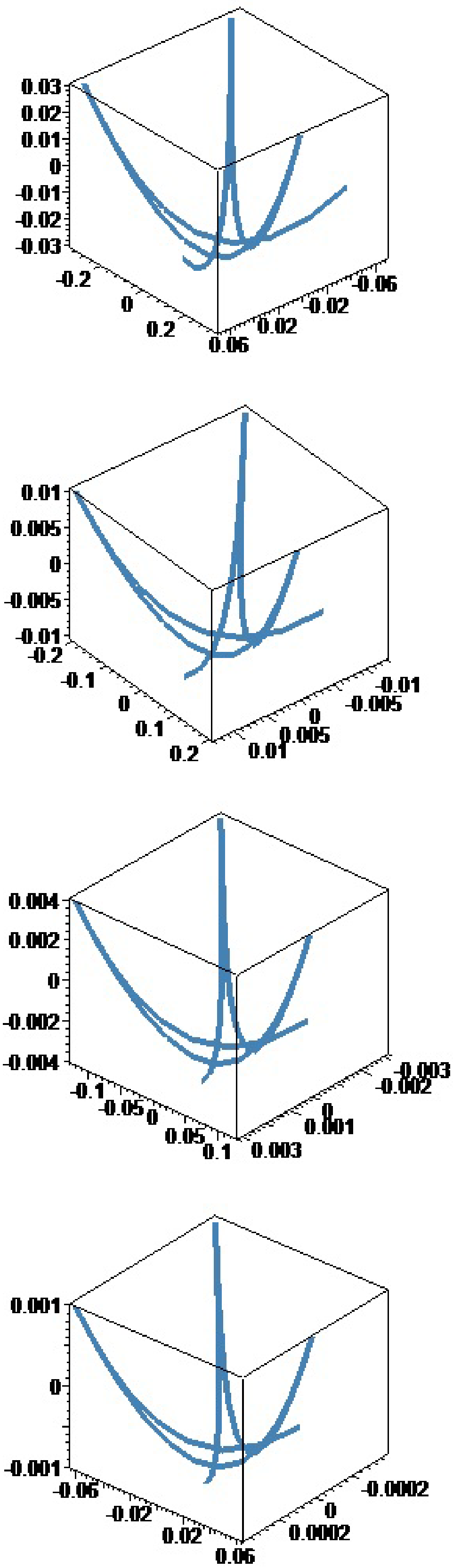

FIGURE 1. $m = \{3, 5, 8, 16\}$ and $y \in [-5, 5]$ for the spacelike Salkowski curve γ_m

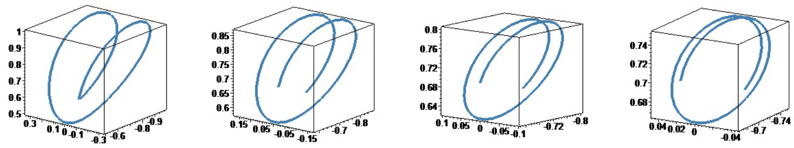

FIGURE 2. $m = \{3, 5, 8, 16\}$ and $y \in [-5, 5]$ for the Smarandache curve β_1

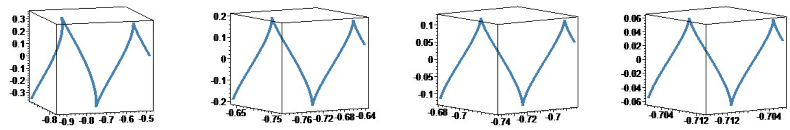

FIGURE 3. $m = \{3, 5, 8, 16\}$ and $y \in [-5, 5]$ for the Smarandache curve β_2

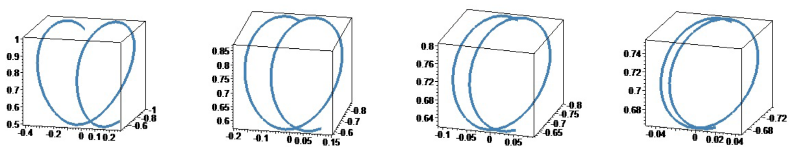

FIGURE 4. $m = \{3, 5, 8, 16\}$ and $y \in [-5, 5]$ for the Smarandache curve β_3

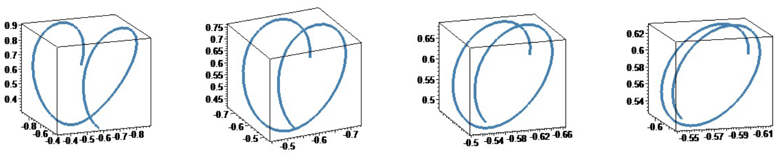

FIGURE 5. $m = \{3, 5, 8, 16\}$ and $y \in [-5, 5]$ for the Smarandache curve β_4